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Dimensional analysis and scaling, with an introduction to fractals 1

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DIMENSIONAL ANALYSIS, SCALING
AND AN INTRODUCTION TO FRACTALS

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- Dimensional analysis refers to the study of the dimensions that characterize physical entities, like mass, force and energy. Classical Mechanics is based on three fundamental entities, with dimensions MLT, the mass M, the length L, and the time T. The combination of these entities gives rise to derived entities, like volume, speed and force, of dimensions L^3, LT^{-1}, MLT^{-2}, respectively. In other areas of Physics, other four fundamental entities are defined, among them the temperature θ and the electrical current I.

- The parameters that characterize physical phenomena are related among themselves by laws, in general of quantitative nature, in which they appear as measures of the considered physical entities. The measure of an entity is the result of its comparison with another one, of the same type, called unit. In this way, an entity (G) is given by two factors, one being the measure (M) and the other the unit (U). When we write V = 50 m^3, V is the entity G, 50 is the ratio between the measures (M), and the unit U is m^3. Therefore:

$$\mathbf{G = M\,(G)\,.\,U\,(G)}$$

Entity (G)	**M (G)**	**U (G)**	**Dimensional symbol**
Area	200	m^2	L^2
Speed	40	$m\ s^{-1}$	LT^{-1}
Force	50	$N = kg\ m\ s^{-2}$	MLT^{-2}
Pressure	1,000	$Pa = kg\ m^{-1}\ s^{-2}$	$ML^{-1}\ T^{-2}$
Flow	5	$m^3\ s^{-1}$	$L^3\ T^{-1}$

To introduce the topic of Dimensional Analysis, let us look at a classical example of the romantic literature, in which Dean Swift, in “The Adventures of Gulliver” describes the imaginary voyages of Lemuel Gulliver to the kingdoms of Liliput and Brobdingnag. In these two places life was identical to that of normal persons, their geometric dimensions were, however, different. In Liliput, man, houses, dogs, trees were twelve times smaller than in the country of Gulliver, and in Brobdingnag, everything was twelve times taller. The man of Liliput was a geometric model of Gulliver in a scale 12:1, and that of Brobdingnag a model in a scale of 1:12.

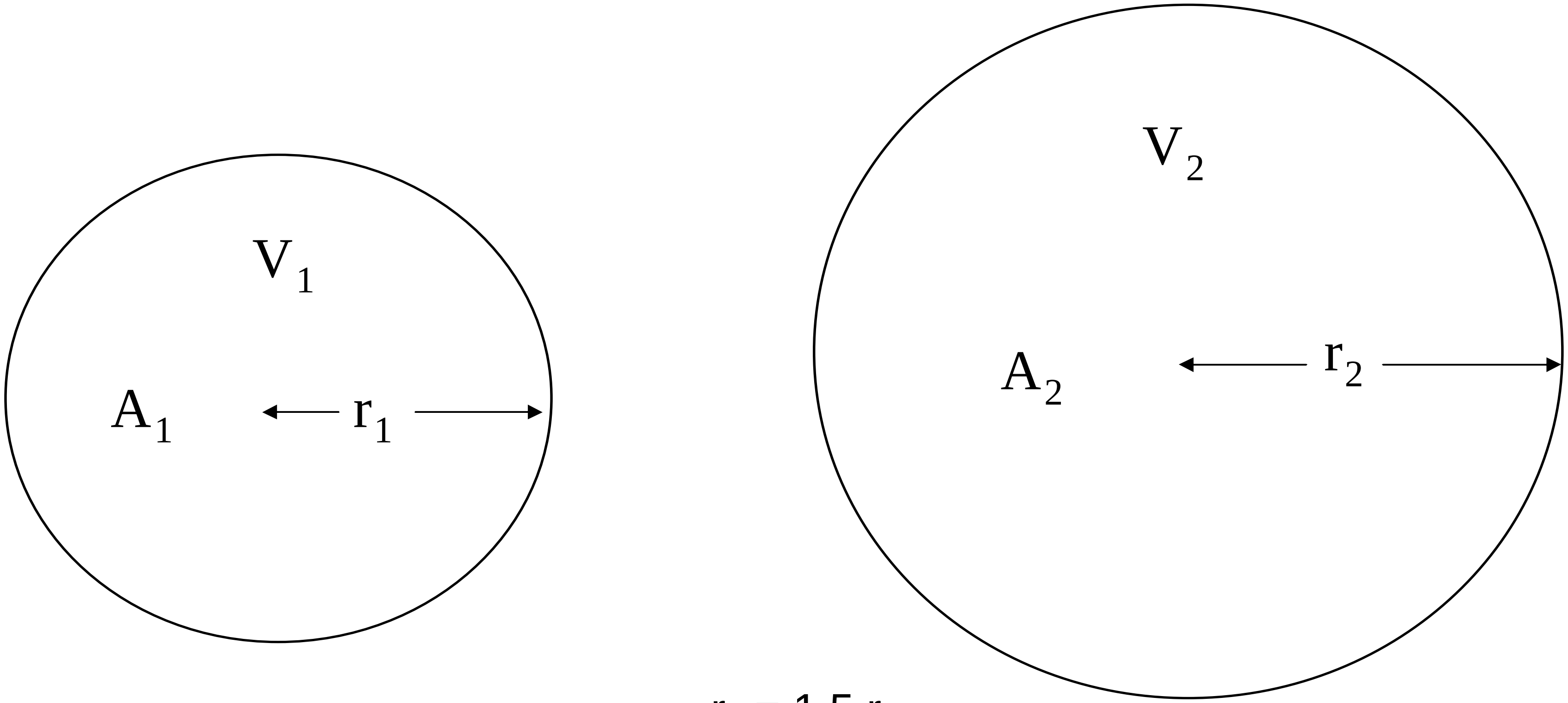
- One can come to interesting observations of these two kingdoms through dimensional analysis. Much time before Dean Swift, Galileus already found out that amplified or reduced models of man could not be like we are. The human body is built of columns, stretchers, bones and muscles. The weight of the body that the structure has to support is proportional to its volume, that is, L^3, and the resistance of a bone to compression or of a muscle for traction, is proportional to L^2.

