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Dimensional analysis and scaling, with an introduction to fractals 2

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DIMENSIONAL ANALYSIS , SCALING and an INTRODUCTION to FRACTALS

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Introduction

Dimensional analysis refers to the study of the dimensions that characterize physical entities, like mass, force and energy. Classical Mechanics is based on three fundamental entities, with dimensions MLT, the mass M, the length L, and the time T. The combination of these entities gives rise to derived entities, like volume, speed and force, of dimensions L^3, LT^{-1}, MLT^{-2}, respectively. In other areas of Physics, other four fundamental entities are defined, among them the temperature θ and the electrical current I.

To introduce the topic of Dimensional Analysis, let us look at a classical example of the romantic literature, in which Dean Swift, in "The Adventures of Gulliver" describes the imaginary voyages of Lemuel Gulliver to the kingdoms of Liliput and Brobdingnag. In these two places life was identical to that of normal persons, their geometric dimensions were, however, different. In Liliput, man, houses, dogs, trees were twelve times smaller than in the country of Gulliver, and in Brobdingnag, everything was twelve times taller. The man of Liliput was a geometric model of Gulliver in a scale 12:1, and that of Brobdingnag a model in a scale of 1:12.

One can come to interesting observations of these two kingdoms through dimensional analysis. Much time before Dean Swift, Galileus already found out that amplified or reduced models of man could not be like we are. The human body is built of columns, stretchers, bones and muscles. The weight of the body that the structure has to support is proportional to its volume, that is, L^3, and the resistance of a bone to compression or of a muscle for traction, is proportional to L^2.

Let´s compare Gulliver with the giant of Brobdingnag, which has all of his linear dimensions twelve times larger. The resistance of his legs would be 144 times larger than that of Gulliver, and his weight 1728 times larger. The ratio resistance/weight of the giant would be 12 times less than ours. In order to sustain its own weight, he would have to make an equivalent effort to that we would have to make to carry other eleven men.

Galileus treated this subject very clearly, using arguments that deny the possibility of the existence of giants of normal aspect. If we wanted to have a giant with the same leg/arm proportions of a normal human, we would have to use a stronger and harder material to make the bones, or we would have to admit a lower resistance in comparison to a man of normal stature. On the other hand, if the size of the body would be diminished, the resistance would not diminish in the same proportion. The smaller the body, the greater its relative resistance. In this way, a very small dog could, probably, carry other two or three small dogs of his size on his back; on the other hand, an elephant could not carry even another elephant of his own size !

Let´s analyze the problem of the liliputans. The heat that a body loses to the environment goes through the skin, being proportional to the area covered by the skin, that is, L^2, considering constant the body temperature and skin characteristics. This energy comes from the ingestion of food. Therefore, the minimum amount of food to be ingested would be proportional to L^2. If Gulliver would be happy with a broiler, a bread and a fruit per day, a liliputan would need a $(1/12)^2$ smaller food volume. But a broiler, a bread, a fruit when reduced to the scale of his world, would have volumes $(1/12)^3$ smaller. He would, therefore, need twelve broilers, twelve breads and twelve fruits to be as happy as Gulliver.

The liliputans should be famine and restless people. These qualities are found in small mammals, like mice. It is interesting to note that there are not many hot blood animals smaller than mice, probably in light of the scale laws discussed above, these animals would have to eat such a large quantity of food that would be difficult to obtain or, that could not be digested over a feasible time.

From all we saw, it is important to recognize that, although being geometric models of our world, Brobdingnag and Liliput could never be our physical models, since they would not have the necessary physical similarity which is found in natural phenomena. In the case of Brobdingnag, for example, the giant would be able to support his own weight having the stature of humans, if he would be living in a planet of gravity (1/12)g.

Physical Entities and Dimensional Analysis

The parameters that characterize physical phenomena are related among themselves by laws, in general of quantitative nature, in which they appear as measures of the considered physical entities. The measure of an entity is the result of its comparison with another one, of the same type, called unit. In this way, an entity (G) is given by two factors, one being the measure (M) and the other the unit (U). When we write $V = 50\ m^3$, V is the entity G, 50 is the ratio between the measures (M), and the unit U is m^3. Therefore:

$$\mathbf{G = M\ (G)\ .\ U\ (G)}$$

M(G) being the measure of G and U(G) the unit of G. In addition, the entity G has a dimensional symbol, which is the combination of the fundamental units that built up the entity. Some examples are given below:

Entity (G)	**M (G)**	**U (G)**	**Dimensional symbol**
Area	200	m^2	L^2
Speed	40	$m\ s^{-1}$	LT^{-1}
Force	50	$N = kg\ m\ s^{-2}$	MLT^{-2}
Pressure	1,000	$Pa = kg\ m^{-1}\ s^{-2}$	$ML^{-1}\ T^{-2}$
Flow	5	$m^3\ s^{-1}$	$L^3\ T^{-1}$

The International Units System has seven fundamental entities:

a) Mass (M): quilogram (kg);

b) Lenght (L): meter (m);

c) Time (T): second (s);

d) Electrical current (I): Ampere (A);

e) Thermodynamic temperature (θ): Kelvin (K);

f) Light intensity (Iv): candela (cd);

g) Quantity of matter (N): mol (mol).

