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UNSATURATED FLOW IN POROUS MEDIA

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UNSATURATED FLOW IN POROUS MEDIA

LECTURE NOTES

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5.3 UNSATURATED FLOW IN RIGID SOILS

By the term rigid soil we designate soils that do not change their bulk volume with a change of water content. We assume that unsaturated flow in soils is governed by the same laws that apply to saturated flow. For unsaturated flow we must consider the fact that a portion of the soil pores filled by air could indeed resaturate or drain. In our discussion of unsaturated flow, capillarity will be quoted as well as the term capillary rise frequently used in the literature. However, general mathematical formulations of physical phenomena should be independent of such simplifying ideas as soil capillaries and consequently, when we mention capillarity, it is just for the sake of modeling approximately some effects occurring in real soils.

5.3.1 Darcy-Buckingham Equation

A simple example of unsaturated flow demonstrated in Fig. 5.8 is analogous to the examples of experiments with saturated flow. The cylinder containing the soil has small openings within its walls leading to the atmosphere. Semipermeable membranes, permeable to water but not to air, separate the soil from free water on both sides of the cylinder. The pools of water are connected to the cylinder with flexible tubes. Full saturation of the soil is first achieved when both pools, lifted to the highest point of the soil, displace the soil air through the openings on the top side of the cylinder. At this moment, there is no flow in the system and the soil is assumed water saturated. With the pool on the left side of the cylinder lowered to the position h_1 and the pool on the right side to position h_2, air enters into the soil through the openings as the soil starts to drain in a manner similar to a soil placed on a tension plate apparatus. The soil on the left side of the cylinder will be drained to a lesser extent than that on the right side with the soil water content distribution from left to right being nonlinear. Although water flows from the left pool to the right pool, the rate of flow is reduced significantly compared with that when the soil is water saturated. If the water level in each of the pools is kept at a constant elevation with time, steady flow will eventually be reached with the water content at each point within the soil remaining invariant. At this time, the flux density q will depend upon the hydraulic gradient and be governed by an equation similar to (5.3)

$$q = -K \frac{\Delta h}{L} \tag{5.30}$$

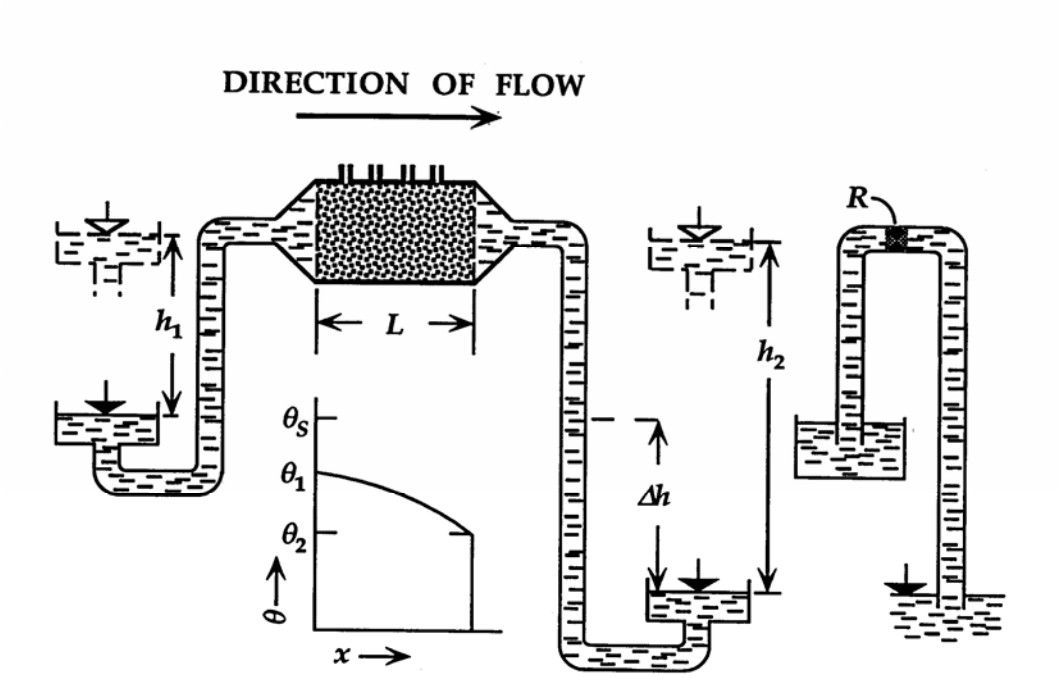

Fig. 5.8. A simple steady flow experiment on an unsaturated soil column. At the right, an analogy of flow in a syphon with an installed resistance R.

where K is the unsaturated hydraulic conductivity [LT^{-1}]. Inasmuch as the soil is not saturated and flow occurs primarily in those pores filled with water, the value of K will be smaller than that of K_S for the same soil. As for saturated flow we commonly take the potential related to the weight of water, i.e. in units of pressure head. For the majority of practical problems, all components of the total potential except those of gravity and soil water are neglected. Hence, (5.30) rewritten to allow the hydraulic conductivity to be a function of the soil water potential head h is

$$q = -K(h)\frac{dH}{dz} \tag{5.31}$$

and for two and three dimensional problems

$$q = -K(h)\,grad\,H \tag{5.32}$$

Equation (5.33) is equivalent to Darcy's equation, and because Buckingham (1907) was the first to describe unsaturated flow dependent upon the potential gradient, equations such as (5.31) and (5.32) are called Darcy-Buckingham equations. The unsaturated hydraulic conductivity K is physically dependent upon the soil water content θ because water flow is realized primarily in pores filled with water. Because the relationship $\theta(h)$ is strongly influenced by hysteresis, $K(h)$ is strongly hysteretic. On the other hand, it follows from percolation theory that $K(\theta)$ is only mildly hysteretic.

Examples of $K(\theta)$ and $K(h)$ demonstrated in Fig. 5.9, show that the more permeable soil at saturation does not necessarily keep its greater permeability throughout the entire unsaturated region. It is also evident in Fig. 5.9 that the hysteretic behavior of $K(h)$ demands that, for a given value of h, the value of K is greater for drainage than for wetting.

The Darcy-Buckingham equation is adequate for describing unsaturated flow only if the soil water content is not changing in time. Unfortunately, this is seldom the case. When θ and q alter in time, we must combine (5.31) with the equation of continuity. The equation of continuity relates the time rate of change of θ to the spatial rate of change of q in a small

elemental volume of soil. The resulting differential equation is strongly non-linear and its solution even for simple conditions is most difficult. Generally, (5.31) is in itself not satisfactory for the solution of such hydrologically important processes as evaporation, infiltration, drainage, subsurface flow etc. Exceptional situations or highly simplified flow conditions are usually the only problems described by the sole use of (5.31).

5.3.2 Unsaturated Hydraulic Conductivity

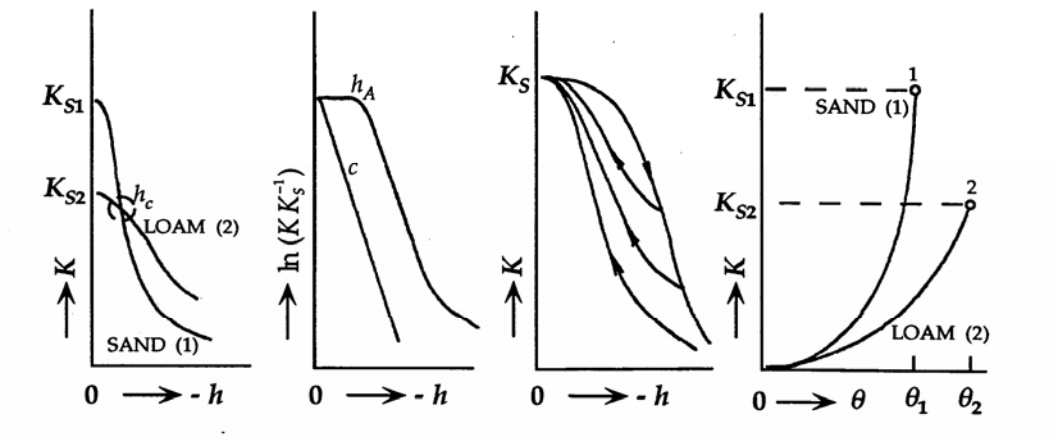

Fig. 5.9. Dependence of unsaturated hydraulic conductivity K upon negative pressure head h (strongly hysteretic) and upon soil water content θ.

