2-Numerical Methods for the Advection Equation

$$\frac{\partial q}{\partial t} + a\frac{\partial q}{\partial x} = 0$$

The Advection Equation: Theory

- 1st order partial differential equation (PDE) in (x,t):

$$\frac{\partial q(x,t)}{\partial t} + a(x,t)\frac{\partial q(x,t)}{\partial x} = 0$$

- Hyperbolic PDE: information propagates across the domain at finite speed → method of characteristics
- Characteristic are the solutions of the equation

$$\frac{dx}{dt} = a(x,t)$$

- So that, along each characteristic, the solution satisfies

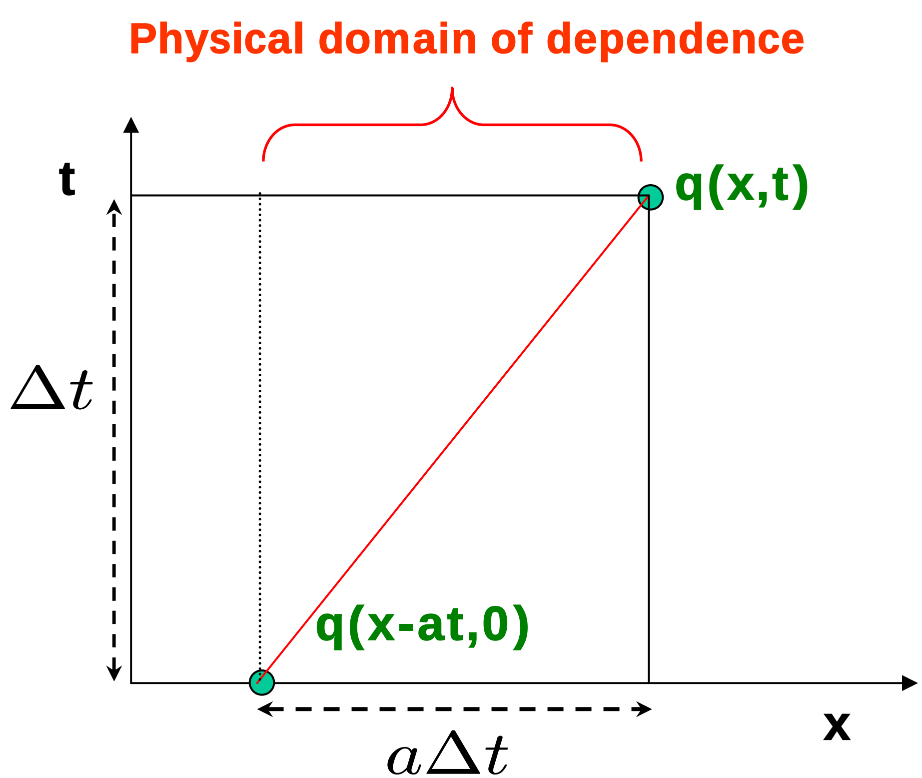
$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + \frac{dx}{dt}\frac{\partial q}{\partial x} = 0$$

The Advection Equation: Theory

$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + \frac{dx}{dt}\frac{\partial q}{\partial x} = 0 \; , \qquad \text{with} \qquad \frac{dx}{dt} = a$$

- The solution is constant along the characteristic curves. The solution at the point (x,t) is found by tracing the characteristic back to some inital point (x,0).