- Let´s compare Gulliver with the giant of Brobdingnag, which has all of his linear dimensions twelve times larger. The resistance of his legs would be 144 times larger than that of Gulliver, and his weight 1728 times heavier. The ratio resistance/weight of the giant would be 12 times less than ours. In order to sustain its own weight, he would have to make an equivalent effort to that we would have to make to carry other eleven men.

Galileus treated this subject very clearly, using arguments that deny the possibility of the existence of giants of normal aspect. If we wanted to have a giant with the same leg/arm proportions of a normal human, we would have to use a stronger and harder material to make the bones, or we would have to admit a lower resistance in comparison to a man of normal stature. On the other hand, if the size of the body would be diminished, the resistance would not diminish in the same proportion. The smaller the body, the greater its relative resistance. In this way, a very small dog could, probably, carry other two or three small dogs of his size on his back; on the other hand, an elephant could not carry even another elephant of his own size !

Let´s analyze the problem of the liliputans in another way . The heat that a body loses to the environment goes through the skin, being proportional to the area covered by the skin, that is, L^2, considering constant the body temperature and skin characteristics. This energy comes from the ingestion of food. Therefore, the minimum amount of food to be ingested would be proportional to L^2. If Gulliver would be happy with a broiler, a bread and a fruit per day, a liliputan would need a $(1/12)^2$ smaller food volume. But a broiler, a bread, a fruit when reduced to the scale of his world, would have volumes $(1/12)^3$ smaller. He would, therefore, need twelve broilers, twelve breads and twelve fruits to be as happy as Gulliver. The liliputans should be famine and restless people. These qualities are found in small mammals, like mice. It is interesting to note that there are not many hot blood animals smaller than mice, probably in light of the scale laws discussed above, these animals would have to eat such a large quantity of food that would be difficult to obtain or, that could not be digested over a feasible time.

From all we saw, it is important to recognize that, although being geometric models of our world, Brobdingnag and Liliput could never be our physical models, since they would not have the necessary physical similarity which is found in natural phenomena. In the case of Brobdingnag, for example, the giant would be able to support his own weight having the stature of humans, if he would be living in a planet of gravity (1/12)g.