We distinguish two approaches for a physical interpretation of the measured hydraulic conductivity K. The first is based on the direct application of the Kozeny equation. The second uses the soil water retention curve to quantify the pore size distribution. With this quantification the Kozeny equation is used for sub-groups of pores. In addition to these two physical approaches, empirical formulations of $K(h)$ are used to merely express observed relationships.

Let us first apply the Kozeny equation. Inasmuch as only a portion of the pores is filled with water in an unsaturated soil, we replace the porosity P by the soil water content θ. We assume that the value of the tortuosity τ is described by Corey (1954, quoted by Corey, 1977) as

$$\frac{\tau_S}{\tau(\theta)} \approx \left(\frac{\theta - \theta_r}{P - \theta_r}\right)^2 \tag{5.33}$$

which is valid for sands where τ_S is the tortuosity in the saturated soil, $\tau(\theta)$ the tortuosity in the soil having water content θ and θ_r a residual water content. When it is assumed that the tortuosity owing to a change of soil water content is ignored in (5.17), Leibenzon (1947) derived the following expression

$$\frac{K}{K_S} = \left(\frac{\theta - \theta_r}{P - \theta_r}\right)^n \tag{5.34}$$

where the exponent n should have values ranging from 3.3 to 4. Averianov (1949) proposed that $n = 3.5$ is a good robust estimate. The value of exponent n is related to the pore size distribution and thus to the soil water retention curve SWRC, see Brooks and Corey (1964) who recommended $n = 2/\lambda + 3$ with λ read from (4.42) when P is replaced by θ_S in (5.34).

Later on, Russo and Bresler (1980) found that values of 1 or 2 fit better than 3 in the exponent n. Childs and Collis-George (1950) obtained the equation

$$K = \alpha \frac{\theta^3}{A_m^2} \tag{5.35}$$

which is comparable to that of Deryaguin et al. (1956)

$$K = \alpha \frac{A_m d^3}{2\mu} \tag{5.36}$$

provided that we assume smooth walls. At small soil water contents and in soils having rough walls, the power function $K \alpha \theta^n$ remains but the exponent $n \neq 3$. A physical interpretation can be obtained using a fractal model of wall roughness (Toledo et al., 1990) The identity of both equations (5.35) and (5.36) is reached when the average thickness of the water film d is taken as functionally dependent upon θ for a given specific surface A_m. In both equations α is an empirical coefficient.

Inasmuch as $\theta(h)$ exists, the dependence of K upon h is also deducible with many empirical formulae quoted in the literature.

Gardner (1958) modified Wind's (1955) empirical proposal

$$K = a\, h^{-m} \tag{5.37}$$

to the relationship

$$K = \frac{a}{|h|^m + b} \tag{5.38}$$

applicable to $h = 0$ where a, b and m are empirical coefficients. Note that for $h = 0$, $a/b = K_S$.

Gardner's exponential relationship (1958)

$$K = K_S \exp(ch) \tag{5.39}$$

is frequently used in analytical solutions. If K/K_S is plotted against h on semi-log paper, a straight line is obtained. This relation usually fits the experimental data well in the range from $h = 0$ ($\theta = \theta_S$) to a certain h_{lim}, see Fig. 5.9, top right. For soils manifesting a distinct air entry value h_A, Gardner and Mayhugh (1958) modified (5.39) to

$$K = K_S \exp\left[c\left(h - h_A\right)\right]. \tag{5.40}$$

The value of the empirical coefficient c with dimension [L^{-1}] is related to soil texture. and most frequently, c = 0.1 to 0.01 cm^{-1}. For δ-function soils in the Green and Ampt approximation of infiltration the value of c is numerically equivalent to the soil water pressure head $|h_f|$ at the wetting front, see Chapter 6. Both (5.39) and (5.40) have been broadly used in analytic and semi-analytic solutions , especially for steady flow problems as we show with some examples in Chapter 6 and as was fully reviewed by Pullan (1990). The reciprocal of c ($= \lambda_c^{-1}$) is sometimes used as one of the soil hydraulic characteristics. In such cases, λ_c is denoted as a microscopic capillary length (Bouwer, 1966; White, 1988).

Because (5.39) and (5.40) are valid in the wet range, (5.38) might be preferred in the dry range. For $h < h_{lim}$, we must use c_2 different from c to extend the applicability of the equation to the dry range. $K(h)$ is often defined as a composite function. For example, the range of $h_A > h > h_{lim}$, (5.39) applies and for $h < h_{lim}$, (5.37) or (5.38) applies in order to simulate the entire soil water regime in some instances.

From studies of capillarity in sands, Brooks and Corey (1964) obtained the frequently used relationship

$$\frac{K}{K_S} = \left(\frac{h_A}{h}\right)^m \tag{5.41}$$

where m depends upon the pore size distribution. Usually, m = 3 to 11.

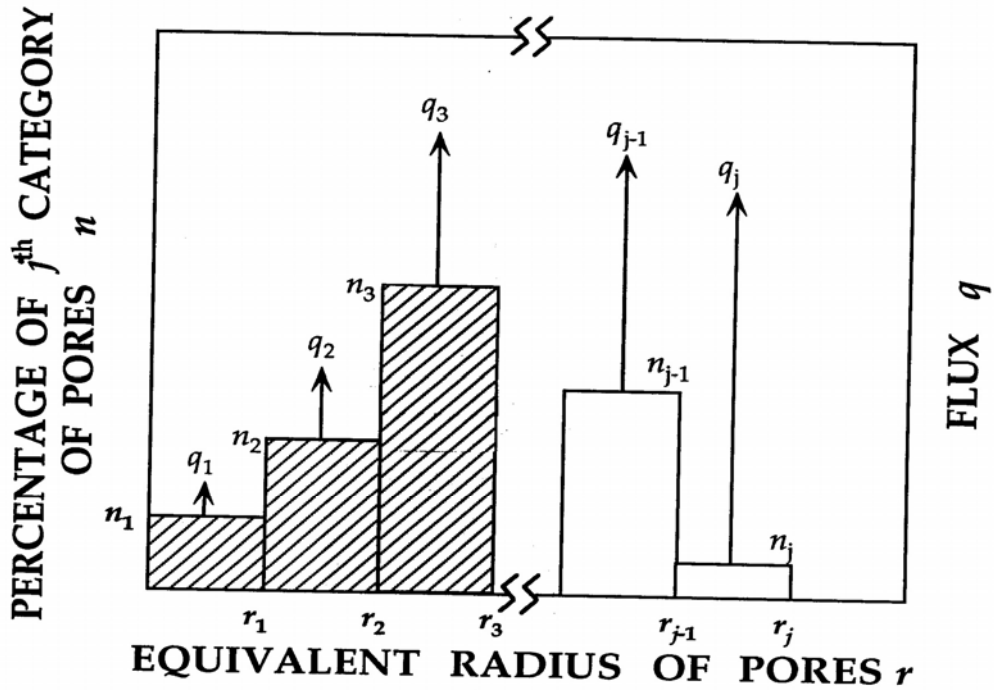

Fig. 5.10. Contribution of soil pore categories with $r_1 < r_2 < \ldots r_j$ and their fluxes $q_1, q_2 \ldots q_j$ to the total flux $q = q_1 + q_2 + \ldots q_j$. If grad H = 1, $q = K$ and $K(r_1) \ldots K(r_3) << K(r_j)$

Physical interpretation of $K(\theta)$ or $K(h)$ must include in addition to the total porosity, the distribution of the pore sizes. Recognizing from (5.10) that the flow rate in a cylindrical capillary is v_p (r^2), drainage of the largest pores drastically reduces the value of K in spite of the relatively small volume of those pores. Childs and Collis-George (1950) were the first to propose a method relating $K(\theta)$ to a pore size distribution function $f(r)$. Using the soil water retention curve to reflect $f(r)$, they obtained (5.35) as a simplified result. Their general approach attracted attention and was further developed and modified. We show those developments here.