- This defines the *physical domain of dependence*

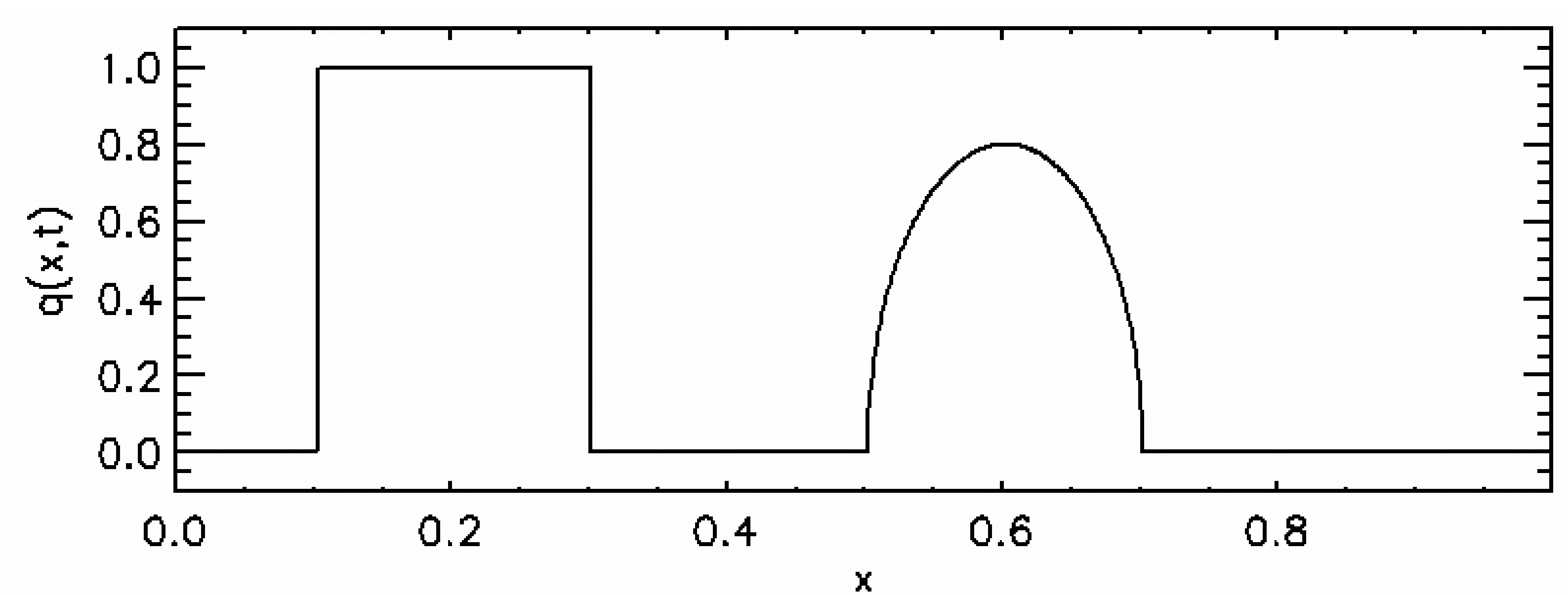


The Advection Equation: Theory

- If a is constant: characteristics are straight parallel lines and the solution to the PDE is a uniform translation of the initial profile:

$$q(x,t) = \phi(x - at)$$

where $\phi(x) = q(x,0)$ is the initial condition



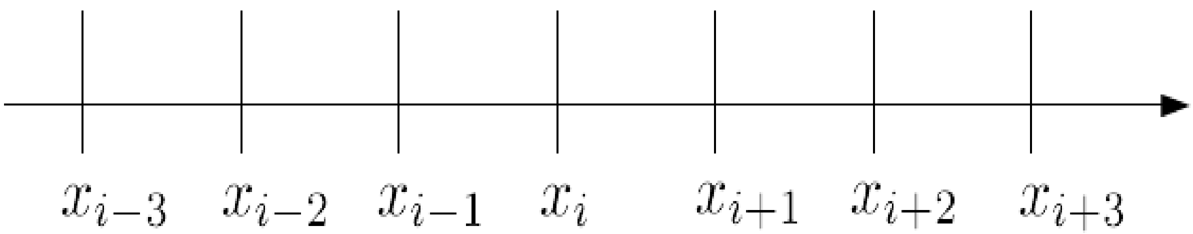
Numerical Methods for the Linear Advection Equation

$$\frac{\partial q}{\partial t} + a\frac{\partial q}{\partial x} = 0$$

- 2 popular methods for performing discretization:
  - Finite Differences
  - Finite Volume
- For some problems, the resulting discretizations look identical, but they are distinct approaches.
- We begin using finite-difference as it will allow us to quickly learn some important ideas

Linear Advection Equation: Finite Difference

- A finite-difference method stores the solution at specific points in space and time.