Derived Physical entities are, in general, expressed by a relation involving the fundamental or derived entities X, Y, Z, ... which take part in their definition:

$$G = k\ X^a . Y^b . Z^c ....$$

where k is a non dimensional constant, and a, b, c, ... constant exponents.

If, for example, we would have doubts on the formula F = m.a, we could make a check and admit, at least, that F is a function of m and a:

$$G = k\,X^{a}.Y^{b} \quad \textbf{or} \quad F = k\,m^{a}.a^{b}$$

since F has dimensions MLT^{-2}, the right hand side member has also to have dimensions MLT^{-2}, that is:

$$MLT^{-2} = k\,M^{a}.\left(LT^{-2}\right)^{b}$$

remembering that the dimension of acceleration is LT^{-2}. So $MLT^{-2} = k\,M^{a}.L^{b}.T^{-2b}$, and we can see that the only possibility is k=1, a=1 and b=1, thus confirming F=m.a.

Products P are any products of the variables that involve a phenomenon. The fall of bodies from an origin 0 with no initial velocity in the vacuum involves the variables space S, acceleration of gravity g and time t, according to:

$$S = \frac{1}{2}g\,.\,t^{2}$$

For this phenomenon we can write an infinite number of products P, as for example:

$P_1 = S^2.\,t^{-2}.g$, with dimensions $L^2.T^{-2}.\,L\,.T^{-2} = L^3.T^{-5}$

$P_2 = S^0.\,t^{2}.g$, with dimensions $1.T^2.\,L\,.T^{-2} = L$

$P_3 = S^{-3}.\,t^{4}.g$, with dimensions $L^{-3}.T^{4}.\,L\,.T^{-2} = L^{-2}.T^{2}$

$P_4 = S^{-2}.\,t^{4}.g^{2}$, with dimensions $L^{-2}.T^{4}.\,(L\,.T^{-2})^{2} = L^{0}.T^{0} = 1$

When a chosen product is non-dimensional, as P_4, it is called a non-dimensional product and is symbolized by π, in this case $P_4 = \pi_4$. A theorem states that: “given n dimensional entities $G_1, G_2, ..., G_n$ generated through products of k fundamental entities, if a phenomenon can be expressed by $F(G_1, G_2, ..., G_n) = 0$, it can also be described by $\phi(\pi_1, \pi_2, ..., \pi_{n-k}) = 0$, a function with less variables.

The problem mentioned in the introduction about the Kingdoms of Liliput and Brobdingnag, is of physical similarity. Every time we work with models of objects in different scales, it is necessary that there is a physical similarity between the model (a prototype, in general smaller) and the real object of study. Depending on the case, we talk about kinematic similarity, which involves relations of velocity and acceleration between model and object; or about dynamic similarity, which involves relations between the forces that act on the model and on the object . In the similarity analysis we use the π products, like the known “numbers” of Euler, Reynolds, Froude and Mach. In this analysis we have:

OBJECT:

$$F(G_1, G_2, ..., G_n) = 0 \longrightarrow \phi(\pi_1, \pi_2, ..., \pi_{n-k}) = 0$$

PROTOTYPE:

$$F(G'_1, G'_2, ..., G'_n) = 0 \longrightarrow \phi(\pi'_1, \pi'_2, ..., \pi'_{n-k}) = 0$$

and the G_i s can be different of G'_i s. There will be physical similarity between object and prototype, only if $\pi_1 = \pi'_1$; $\pi_2 = \pi'_2$; ...; $\pi_{n-k} = \pi'_{n-k}$.

This analysis is frequently used in hydrodynamics, studies of machines, engineering, etc., and it has not many applications in Soil-Plant-Atmosphere systems. The study of Shukla et al. (2002) which utilizes the non dimensional products π to describe miscible displacement, is an exception. Texts of Maia (1960), Fox & McDonald (1995) e Carneiro (1996) are good references on this subject.

Non dimensional entities, like the π products, have a numerical value k of dimension 1:

$$M^o L^o T^o K^o = 1$$

It is also common to produce non-dimensional variables through the ratio of two entities G_1 and G_2 of the same dimension: $G_1/ G_2 = \pi$. This is the case of the number π = 3,1416..... which is the result of the ratio of the length of any circle (πD, of dimension L) and the respective diameter (D, also dimension L).