In its simplest form the porous system is composed of j categories of pores with j = 1 for the category of smallest pores. In each category the pore radii are in ranges r_{j-1} to r_j. In each category the flux is $q_j(\bar{r}_j^4, n)$ where $\bar{r}_j$ is the mean radius and n is the percentage of the category and frequently $q_{j-1} < q_j$ even if $n_{j-1} > n_j$. Assuming ∇H = - 1, the unsaturated hydraulic conductivity $K = \Sigma q_j$. When the soil is only partially saturated with water, contributions of fluxes q_j, q_{j-1} et al. from the larger , empty pores of radii j, (j-1) et al. do not exist, see Fig. 5.10. In a more exact derivation, we start with the mean flow rate v_p in pores of radius r according to the Hagen-Poiseuille equation

$$v_p(r) = a r^2 I_h \tag{5.42}$$

where $a = \rho_W g/8\mu$, see (5.10). The flux density in a porous system with a continuous distribution function of pores $f(r)$ and a tortuosity τ is

$$q = \frac{1}{\tau}\int_0^r v_p(r) f(r) dr. \tag{5.43}$$

Or, with (5.42) for $I_h = 1$, $q = K$ and

$$K = \frac{1}{\tau}\int_0^r ar^2 f(r)dr\,. \tag{5.44}$$

When $f(r)dr$ is approximated by $d\theta_E(h)$, i.e. by the derivative of the SWRC and for the relation between the pore radius and the pressure head ($r = c/h$), we obtain

$$K = \frac{ac}{\tau}\int_0^{\theta_E} \frac{1}{h^2(\theta_E)} d\theta_E(h). \tag{5.45}$$

For relative hydraulic conductivity K_r

$$K_r = K / K_S \tag{5.46}$$

and with the tortuosity from (5.33) modified to

$$\tau_S / \tau = \theta_E^{\,b}, \tag{5.47}$$

we have

$$K_r = \theta_E^b \int_0^{\theta_E} \frac{d\theta_E}{h^2(\theta_E)} \Big/ \int_0^1 \frac{d\theta_E}{h^2(\theta_E)}\,. \tag{5.48}$$

Various authors have not found a unique interpretation for exponent b in the above equation. Marshall (1958) and Millington and Quirk (1961) defined b as the probability of occurrence of continuous pores. For isotropic and homogeneous media $b = 2P_f$ with P_f denoting that portion of the porosity within which water is flowing. Marshall assumed $P_f = 1$ and hence, b was 2. Millington and Quirk used P_f as 2/3 and hence, b was 4/3. Burdine (1953) interpreting the tortuosity with (5.33) evaluated b as 2.

Inasmuch as the microscopic pore size distribution is used to characterize the macroscopic flux in a soil, Mualem (1976) classified such models as microscopic models. After evaluating about 50 soils on a macroscopic scale, Mualem decided that b was 0.5 and (5.48) is modified to

$$K_r = \theta_E^{0.5}\left[\int_0^{\theta_E} \frac{d\theta_E}{h(\theta_E)} \Big/ \int_0^1 \frac{d\theta_E}{h(\theta_E)}\right]^2. \tag{5.49}$$

If the van Genuchten soil water retention curve (4.43)

$$\theta_E = \frac{1}{\left[1 + (\alpha|h|)^n\right]^m} \tag{5.50}$$

and (5.48) are combined, we obtain

$$K_r(\theta_E) = \theta_E^b \left[1 - \left(1 - \theta_E^{1/m}\right)^m\right]^2 \tag{5.51}$$

with $m = 1 - c/n$, and $n > 1$. These specific relations are suggested for simple evaluation of the integrals in (5.49). For the model of Burdine, $a = 1$, $b = 2$ and $c = 2$. For the model of Mualem, $a = 2$, $b = 0.5$ and $c = 1$. Let us note that Mualem's database consisted mainly of repacked laboratory soils and $b = 0.5$ does not hold for all field soils where the deviation may vary from less than -10 to more than 10 (van Genuchten et al., 1989).

Mualem's model of $K(h)$ is

$$\frac{K(h)}{K_S} = \frac{\left\{1 - (\alpha|h|)^{n-1}\left[1 + (\alpha|h|)^n\right]^{-m}\right\}^2}{\left[1 + (\alpha|h|)^n\right]^{m/2}}\,. \tag{5.52}$$

Similarly, using the soil water retention curve $\theta_E = (h_A/h)^\lambda$ of Brooks and Corey (4.42) the relative hydraulic conductivity is

$$\frac{K(\theta_E)}{K_S} = \theta_E^{\,b+a/\lambda} \tag{5.53}$$

and

$$\frac{K(h)}{K_S} = \left(\frac{h_A}{h}\right)^{a+b/\lambda} \tag{5.54}$$

For the model of Childs and Collis-George, $a = 2$ and $b = 2$. For that of Burdine, $a = 2$ and $b = 3$. For that of Mualem, $a = 2$ and $b = 2.5$.

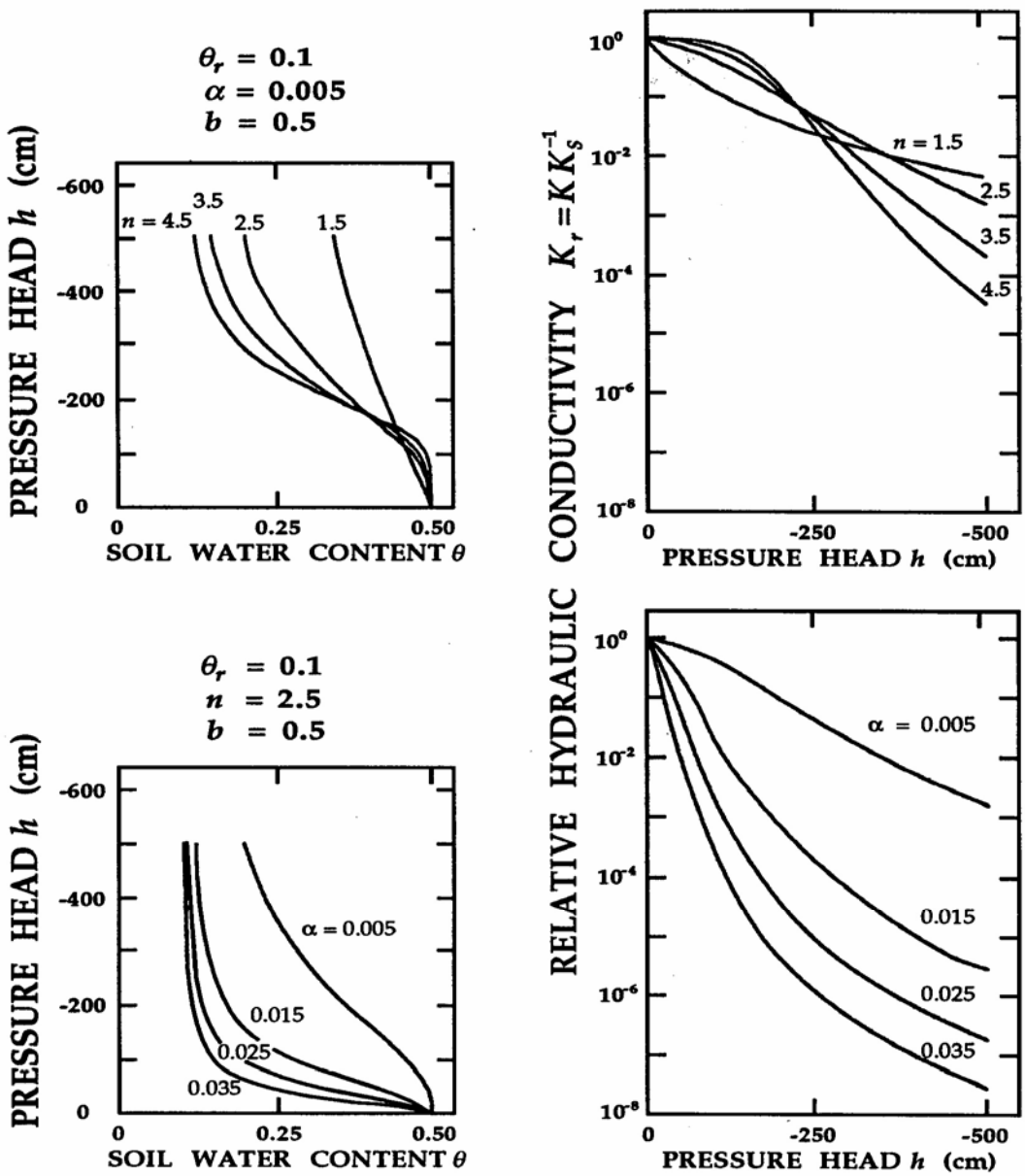

Fig. 5.11. Sensitivity analysis of (5.50) and (5.52) by Wösten and van Genuchten (1988) shows the dependence of h(θ) and K(h) upon parameters α and n.