- Associated with each grid point is a function value,

$$q_i = q(x_i)$$

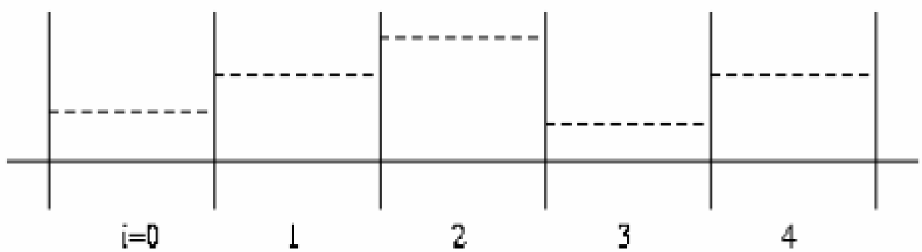
- We replace the derivatives in out PDEs with differences between neighboring points.

Linear Advection Equation: Finite Volumes

- In a finite volume discretization, the unknown is the average value of the function:

$$\langle q \rangle_i = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} q(x) dx$$

where $x_{i-1/2}$ is the position of the left edge zone i



- Solving out conservation laws involves computing fluxes through the boundaries of these control volumes.

Linear Advection Equation:

- We start with the linear advection equation

$$\frac{\partial q(x,t)}{\partial t} + a\frac{\partial q(x,t)}{\partial x} = 0$$

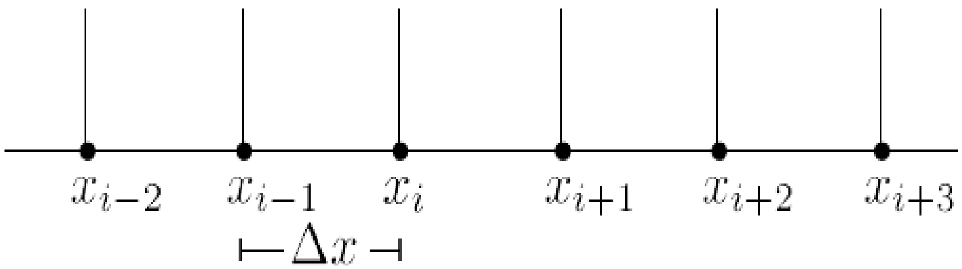
- with initial conditions (i.c.) $q(x,0) = q_0(x)$

- and boundary conditions (b.c.) $\begin{cases} q(0,t) = q_l(t) \\ q(L,t) = q_r(t) \end{cases}$

- Actually, only one b.c. is needed since this is a 1st order equation. Which boundary depends on the sign of a.

Linear Advection Equation:

- We use a finite difference mesh:



- We discretize the function q(x,t) by storing its value at each point in the finite-difference grid

$$q_i^n = q(x_i, t^n)$$

- Subscript "i" → grid location
- Superscript "n" → time level
- In addition to discretizing in space, we introduce time discretization. Thus $\Delta t^n = t^{n+1} - t^n$

Linear Advection Equation:

- We need to approximate the derivatives in our PDE

$$\frac{\partial q(x,t)}{\partial t} + a\frac{\partial q(x,t)}{\partial x} = 0$$

- In time, we use fwd derivative $\frac{\partial q(x,t)}{\partial t} \approx \frac{q_i^{n+1} - q_i^n}{\Delta t}$ since we want to use information from the previous time level
- In space, we use centered derivative, since it is more accurate:

$$\frac{\partial q(x,t)}{\partial x} \approx \frac{q_{i+1}^n - q_{i-1}^n}{2\Delta x}$$

Linear Advection Equation:

- Putting all together: $$\frac{q_i^{n+1} - q_i^n}{\Delta t} + a\left(\frac{q_{i+1}^n - q_{i-1}^n}{2\Delta x}\right) = 0$$
- and solving with respect to q_i^{n+1}

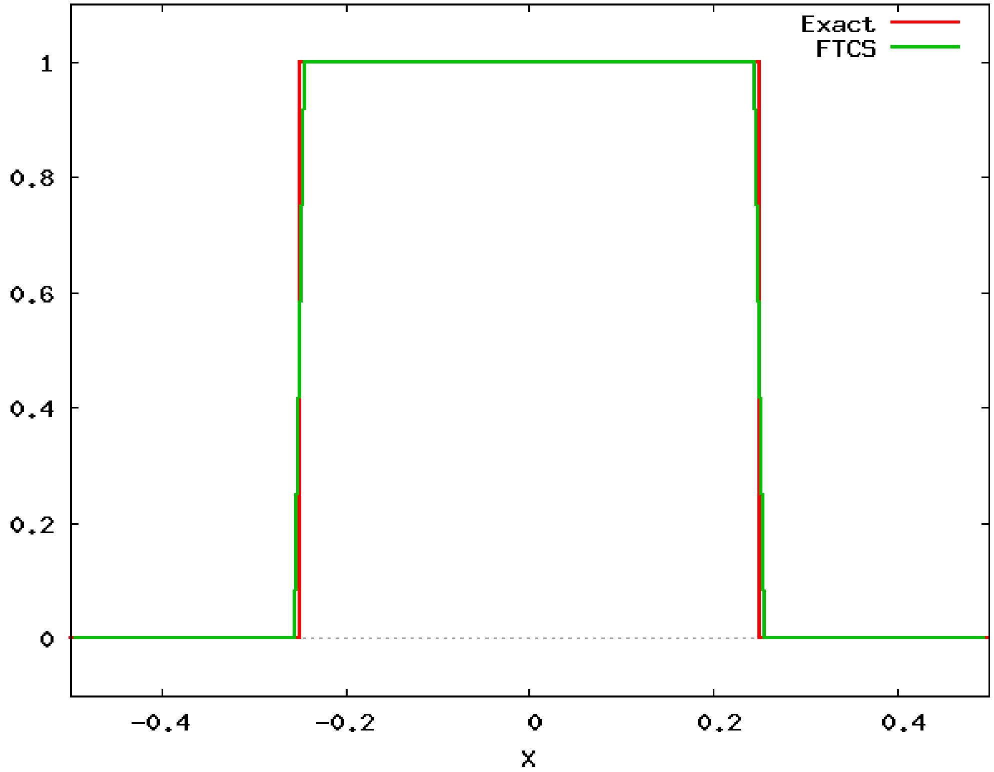
$$q_i^{n+1} = q_i^n - \frac{C}{2}\left(q_{i+1}^n - q_{i-1}^n\right)$$

where $C = a\frac{\Delta t}{\Delta x}$ is called the Courant number or the Courant-Friedrichs-Lewy (CFL) number.

- We call this method *FTCS* for forward in time, center in space.
- The value at the new time level depends only on quantities at the old time step → explicit method

Linear Advection Equation:

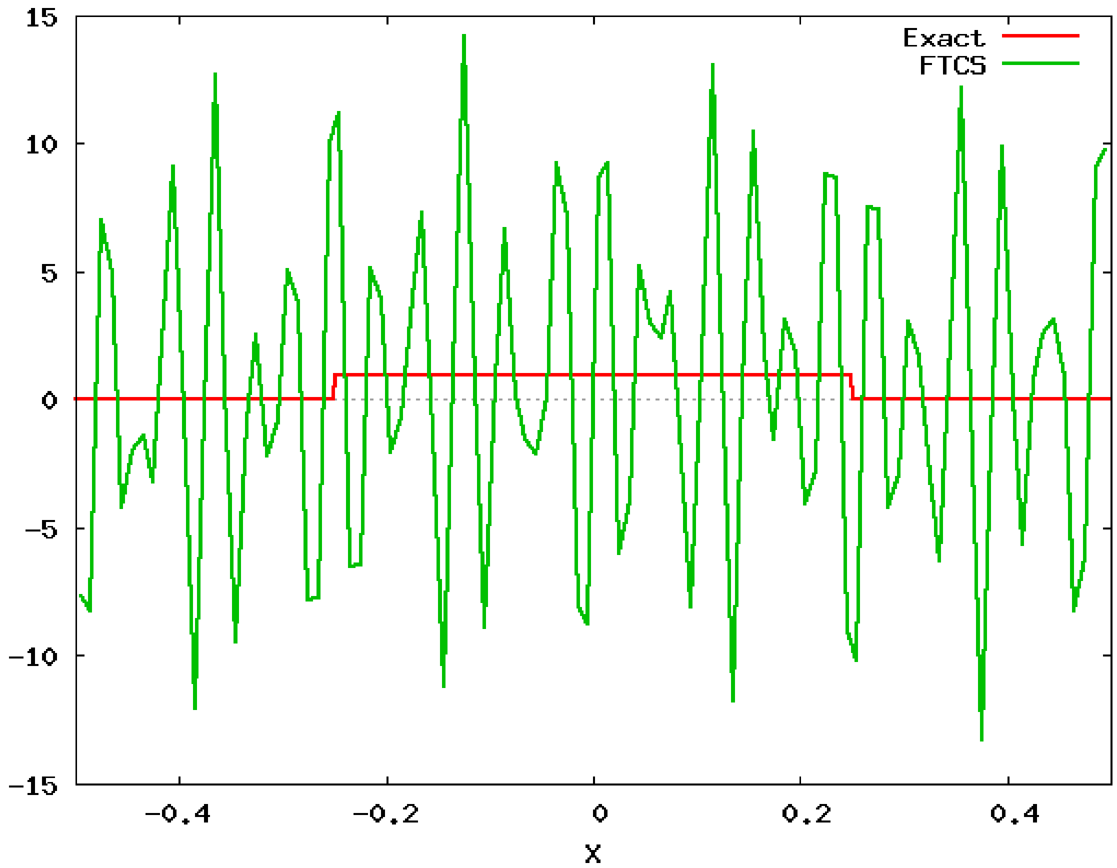
- At t = 0, we prescribe a square pulse:



- and prescribe periodic b.c.

Linear Advection Equation:

- After one period, the solution looks like:



- Oops!! Something isn’t right… *WHY ??*

Linear Advection Equation: stability analysis

- Let's perform an analysis of FTCS by expressing the solution as a Fourier series. Since the equation is linear, we only need to examine the behavior of a single mode. Consider a trial solution of the form:

$$q_i^n = A^n e^{Ii\theta}\,, I = (-1)^{1/2}\,, \theta = k\Delta x$$

- This is a spatial Fourier expansion. Plugging in the difference formula:

$$q_i^{n+1} = q_i^n - \frac{C}{2}\left(q_{i+1}^n - q_{i-1}^n\right) \;\rightarrow\; A^{n+1} = A^n - \frac{C}{2}A^n\left(e^{I\theta} - q^{-I\theta}\right)$$

Linear Advection Equation: stability analysis

- Defining the amplification factor $\boxed{\frac{A^{n+1}}{A^n}}$ one obtains

$$\frac{A^{n+1}}{A^n} = 1 - \frac{C}{2}\left(e^{I\theta} - e^{-I\theta}\right) = 1 - IC\sin\theta$$

- A method is well-behaved or *stable* if $\left|\frac{A^{n+1}}{A^n}\right| \le 1$
- But for FTCS one gets $\left|\frac{A^{n+1}}{A^n}\right| = 1 + C^2\sin^2\theta \ge 1$
- Indipendently of the CFL number all Fourier modes increase in magnitude as time advances.
- This method is *unconditional unstable!!*.