$r_2 = 1.5\ r_1$

$r_1 = 3$ cm

$r_2 = 4{,}5$ cm

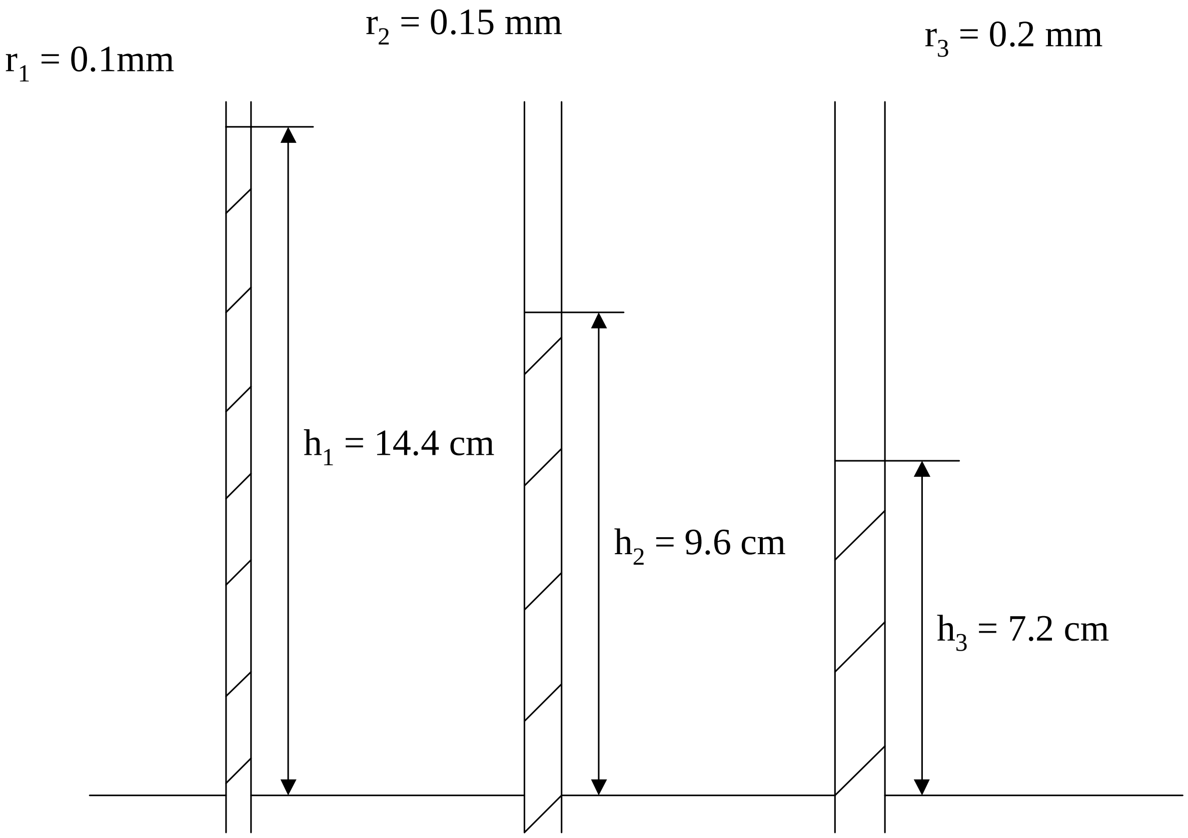
$$A_1 = \pi\, r_1^2 = 28.27\ \text{cm}^2$$

$$V_1 = \frac{3\pi\, r_1^3}{4} = 63.62\ \text{cm}^3$$

$$A_2 = \pi\, r_2^2 = 63.62\ \text{cm}^2$$

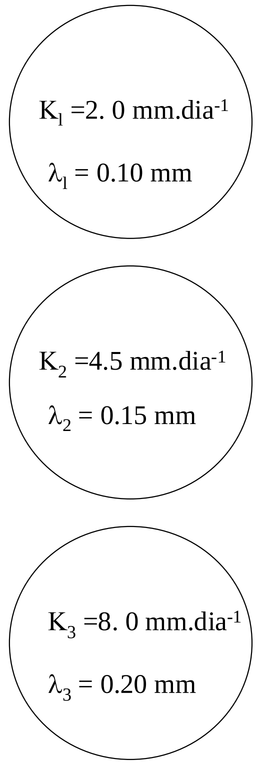
$$V_2 = \frac{3\pi\, r_2^3}{4} = 214.71\ \text{cm}^3$$