The exponent $(a + b/\lambda)$ in (5.54) is identical to the exponent m in (5.41). Although the value of the exponent should theoretically be in a narrow range between 2.5 and 4.5, experimental data yield values that extend to about 11. This discrepancy can be explained by the over-simplification of the porous body in the model. In the derivation of the above

equations, several approximations were made. First, the soil porous system was modeled by a bundle of cylindrical capillary tubes. Second, the pore size distribution function was approximated from the soil water retention curve. And third, the value of b was empirically evaluated. However, in spite of these approximations for the derivation of $K(\theta_E)$ and $K(h)$, the most problematic is the proper interpretation of the soil water retention curve close to θ_S.

A formal sensitivity analysis of (5.52) by Wösten and van Genuchten (1988) showed that differences in K_r increase with a decrease in h (i.e. with the soil drying) as the parameter α is altered, see Fig. 5.11. On the other hand, the influence of the exponent n also brings about a great potential error in the wet region. In Fig. 5.11, θ_E has been replaced by θ with the soil water retention curve (4.43) having the form

$$\theta = \theta_r + \frac{\theta_S - \theta_r}{\left[1+\alpha|h|^n\right]^m}. \tag{5.55}$$

Additional sensitivity analyses made by Šír et al. (1985) and Vogel and Císlerová (1988) show the role of an error $\delta h(\theta_E)$ in the experimental determination of $h(\theta_E)$. If $\delta h(\theta_E)$ is a constant in the range $0 \leq \theta_E \leq 1$, the absolute error of $K(\theta)$ rises steeply with an increase of θ_E.

These derived equations are valid for the laboratory Darcian scale assuming microscopic homogeneity of the porous system. The assumption of microscopic homogeneity does not hold if the soil is aggregated, penetrated by plant roots and earthworms or dissected by fissures. In such cases the Darcian scale must be adapted to the pedon scale, i.e. to the reality of a field soil. On the pedon scale, microscopic homogeneity of the porous system may or may not exist.

The hydraulic conductivity function $K(h)$ of soil aggregates is about two orders of magnitude less than that of the bulk soil. The difference in $K(h)$ between aggregates of the soil and bulk soil decreases only slowly in the wet interval for $h > -800$ cm for well developed aggregates (Gunzelmann et al., 1987). In aggregated soils we deal with two different domains of velocity fields. One domain is related to interpedal (structural) pores and characterized by an accelerated flux. The second domain conducts water and solutes at relatively small flow rates and is found in intrapedal (matrix) pores.

When the soil porous system is characterized by a bi-modal pore size distribution curve (Fig. 2.4), the relation $K(h)$ shows two distinct regions. For $0 > h > h_1$, only the by-pass pores belonging to the secondary peak are considered with Mualem's model applied to the soil water retention curve of the inter-pedal (by-pass) pores. For $h < h_1$ we use the remaining portion of the SWRC representing only the intrapedal (matric) pores, see Fig. 5.12 and Othmer et al. (1991). Hence, two matching factors are needed. For the region $0 > h > h_1$ the matching factor is K_S. For the region $h < h_1$ it is a measured value of $K(h < h_1)$. Although this mechanistic separation of the two porous systems uses the same basic equations, the accelerated fluxes through the by-pass pores are conveniently described.

Up to now we have discussed the problems related to $K(\theta)$ in a wet soil. In a dry soil, the probable errors in modeling $K(\theta)$ are related to θ_r. The residual soil water content θ_r in θ_E of (5.50) and further on in other $K(\theta)$ models leads to $K(\theta \leq \theta_r) = 0$. This zero value of hydraulic conductivity for $\theta > 0$ is in agreement with our description of SWRC in Section 4.3 where we used θ_{Wr} to denote the boundary between coherent and incoherent water phase distributions. However, we have shown in the same Section 4.3 that θ_r is obtained as a fitting parameter which we are not allowed to interpret physically. Thus, the physically observed θ_{Wr} in $K(\theta)$ may not coincide with θ_r obtained by fitting (4.42) or (4.43) to experimental

SWRC data. A simple method for independently estimating θ_{Wr} for $K(\theta)$ models has not yet been proposed and tested on a broad scale.

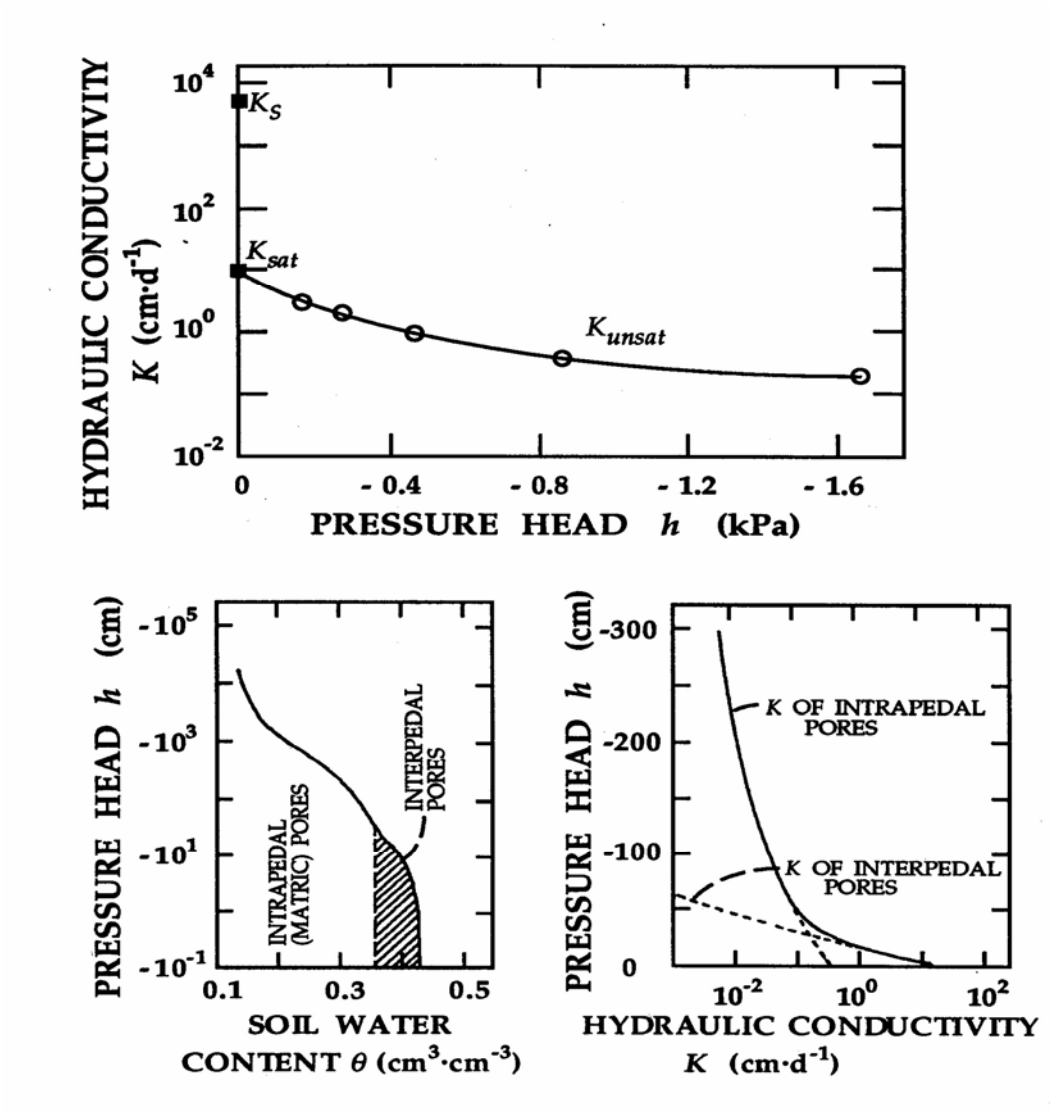

Fig. 5.12. Saturated hydraulic conductivity of a soil with macropores K_S and saturated hydraulic conductivity with macropores excluded K_{sat} which smoothly continues to K(h). Top graph is for Fluvaquent clay (Boolting et al., 1991). Unsaturated hydraulic conductivity K(h) has frequently two branches owing to a bi-modal porosity. The example given in the bottom graphs is from the Bt horizon of a loamy soil (Othmer et al., 1991).