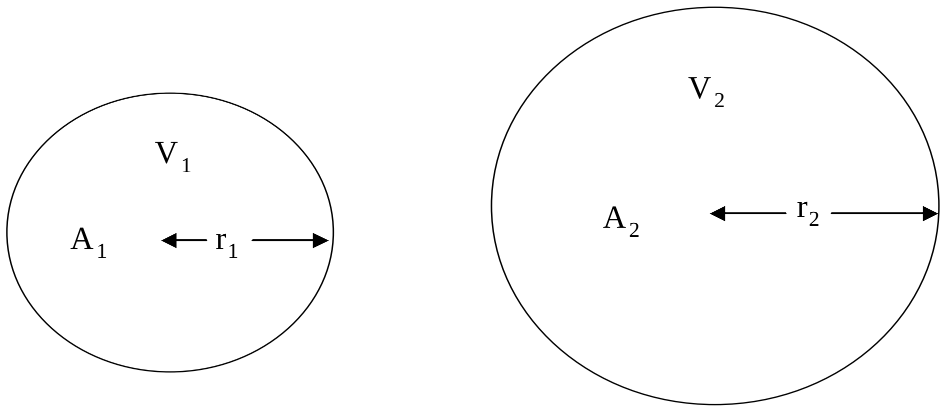
In the Soil-Plant-Atmosphere system, several variables are non dimensional by nature (or definition), and are represented in % or parts per million (ppm). Soil water content u (on mass basis), θ (on volume basis), porosities, etc., are examples of π products. Important is the procedure of turning dimensional variables into non dimensional ones. The simplest case is dividing the variable by itself, in two different conditions. For instance, in experiments using soil columns, each researcher uses a different column length L. How can we compare results ? If the space coordinate x or z (along the column) is divided by its maximum value L, we have a new variable: X = x/L, with the advantage that, for any L, at x = 0, X = 0; at x = L, X =1, varying, therefore, within the interval 0 to 1.

This procedure can also be used for variables which already are dimensionless, like the soil water content θ. If we divide $(\theta - \theta_s)$ by its largest interval $(\theta_o - \theta_s)$, where θ_s e θ_o are, respectively, initial and saturation values, we obtain a new variable $\Theta = (\theta - \theta_s)/(\theta_o -$

θ_s), for which $\Theta = 0$ for $\theta = \theta_s$ (dry soil) and $\Theta = 1$ for $\theta = \theta_o$ (saturated soil). In this way, for any type of soil, Θ varies from 0 to 1 and comparisons can be made more adequately.

Scales and Scaling

We already mentioned scales when presenting the "Adventures of Gulliver" and discussing physical similarity between object and prototype. Maps are also drawn in scale, for example, in a scale of 1:10,000, 1 cm^2 of paper can represent 10,000 m^2 in the field. Entities that differ in scale cannot be compared in a simple way. As we have seen, there is the problem of physical similarity, but if we desire to make a comparison without changing the scale of each one ? One technique to do this is called "scaling", frequently used in Soil Physics. It was introduced into Soil Science by Miller & Miller (1956) through the concept of similar media applied to "capillary flow" of fluids in porous media. According to these authors, two media M_1 and M_2 are similar when the variables that describe the physical phenomena that occur within them, differ of a linear factor λ, called microscopic characteristic length, which relates their physical characteristics. The best way to visualize this concept is to consider M_2 as an amplified (or reduced) photography of M_1 by a factor λ. For these media, the particle diameter of one is related to the other by: $D_2 = \lambda D_1$. The surface of this particle by: $S_2 = \lambda^2 S_1$, and its volume by $V_2 = \lambda^3 V_1$ (Figure 1). Under these conditions, if we know the flow of water through M_1, would it be possible to estimate the flow through M_2, based only on λ ? Using artificial porous media (glass beads **)**, Klute & Wilkinson (1958) and Wilkinson & Klute (1959) obtained results on water retention and hydraulic conductivity that validated the similar media concept.



$$r_2 = 1.5\ r_1$$

$$r_1 = 3\ \text{cm} \qquad\qquad r_2 = 4{,}5\ \text{cm}$$

$$A_1 = \pi\, r_1^2 = 28.27\,\text{cm}^2 \qquad\qquad A_2 = \pi\, r_2^2 = 63.62\,\text{cm}^2$$

$$V_1 = \frac{3\pi\, r_1^3}{4} = 63.62\,\text{cm}^3 \qquad\qquad V_2 = \frac{3\pi\, r_2^3}{4} = 214.71\,\text{cm}^3$$

$$\frac{A_2}{A_1} = 2.25 \quad \rightarrow \quad \sqrt{2.25} = 1.5 \quad \text{ou} \quad A_2 = (1.5)^2 A_1$$

$$\frac{V_2}{V_1} = 3.37 \quad \rightarrow \quad \sqrt[3]{3.37} = 1.5 \quad \text{ou} \quad V_2 = (1.5)^3 V_1$$

Figure 1 – Spheres seen under the similar media concept.