Figure 1 – Spheres seen under the similar media concept.



$$h_1r_1 = h_2r_2 = h_3r_3 = \text{constant}$$
$$14.4 \times 0.1 = 9.6 \times 0.15 = 7.2 \times 0.2 = 1.44$$

Figure 2 – Similar capillaries in water.



$$\frac{K_1}{\lambda_1^2} = \frac{K_2}{\lambda_2^2} = \frac{K_3}{\lambda_3^2} = \text{constant}$$

$$\frac{2}{(0.10)^2} = \frac{4{,}5}{(0.15)^2} = \frac{8}{(0.20)^2} = 200$$

Figure 3 – Cross-sections of soil columns with their respective conductivities.

$$h^* = \frac{\lambda_1 \rho g h_1}{\sigma} = \frac{\lambda_2 \rho g h_2}{\sigma} = ... = \frac{\lambda_i \rho g h_i}{\sigma} \quad (6)$$

$$K^* = \frac{\eta K_1}{\lambda_1^2 \rho g} = \frac{\eta K_2}{\lambda_2^2 \rho g} = ... \frac{\eta K_i}{\lambda_i^2 \rho g} \quad (7)$$

$$D^* = \frac{\eta D_1}{\lambda_1 \sigma} = \frac{\eta D_2}{\lambda_2 \sigma} = ... = \frac{\eta D_i}{\lambda_i \sigma} \quad (8)$$

$$t^* = \frac{\lambda_1 \sigma t_1}{\eta \left(x_{1max}\right)^2} = \frac{\lambda_2 \sigma t_2}{\eta \left(x_{2max}\right)^2} = ... = \frac{\lambda_i \sigma t_i}{\eta \left(x_{i\,max}\right)^2} \quad (9)$$

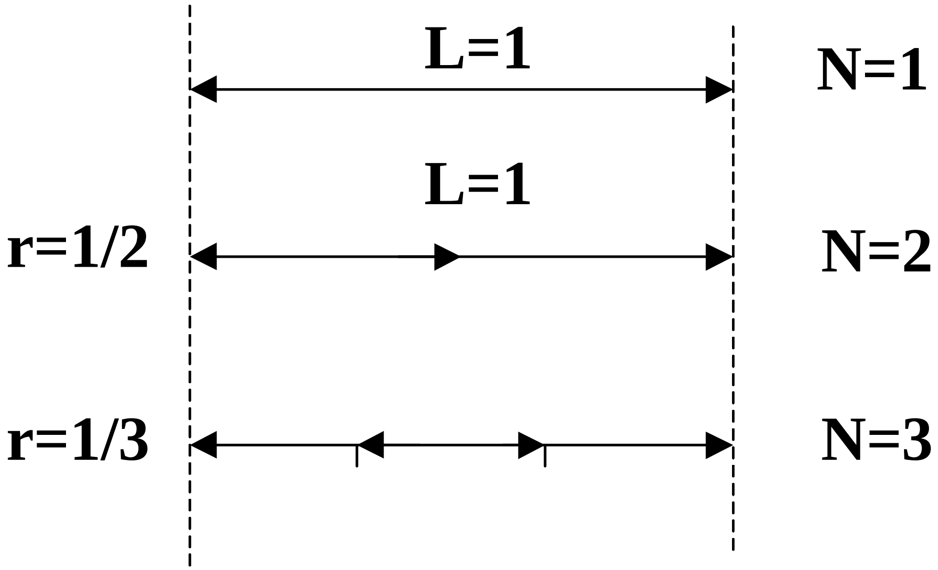
$$\frac{\partial\Theta}{\partial t^*} = \frac{\partial}{\partial X}\left[D^*(\Theta)\frac{\partial\Theta}{\partial X} \right] \tag{10}$$

$$\Theta = 0 \ , \ X \geq 0 \ , \ t^* = 0 \tag{11}$$

$$\Theta = 1 \ , \ X = 0 \ , \ t^* > 0 \tag{12}$$

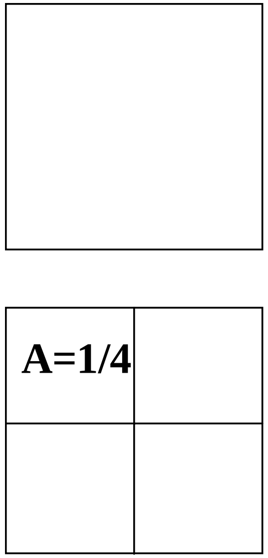
$$X = \phi^*(\Theta).\left(t^*\right)^{1/2} \tag{13}$$