Attempts to physically interpret the unsaturated hydraulic conductivity function lead to more realistic models of porous media than the parallel capillary tube model considered up to now. From soil morphology and from macroscopic measurements of the SWRC, the topological structures within a porous system can be deduced by either percolation theory or procedures of fractal geometry – or by a combination of both. The soil water flux is then related to the flow within individual pores and their fractal dimensions. Subsequently, the flow is formulated by the Hagen-Poiseuille equation with a procedure formally analogous to that developed by Childs and Collis-George (see e.g. Rieu and Sposito, 1991).

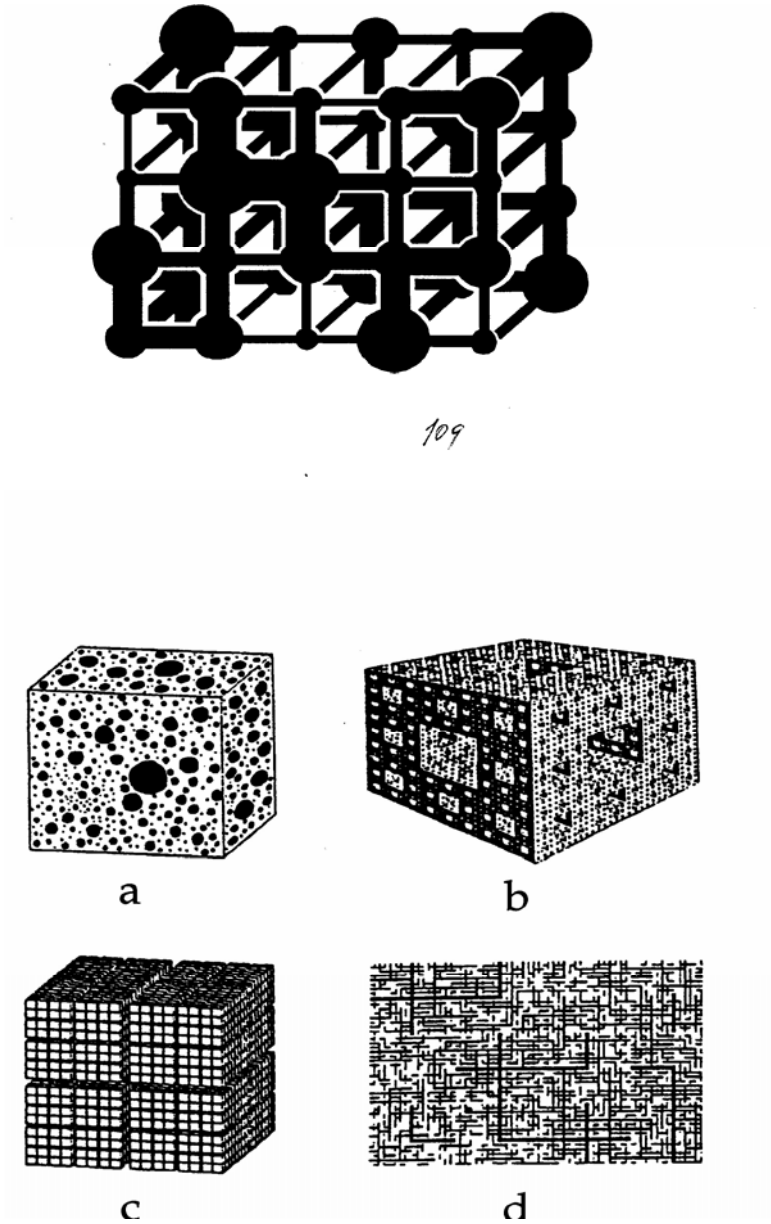

Fig. 5.12.a. Example of percolation (top) and fractal (bottom, a,b,c,d) models for estimation of soil water retention curve and unsaturated hydraulic conductivity.

In addition to capillary pores, soil contains macropores where water is not influenced by meniscus forces. Such macropores originate owing to the growth and decay of plant roots, activities of soil edaphon and shrinkage in loam and clay soils.

Macropores play a special role in the flow of water especially during infiltration. When the soil water pressure is positive or when an unsaturated soil is ponded with water, water flows in the so-called "macropores". The mechanism of the flow in this case may be different from that of the capillary porous system. Water may flow either along the walls of the pores like a thick film, or through the entire cross-sectional area of the pore. When water conduction in cracks is combined with absorption, the kinetic wave approximation (German and Beven, 1985) can be used. The flux density is restricted just to macropores and it is generally reduced by absorption. The theory describes the transformation of both flux density and front velocity when water is transported in macropores as a pulse. It is applicable only

under the provision that the macropores do not change during the transport of water. The theory cannot be used for modeling water flow in the fissures of shrinking-swelling soils.

When the soil is fully water-saturated and the flux exists at positive pressure, the saturated hydraulic conductivity comprises the flux in macropores together with flow in the soil matrix. If the flux in macropores is hindered, or if macropores are absent with the soil matrix not changed and remaining still water-saturated, the resulting value of the hydraulic conductivity may be decreased by two to four orders of magnitude (Boolting et al., 1991; Liu et al., 1994).

In this book, we use the term macropores only for pores without capillarity. In some of the literature a confusion exists inasmuch as coarse capillary pores and by-pass pores are also called macropores just to emphasize the large flux in those pores. However, if capillarity is manifested with the flow realized by the gradient of the negative soil water pressure, the Darcy-Buckingham equation is still appropriate with no need to replace it.

Up to this point we have assumed that Darcy's equation is fully applicable to unsaturated flow. However, when the validity of Darcy's equation is doubted for saturated flow in clays, non-Darcian pre linear flow should be even more pronounced for unsaturated flow in clays. Experiments indicating this possibility (Swarzendruber, 1963) have been theoretically explained (Bolt and Groenevelt, 1969).

The influence of the temperature upon $K(\theta)$ is usually expressed by $\mu_W(T)$ in

$$K(\theta) = K_r(\theta)\, K_p \rho_W g / \mu_W . \qquad (5.56)$$

However, Constanz (1982) provided experimental evidence that in some instances (5.56) was only approximate.

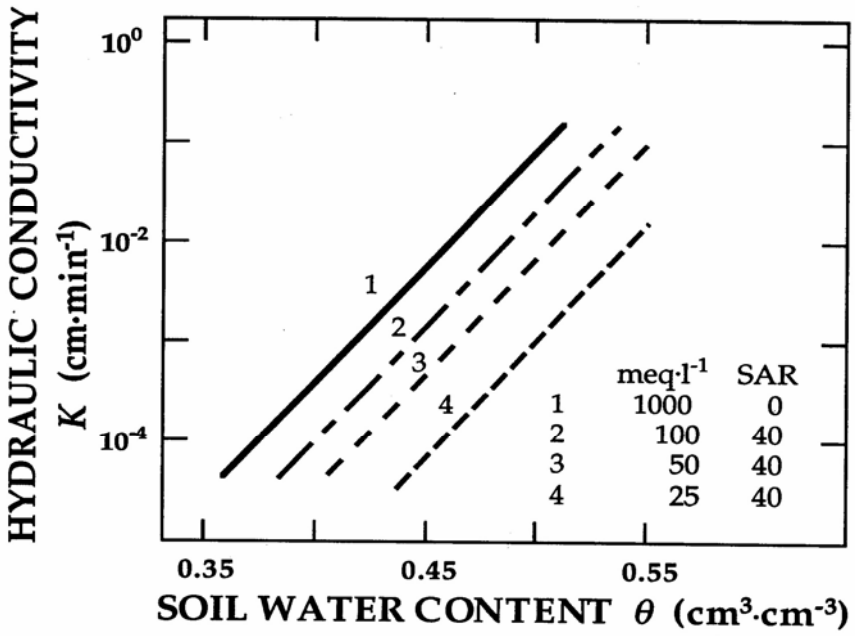

Fig. 5.13. The influence of SAR and the concentration of the soil solution upon $K(\theta)$ according to Dane and Klute (1977).