Linear Advection Equation:

- Let's try a different approach. Consider the backward derivative:

$$\frac{\partial q(x,t)}{\partial x} \approx \frac{q_i^n - q_{i-1}^n}{\Delta x}$$

- Let's apply the von Neumann stability analysis on the resulting discretized equation:

$$\frac{q_i^{n+1} - q_i^n}{\Delta t} + a\left(\frac{q_i^n - q_{i-1}^n}{\Delta x}\right) = 0 \quad \text{with} \quad q_i^n = A^n e^{Ii\theta}$$

- Solving for the amplification factor gives

$$\frac{A^{n+1}}{A^n} = 1 - C + C\cos\theta - I\sin\theta$$

Linear Advection Equation:

- Taking the norm, $\left|\dfrac{A^{n+1}}{A^n}\right| = 1 - 2C(1-C)(1-\cos\theta)$
- Recall that for stability one needs $\left|\dfrac{A^{n+1}}{A^n}\right| \leq 1$
- But $1 - \cos\theta \geq 0$ so the stability condition is met when

$$2C(1-C) \geq 0$$

- Recalling the definition $C = a\dfrac{\Delta t}{\Delta x}$, one has for a > 0

$$0 \leq a\frac{\Delta t}{\Delta x} \leq 1$$

Condition for stability

Linear Advection Equation:

- Since the advection speed a is a parameter of the equation, Δx is fixed from the grid, this is a constraint on the time step:

$$\Delta t \leq \frac{\Delta x}{a}$$

- Δt cannot be arbitrarily large.

- In the case of nonlinear equations, the speed can vary in the domain and the maximum of a should be considered.

Linear Advection Equation:

- Repeating the argument for the fwd derivative,

$$\frac{q_i^{n+1} - q_i^n}{\Delta t} + a\left(\frac{q_{i+1}^n - q_i^n}{\Delta x}\right) = 0 \quad \text{with} \quad q_i^n = A^n e^{Ii\theta}$$

- Gives

$$\left|\frac{A^{n+1}}{A^n}\right| = 1 + 2C(1 - C)(1 - \cos\theta)$$

- If a > 0, the method will always be *unstable*
- However, if a is negative, then this method is *stable* and the previous is unstable.

Linear Advection Equation: What Have We Learned ?

- The stable method is the one with the difference that makes use of the grid point where information is coming from.