$$D(\Theta) = 1{,}462 \ x \, 10^{-5} \, a_i^2 \, \exp\left(8{,}087.\,\Theta\right) \tag{14}$$



$$\boldsymbol{D}_L = \frac{\log \boldsymbol{N}}{\log(1/\boldsymbol{r})} = \frac{\log 2}{\log 2} = \frac{\log 3}{\log 3} = 1$$

– Generalization of the relation $N.r^D = 1$, for the case D = 1, i.e., $N.r^1 = 1$.



N=1
A=1

r(linear) =1/2
r(área) =1/4

N=4
A=1/4

r(linear) =1/4
r(área) =1/16

→

N=16
A=1/16

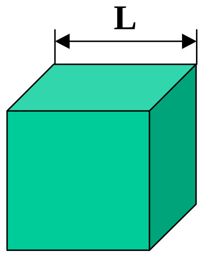
$$D_A = \frac{\log N}{\log(1/r)} = \frac{\log 4}{\log 2} = \frac{\log 16}{\log 4} = 2$$

$N.r^2=1$

$$r = \frac{1}{\sqrt{N}}$$

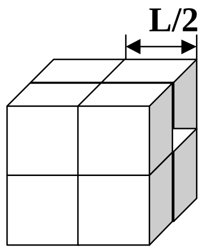
Bidimensional objects.

$$D_L = D_A - 1$$



N=1
V=1

r(linear) =1/2
r(volume)=1/4



N=8
V=1/4

r(linear) =1/4 → **N=64**
r(volume)=1/16 **V=1/16**

$$D_v = \frac{\log N}{\log(1/r)} = \frac{\log 8}{\log 2} = \frac{\log 64}{\log 4} = 3$$