The influence of the concentration of the soil solution and of the exchangeable cations is similar to that already mentioned for K_S. Dane and Klute (1977) reported that a decrease of the concentration in the soil solution when the SAR was kept constant resulted in roughly the same decrease of K in the whole range of θ, see Fig. 5.13. It is also expected that the function $K(\theta)$ would change with ESP (Kutílek, 1983).

Measuring techniques for determining $K(\theta)$ are usually related to the solution of specified unsteady flow processes, and will be discussed in Chapter 6.

5.3.3 Richards' Equation

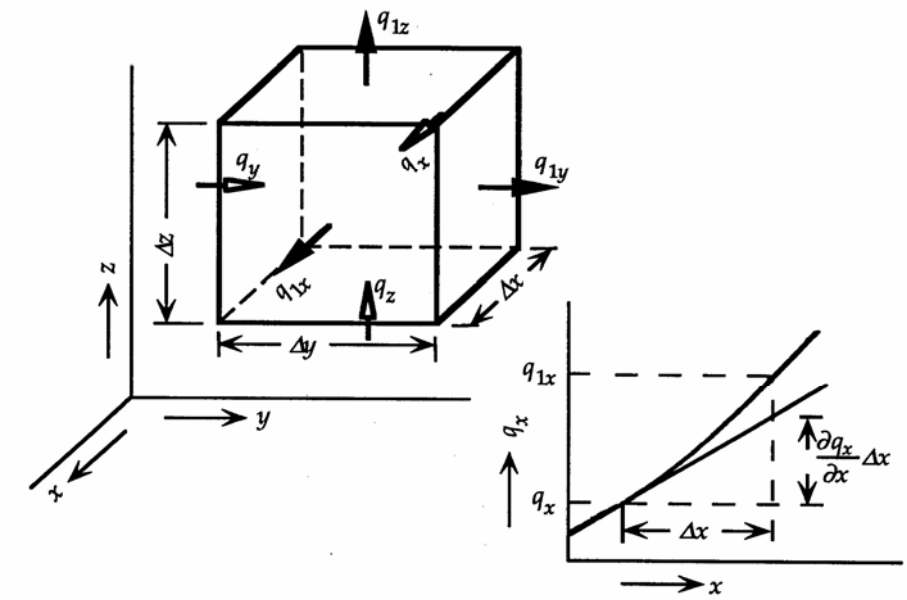

Fig. 5.14. Derivation of the equation of continuity (5.62).

Equation (5.32) is fully applicable to steady unsaturated flow when $\nabla \cdot q = 0$, $dq/dt = 0$ and $d\theta/dt = 0$. In practical situations, unsteady flow frequently exists with $d\theta/dt \neq 0$. In these situations, two equations are needed to describe the flux density and the rate of change of θ in time. The flux density is described by the Darcy-Buckingham equation and the rate of filling or emptying of the soil pores is described by the equation of continuity. Consider the prism element having edges of length Δx, Δy and Δz given in Fig. 5.14. The difference between the volume of water flowing into the element and that flowing out of the element is equal to the difference of water content in the element in time Δt. The rate of inflow (macroscopic) in the direction of the x axis is q_x. If we assume the change in q_x is continuous, the rate of outflow is $[q_x + (\partial q_x/\partial x)\Delta x]$. The inflow volume is $q_x\ \Delta y \Delta z \Delta t$ and the outflow volume is $[q_x + (\partial q_x/\partial x)\Delta x]\Delta y \Delta z \Delta t$. The difference between inflow and outflow volumes is

$$\left\{q_x \Delta y \Delta z \Delta t - \left[q_x + \left(\partial q_x / \partial x\right)\Delta x\right]\Delta x \Delta y \Delta z\right\} \tag{5.57}$$

or

$$-\left(\frac{\partial q_x}{\partial x}\right)\Delta x \Delta y \Delta z \Delta t. \tag{5.58}$$

Similarly in the direction of the y axis, the difference between the inflow and the outflow volumes is

$$-\left(\frac{\partial q_y}{\partial y}\right)\Delta x \Delta y \Delta z \Delta t \tag{5.59}$$

and that in the direction of the z axis

$$-\left(\frac{\partial q_z}{\partial z}\right)\Delta x\Delta y\Delta z\Delta t. \qquad (5.60)$$

The sum of the above differences equals the change of the water content of the element. Provided that $\theta(t)$ has a continuous derivative for $t>0$,

$$\frac{\Delta\theta}{\Delta t}\Delta x\,\Delta y\,\Delta z\,\Delta t = -\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)\Delta x\,\Delta y\,\Delta z\,\Delta t\ . \qquad (5.61)$$

Taking the limit as $t \to 0$, we obtain the equation of continuity

$$\frac{\partial\theta}{\partial t} = -\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)\ . \qquad (5.62)$$

If we insert for q_x, q_y and q_z from (5.32), we have

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left[K(h)\frac{\partial H}{\partial x}\right]+\frac{\partial}{\partial y}\left[K(h)\frac{\partial H}{\partial y}\right]+\frac{\partial}{\partial z}\left[K(h)\frac{\partial H}{\partial z}\right] \qquad (5.63)$$

provided that the soil is isotropic. In one-dimensional form for $H = h + z$ the above equation becomes

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left[K(h)\frac{\partial h}{\partial z}\right]+\frac{\partial K}{\partial z}\ . \qquad (5.64)$$

Equations (5.63) and (5.64) are called Richards' equations in the name of the author who first derived them (1931).

If the soil is either wetting or drying, θ will be uniquely dependent upon only h and

$$\frac{\partial\theta}{\partial t} = \frac{d\theta}{dh}\frac{\partial h}{\partial t}\ . \qquad (5.65)$$

Hence, the capacitance form of Richards' equation is obtained as

$$C_W(h)\frac{\partial h}{\partial t} = \frac{\partial}{\partial z}\left[K(h)\frac{\partial h}{\partial z}\right]+\frac{\partial K}{\partial z} \qquad (5.66)$$

where soil water capacity $C_W = d\theta/dh$ [L^{-1}] is illustrated in Fig. 5.15.

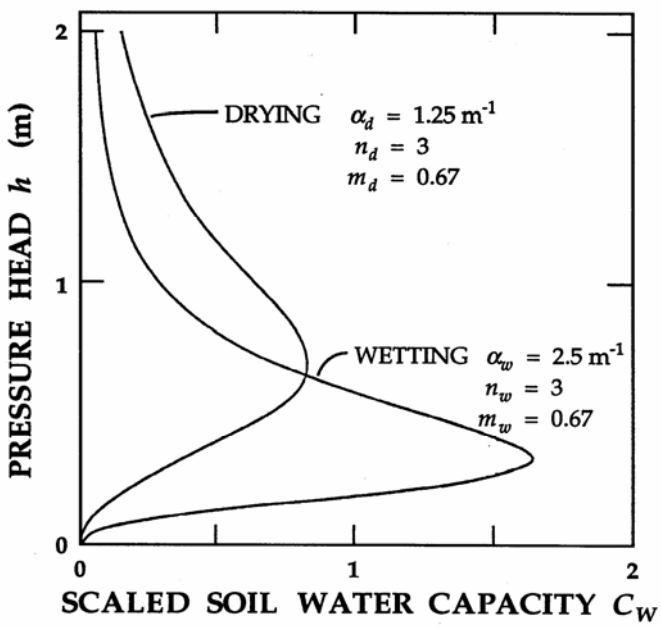

Fig. 5.15. Soil water capacity C_W as a dfunction of pressure head h for drying and wetting (Luckner et al., 1989).