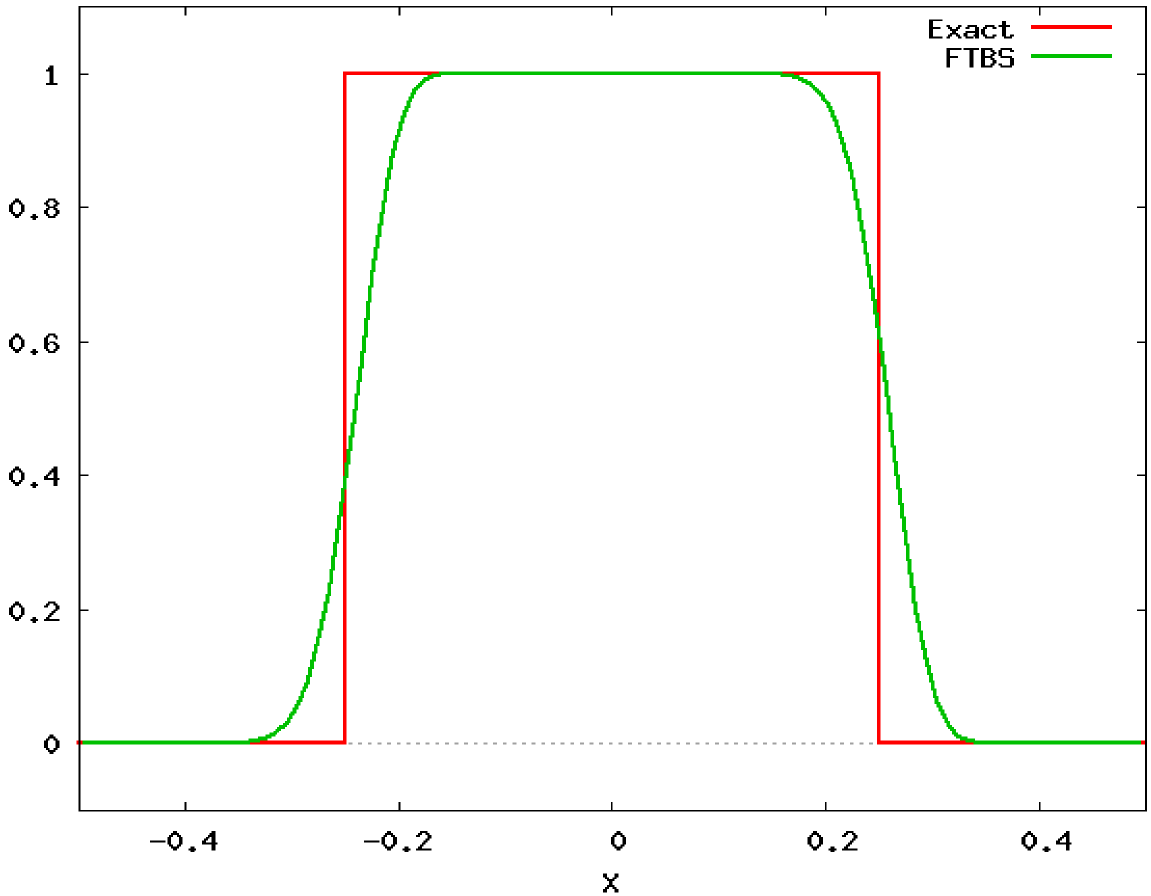
- This type of discretization goes under the name "upwind":

  - **For a > 0 we want** $$q_i^{n+1} = q_i^n - \frac{a\Delta t}{\Delta x}\left(q_i^n - q_{i-1}^n\right)$$
  - **The a < 0 we want** $$q_i^{n+1} = q_i^n - \frac{a\Delta t}{\Delta x}\left(q_{i+1}^n - q_i^n\right)$$

- This is the *first-order Godunov Method*.

Linear Advection Equation:

- After one period, the solution looks like:



- Much better now…
- *But we still see some smearing…*

Equivalent Advection/Diffusion Equation

- A discretized P.D.E gives the exact solution to an equivalent equation with a diffusion term:

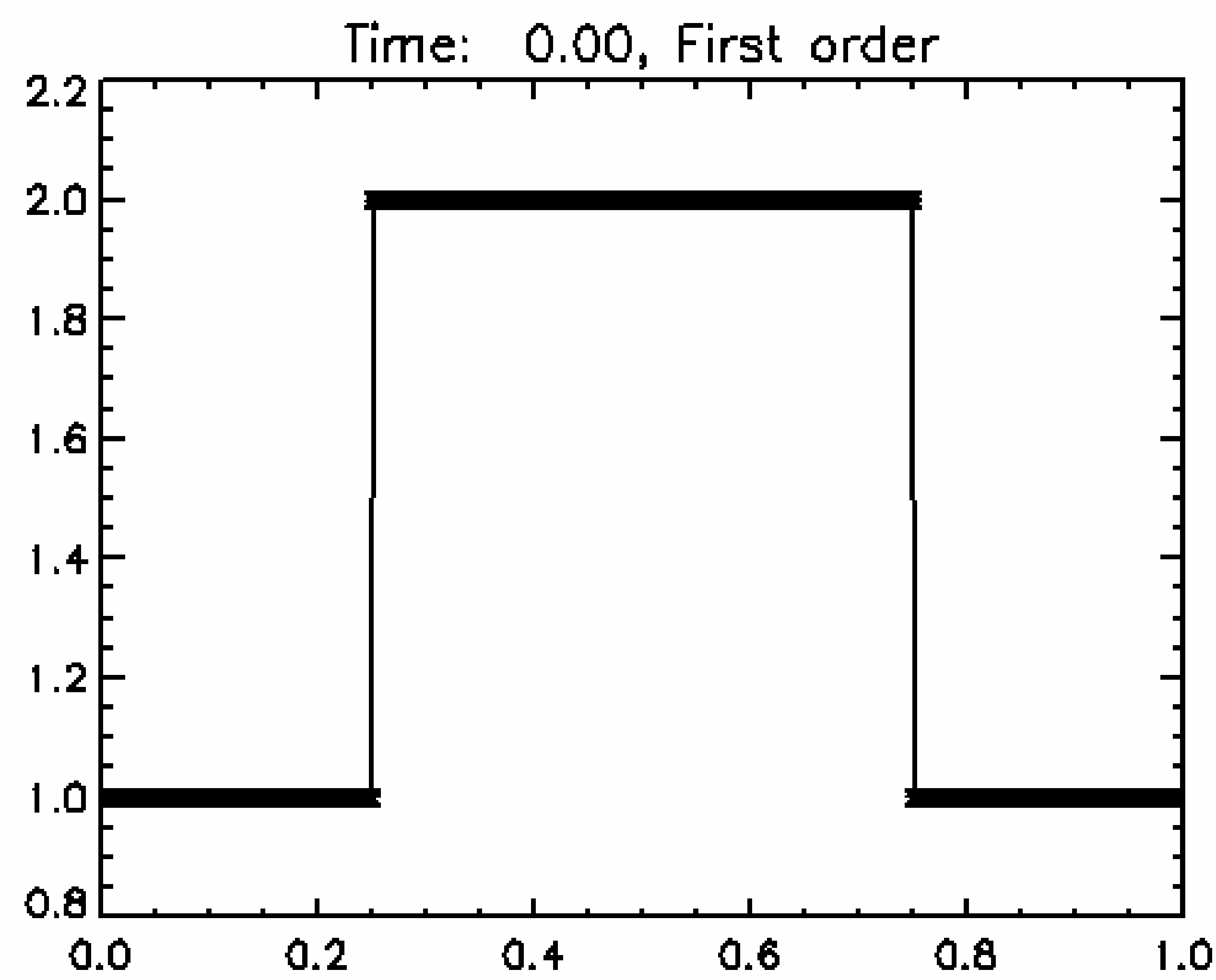
- Consider $$\frac{\partial q}{\partial t} + a\frac{\partial q}{\partial x} = 0\,, \quad a > 0$$

  - discretize w/ upwind $$\frac{q_i^{n+1} - q_i^n}{\Delta t} + a\frac{q_i^n - q_{i-1}^n}{\Delta x} = 0$$

  - do Taylor expansion on q_i^{n+1} and q_{i-1}^n
  - The solution to the discretized equation is **<u>also</u>** the solution of

$$\frac{\partial q}{\partial t} + a\frac{\partial q}{\partial x} = \frac{a\Delta x}{2}\left(1 - a\frac{\Delta t}{\Delta x}\right)\frac{\partial^2 q}{\partial x^2} + H.O.T.$$

Linear Advection Equation:



Linear Advection Equation: Conservative Form

- Godunov method can be cast in conservative form, i.e.

$$q_i^{n+1} = q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$$

by defining the "flux" function

$$\boxed{F_{i+1/2}^n = \frac{a}{2}\left(q_{i+1}^n + q_i^n\right) - \frac{|a|}{2}\left(q_{i+1}^n - q_i^n\right)}$$

- In fact for a > 0, one has $$q_i^{n+1} = q_i^n - \frac{a\Delta t}{\Delta x}\left(q_i^n - q_{i-1}^n\right)$$

- and for a < 0 $$q_i^{n+1} = q_i^n - \frac{a\Delta t}{\Delta x}\left(q_{i+1}^n - q_i^n\right)$$

C Implementation

- Look → advection.c