N.r^3=1

$$r = \frac{1}{\sqrt[3]{N}}$$

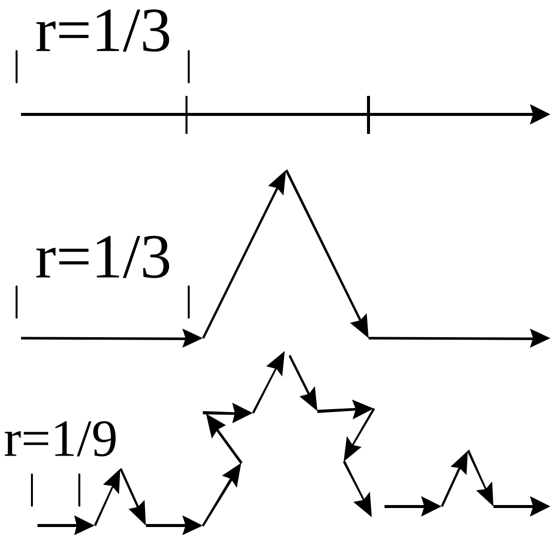
$$D_L = D_V - 2$$

– **Tridimensional objects.**

a) $L=1 \; ; \; N=1$

b) $L=4/3 \; ; \; N=4$

c) $L=16/9=(4/3)^2 \; ; \; N=16$

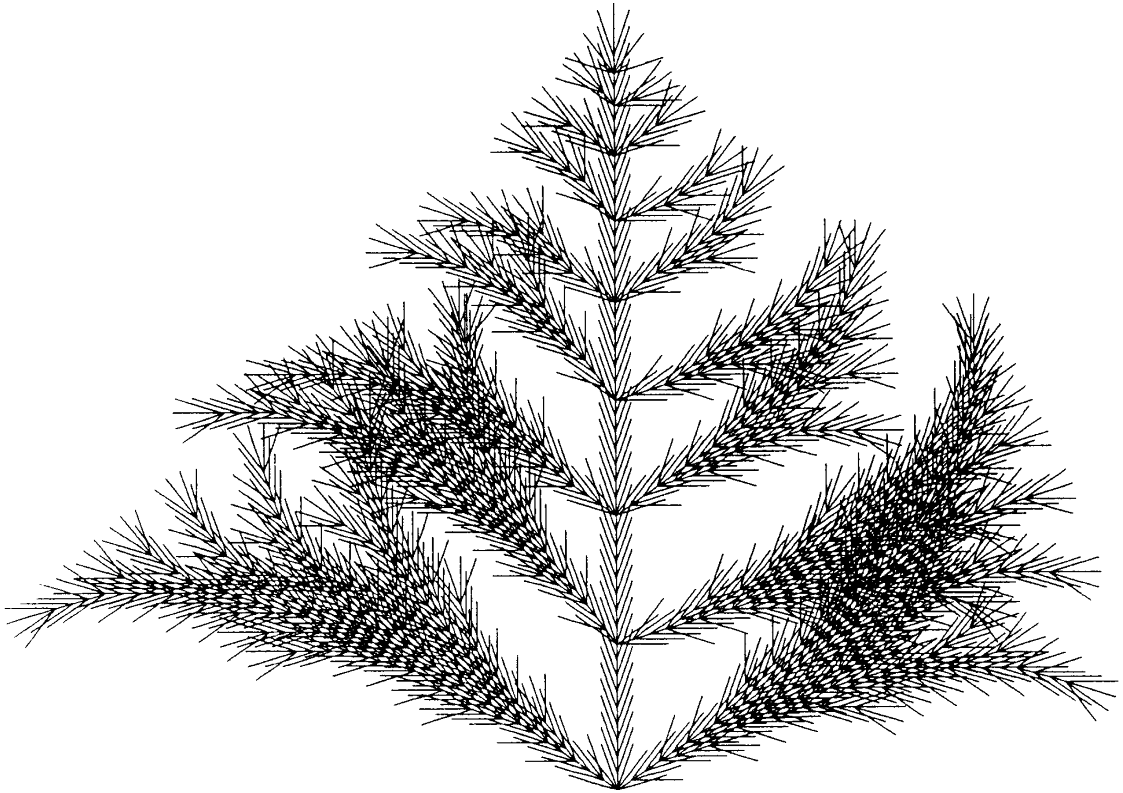


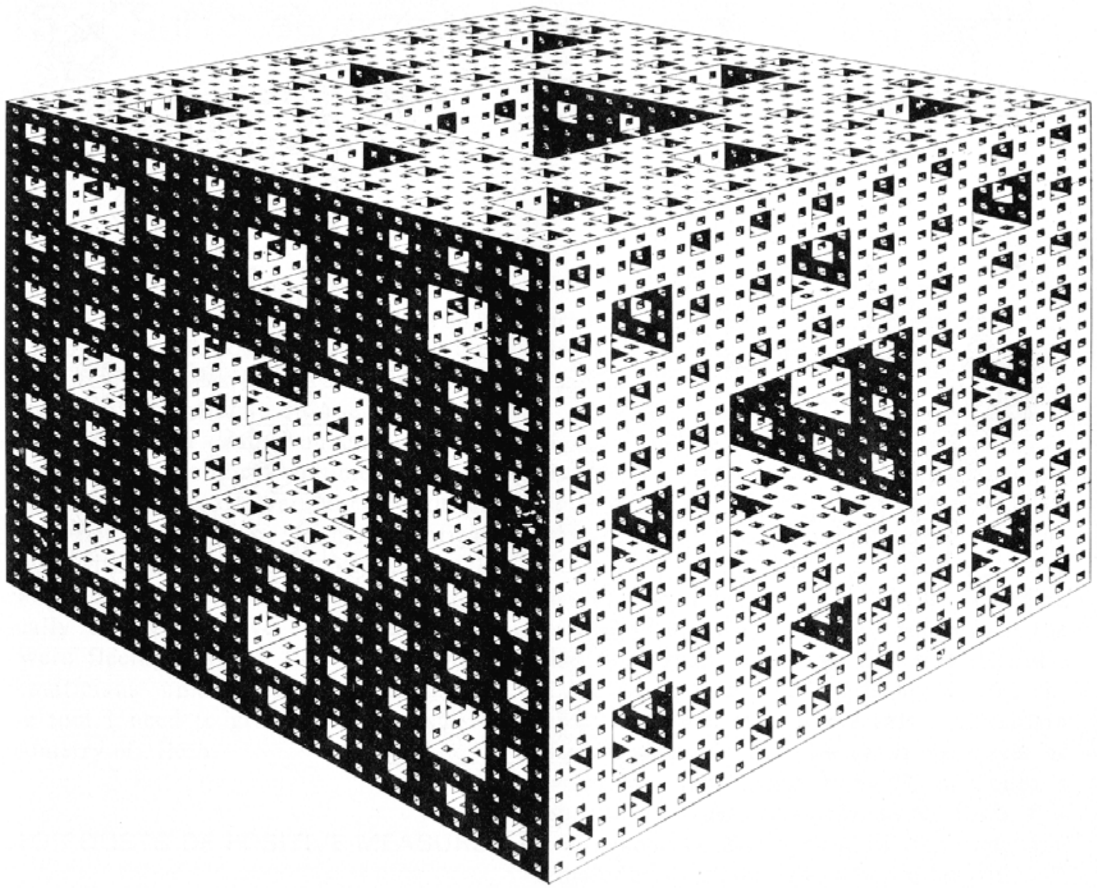
d) Próximo estágio → $L = 64/27=(4/3)^3 \; ; \; N=64$