An alternative development using

$$\frac{\partial h}{\partial z} = \frac{dh}{d\theta}\frac{\partial \theta}{\partial z} \tag{5.67}$$

leads to the diffusivity form of Richards' equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial \theta}{\partial z}\right] + \frac{\mathrm{d}K}{\mathrm{d}\theta}\frac{\partial \theta}{\partial z} \tag{5.68}$$

where the Darcy-Buckingham equation has the diffusivity form

$$q = -D(\theta)\frac{\partial \theta}{\partial z} + K(\theta) \tag{5.69}$$

and the soil water diffusivity D is the term derived from

$$D(\theta) = K(\theta)\frac{dh}{d\theta}. \tag{5.69a}$$

The main reason for the derivation of either the capacitance equation (5.66) or the diffusivity equation (5.68) is the reduction of the number of variables from 4 to 3.

Both equations (5.66) and (5.68), strongly non-linear owing to functions $C_W(h)$, $K(h)$ and $D(\theta)$, are sometimes called Fokker-Planck equations. The name of (5.68) was derived from its resemblance (when its second term on the right hand side is omitted) to that for molecular diffusion. The units of D in (5.68) are identical to those of the diffusion coefficient. Many analytical and semi-analytical solutions for the diffusivity equation for various boundary conditions are known from the theories of diffusion (Crank, 1956) and heat flow (Carslaw and Jaeger, 1959). They have been profitably applied for the solution of many processes of unsaturated flow in soils. Whenever there is a region of positive pressure in the soil (5.68) is not applicable and (5.66) should be used.

Sometimes, Kirchhoff's transformation

$$U = \int_{h_0}^{h} K(h)\,dh \tag{5.70}$$

is used with (5.64) to yield

$$\frac{C_W(h)}{K(h)}\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial z^2} - \frac{1}{K(h)}\frac{dK}{dh}\frac{\partial U}{\partial z} \tag{5.71}$$

or

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 U}{\partial z^2} - \frac{\partial K}{\partial z}. \tag{5.72}$$

Because the last term of (5.64), (5.66) and (5.68) originated from the gravitational component z of the total potential H, it is frequently referred to as the gravitational term of the Richards' equations. The first term of the right hand side of each of those equations expresses the flow of water in the soil owing to the gradient of the soil water (matric) potential component h. In some instances when the gravitational term is neglected, the solution of the resulting non-linear diffusion equation with its non-constant diffusivity

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial \theta}{\partial z}\right] \tag{5.73}$$

offers approximate results. If the flow is horizontal, solutions of (5.73) are exact.

5.3.4 Soil Water Diffusivity

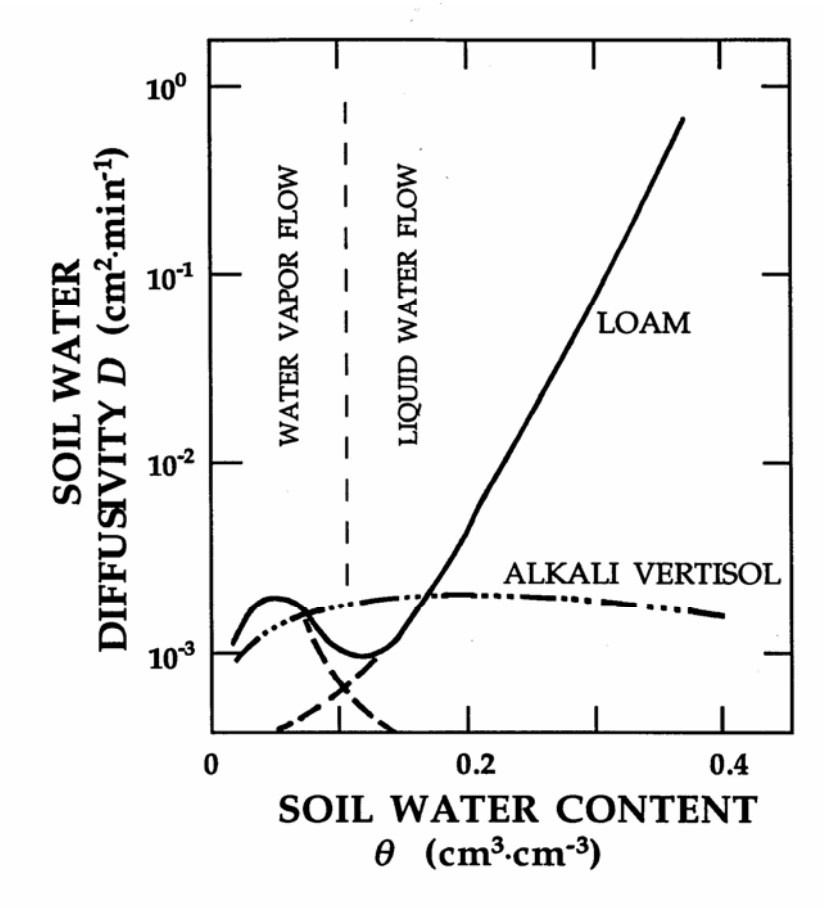

Fig.5.16. Dependence of soil water diffusivity D upon the soil water content θ. For liquid flow the function is nearly exponential for the majority of soils except for swelling clays. The secondary peak in the dry region near θ = 0 is owing to the dominance of water vapor flow.

The most common $D(\theta)$ relationship is demonstrated in Fig. 5.16. With the exception of the region of very small soil water contents less than θ_H ($h < -10^5$ cm), the curve steeply rises with θ. Soil water diffusivity $D(\theta)$ in the wet range above θ_H is typically less steep in its relation to θ as compared with $K(\theta)$. In this wetter range of θ, D changes about five orders of magnitude compared with seven orders of magnitude for K.

In the dry region of $0 \leq \theta < \theta_H$ with a great portion of pores filled with air, water vapor flow is enhanced while liquid water flow is limited to that of very thin water films on the soil solid surfaces. The rate of liquid flow, strongly dependent upon the thickness of the film, has already been demonstrated by (5.36). Here, the vapor flux exceeds the liquid flux. A more detailed discussion on water vapor flux will be given in Section 5.3.5. Now, we shall study in detail the monotonically rising part of $D(\theta)$, i.e. for $\theta > \theta_H$.

Among the well known and frequently used empirical equations is the exponential form (Gardner and Mayhugh, 1958)

$$D = D_o \exp\left[\beta\left(\theta - \theta_o\right)\right] \tag{5.74}$$

where D_O corresponds to θ_O and β ranges approximately between 1 and 30. Or,

$$D = \alpha \exp\left[\beta\left(\theta_E^*\right)\right] \tag{5.75}$$

where θ_r is replaced by θ_H in θ_E^* and at θ_H, $D = \alpha$. A physically more exact equation should be derived from the soil water retention curve and from $K(\theta)$. Using (4.43) and (5.51) in (5.69a), van Genuchten (1980) obtained

$$D(\theta_E) = \frac{K_S(1-m)\theta_E^{1/2-1/m}}{\alpha m\,(\theta_S-\theta_r)}\left[\left(1-\theta_E^{1/m}\right)^{-m} + \left(1-\theta_E^{1/m}\right)^{m} - 2\right]. \quad (5.76)$$

If the simpler (4.42) is used instead of (4.43) we have

$$D(\theta_E) = \frac{K_S\, h_A\, \theta_E^{(b-1)+(a-1)/\lambda}}{\lambda\,(\theta_S-\theta_r)} \quad (5.77)$$

with the values of a and b being those given earlier for (5.53) and (5.54).

In some clays, mainly alkali Vertisols, the value of D decreases with an increase of θ, if the soil is confined and not allowed to swell, see Fig. 5.16 (Kutílek, 1983, 1984). For some undisturbed soils as well as for disturbed repacked soil columns in the laboratory (Clothier and White, 1981), D does not vary as strongly with θ as discussed above. If (D_{max} - D_{min}) is less than one-half an order of magnitude, the linearized form of (5.73)

$$\frac{\partial\theta}{\partial t} = \overline{D}\,\frac{\partial^2\theta}{\partial x^2} \quad (5.78)$$

serves as an excellent approximation where the mean-weighted diffusivity $\overline{D}$ for the wetting process (Crank, 1956) is

$$\overline{D} = \frac{5}{3(\theta_o-\theta_i)^{5/3}}\int_{\theta_i}^{\theta_o}(\theta-\theta_i)^{2/3}D(\theta)d\theta \quad (5.79)$$

and for the drainage process is

$$\overline{D} = \frac{1.85}{(\theta_o-\theta_i)^{1.85}}\int_{\theta_i}^{\theta_o}(\theta-\theta_i)^{0.85}D(\theta)d\theta \quad (5.80)$$

where θ_i is the initial soil water content and θ_o is θ at $x = 0$ for $t > 0$.

Soils manifesting values of D that are constant or nearly so are called "linear soils" because (5.78) is a linear equation. If the Brooks and Corey soil water retention curve (4.42) is used, $K_r(\theta_E)$ is described by (5.53) and $D(\theta_E)$ by (5.77). The condition of a "linear soil" is satisfied if in these equations [λ = - (a - 1)/(b - 1)] or [$a = b = 1$]. If the first condition is applied to the Burdine equation, we obtain $h = h_A\theta_E^2$ and $K_r = \theta_E^{-1}$. Neither of these equations describe physical reality. Similarly, equations of Childs and Collis-George or those of Mualem lead to unacceptable results. The second condition leads to (Kutílek et al., 1985)

$$h = h_A\,\theta_E^{-1/\lambda} \quad (5.81)$$

$$K_r = \theta_E^{1/\lambda+1} \quad (5.82)$$

and

$$D = -\frac{h_A}{\lambda}\frac{K_S}{(\theta_S-\theta_r)} \quad (5.83)$$

This discussion shows the restrictions in the definition of a strictly linear soil when we require that D has a constant value and K_r is linearly dependent upon θ_E. If the second condition is not satisfied, we speak of "linear" soils. In general, there exists a family of "linear soils" described by the above equations. If λ = 1, the hydraulic conductivity function (5.82) is quadratic and meets the requirements of the solutions of Burgers' equation (Clothier et al., 1981).

Concluding, we should keep in mind that the soil water diffusivity is used in Richards' equation in order to reduce the number of variables. It has no direct physical meaning and is only defined mathematically, see (5.69). Moreover, inasmuch as $D(\theta)$ is dependent upon the derivative of the soil water retention curve, it has different values for wetting and drying processes. The temperature dependence of $D(\theta)$ is in accordance with changes of surface tension and viscosity with T. However, its prediction is only approximate owing to some not well understood phenomena that associates the temperature dependence of $h(\theta)$ and $K(\theta)$.

5.3.5 Diffusion of Water Vapor

In section 5.3.4 we have already shown that the relative maximum in the $D(\theta)$ relationship in the dry region is caused by water vapor flow. Indeed, the soil water diffusivity D contains two components: D_L the diffusivity of liquid water, and D_G the diffusivity of water vapor, i.e. the gaseous phase. Hence, $D = D_L + D_G$ (Philip, 1957a). Jackson (1964) derived D_G as analogous to the earlier introduced soil water diffusivity

$$D_G = D_p \frac{d\rho_{rG}}{d\theta} \tag{5.84}$$

where ρ_{rG} is the relative density (concentration) of water vapor and D_p the diffusion coefficient of water vapor in soil which is approximated by

$$D_p = D_a \alpha \left(P - \theta\right)^{\mu} \tag{5.85}$$

where D_a is the diffusion coefficient of water vapor in free air and α and μ are factors that account for the tortuosity and complexity of the soil porous system. Detailed information about (5.85) is provided by Currie (1960).The term $d\rho_{rG}/d\theta$ is actually the slope of the adsorption isotherm and its inflection point corresponds to the relative maximum of $D_G(\theta)$. The water vapor diffusivity rises to this maximum from $\theta = 0$ and reaches this value at a relative water vapor pressure $p/p_o = 0.3$ to 0.4 in the majority of soils. The maximum value of D_G, having a wide range between 10^{-4} to 10^{-3} $cm^2 \cdot s^{-1}$, depends upon soil texture, mineralogy of the clay fraction and organic matter content, see Fig. 5.17. At greater soil water contents D_G decreases as D_L increases. Values of D_L exceed those of D_G at $p/p_o \approx 0.5$ to 0.8. In terms of the average thickness of the adsorbed water films on the soil solid surface, D_G reaches a maximum value after the first molecular layer is completed and before or at least when the second molecular layer is formed. D_L exceeds D_G when about 4 to 6 molecular layers of adsorbed water exist.

For structured soils, (5.84) and (5.85) are far from reality. For more realistic descriptions 2- or 3-modal porosity models should be considered (Currie and Rose, 1985). The effect of such consideration is similar to that of a bi-modal porosity model upon $K(h)$ in the wet region.

It is worthwhile to repeat here at the end of this chapter the principal theoretical gain of all discussions. Without a knowledge of the hydraulic functions of soils – $h(\theta)$, K_S, $K(\theta)$ and $D(\theta)$ – a quantitative description of water flow in soils is not feasible.

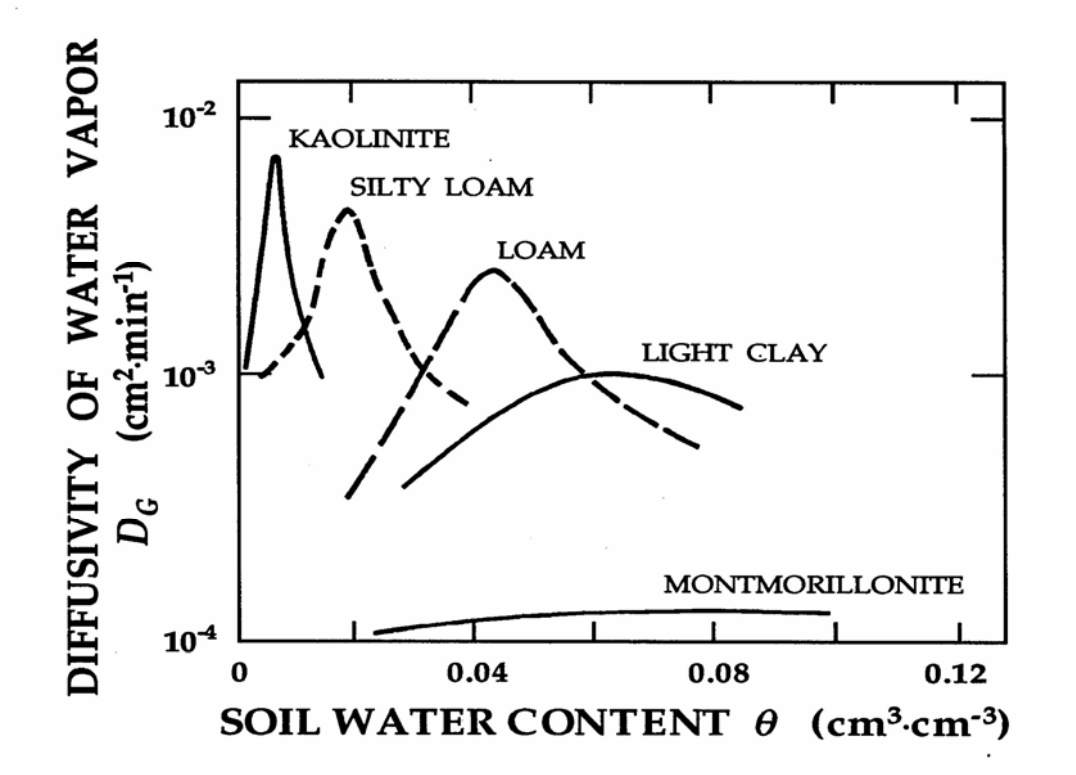

Fig. 5.17. Diffusivity of soil water vapor versus soil water content close to $\theta = 0$ for soils and clay minerals (Kutílek, 1978).