$N.r^D=1$

$$D = \frac{\log N}{\log(1/r)} = \frac{\log 4}{\log 3} = \frac{\log 16}{\log 9} = \frac{\log 64}{\log 27} = 1{,}26..$$

von Koch's curve.





In Soil Physics, since the path followed by the water , the íons os the gases, floowing the particle distribution are all tortuous, the concept of fractals could be good option for modelling. Along these lines, Tyler & Wheatcraft (1989) measured the volumetric fractal dimension of a soil using as a basis the particle distribution, measuring the slope of the realions log N versus log R, where N is the numeber of particles of radia less than R. Later , Tyler & Wheatcraft (1992) recognized the dificulty of measuring the number of particles N and used the mass os particles in a non-dimensional way M (R $<$ Ri)/Mt and the radia were also made non-dimensional Ri/Rl.

Bacchi & Reichardt (1993) used these concepts to model soil water retention curves, estimating the pore length Li corresponding to a given textural class, employing the empiric expression of Arya & Paris (1981): Li = 2RiNiα, where 2Ri is the diameter of the particles of class i and Ni the number of particles of this class. No success was obtained for this reseach line and it is still open for new thoughts. Bacchi et al. (1996) compared the use of the particle distribution and of the pore distribution to measure the soil fractal dimension Dv and applied their effects on soil hydraulic conductivity data.

Still among the Brazilans, Guerrini (1992, 2000) aplied the fractal geometry with success in agronomy. The basic text for fractal geometry is Mandelbrot (1982) and in addition to the already cited papers, the following should be of great interest: Puckett et al. (1985), Turcotte (1986), Tyler & Wheatcraft (1990), Guerrini & Swartzendruber (1994, 1997) e Perfect & Kay (1995).

Comments on "Fractal Fragmentation, Soil Porosity, and Soil Water Properties: I. Theory"

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A recent comment (Yu, 2007) and response (Sposito, 2007) evaluate the relative merits of the fractal model of porous media proposed by Rieu and Sposito (1991), as opposed to that of Katz and Thompson (1985), or related work by Nigmatullin et al. (1992). The chief contrast alleged lies in the two results for the porosity,

$$\phi = \left(\frac{r_0}{r_m}\right)^{3-D} \quad [1]$$

and

$$\phi = 1 - \left(\frac{r_0}{r_m}\right)^{3-D} \quad [2]$$

In these two formulas ϕ is the porosity, D is the fractal dimensionality, and r_0 and r_m are minimum and maximum radii, respectively [note the difference from the notation of Rieu and Sposito (1991)]. Whether pore or solid space is meant depends on the author and the context. On the face of it this difference appears troublesome. The response to the comment (Sposito, 2007) notes that the Rieu and Sposito (1991) model was developed to describe soil aggregates and the tendency for the porosity of aggregates to increase with increasing aggregate size. But this is unnecessarily restrictive, and the model can be applied to the textural pore space, and to explain associated water retention curves (Hunt and Gee, 2002). In this case it is possible to use Eq. [1] and [2] to describe the same object.