After this, contributions that appeared in the literature did not significantly push ahead this concept. More than 10 years later, Reichardt et al. (1972) reappear with the subject, having success even with natural porous media, i.e., soils of a wide range in texture. They assumed that soils can be considered similar media, each one characterized by its factor λ which, at the beginning, they did not know how to measure. They tested the concept on horizontal water infiltration studies, using homogeneous soil columns of initial soil water content θ_i, applying free water at the entrance (x = 0) so that at this point the saturation water content θ_o was maintained thereafter:

$$\theta = \theta_i \ , \ x > 0 \ , \ t = 0 \qquad (1)$$

$$\theta = \theta_o \ , \ x = 0 \ , \ t > 0 \qquad (2)$$

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[D(\theta) \frac{\partial \theta}{\partial x} \right] \qquad (3)$$

where $D(\theta) = K(\theta).dh/d\theta$; $K(\theta)$ is the soil hydraulic conductivity and h the soil water matric potential.

Since for any soil the solution of this boundary value problem BVP is of the same type: $x = \phi(\theta).t^{1/2}$, in which $\phi(\theta)$ depends on the characteristics of each porous media, would it not be possible to find a generalized solution for all media (considered similar) if λ of each would be known ? The procedure they used included the process of making all involved variables dimensionless, using the similar media theory applied to each of the i soils, each with its λ_1, λ_2, ...λ_i. The soil water content θ and the space coordinate x were transformed as indicated above**:**

$$\Theta = \frac{(\theta - \theta_i)}{(\theta_o - \theta_i)} \qquad (4)$$

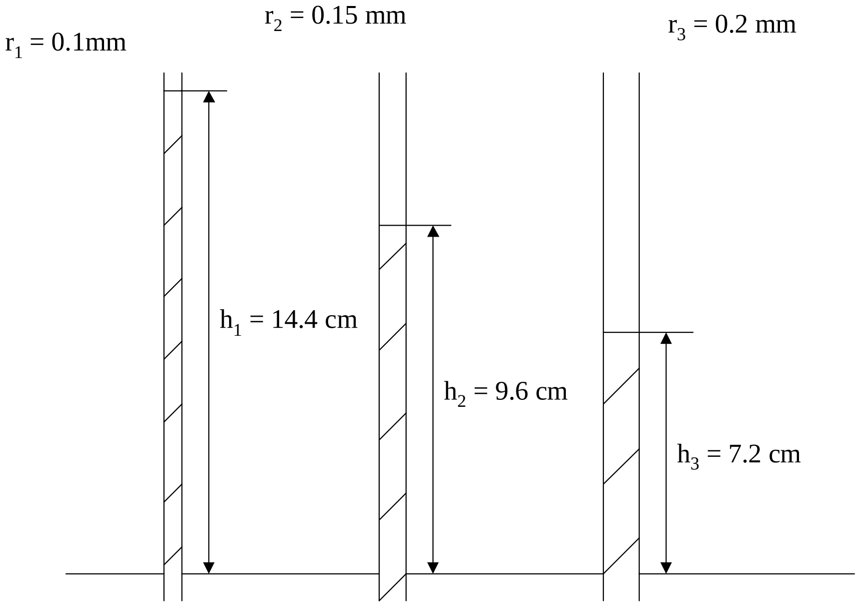
$$X = \frac{x}{x_{max}} \qquad (5)$$

The matric soil water potential h was considered to be only the result of capillary forces: $h = 2\sigma/\rho gr$ or $hr = 2\sigma/\rho g$ = constant. If each soil i would have only capillaries of radius r_i, and if the characteristic length λ_i would be proportional to r_i, we would have:

$$h_1 r_1 = h_2 r_2 = \ldots\ldots = h_i r_i = \text{constant}$$

If, among the i soils, we choose one as a standard soil, for which we make, arbitrarily, $\lambda^* = r^* = 1$ (one µm, or any other value), the constant above becomes $h^*r^* = h^*$, which is the matric potential h^* of the standard soil (Figure 2). Through dimensional analysis we can also make h^* non-dimensional:

$$h^* = \frac{\lambda_1 \rho g h_1}{\sigma} = \frac{\lambda_2 \rho g h_2}{\sigma} = \ldots\ldots = \frac{\lambda_i \rho g h_i}{\sigma} \quad (6)$$



$$h_1r_1 = h_2r_2 = h_3r_3 = \text{constant}$$

$$14.4 \times 0.1 = 9.6 \times 0.15 = 7.2 \times 0.2 = 1.44$$

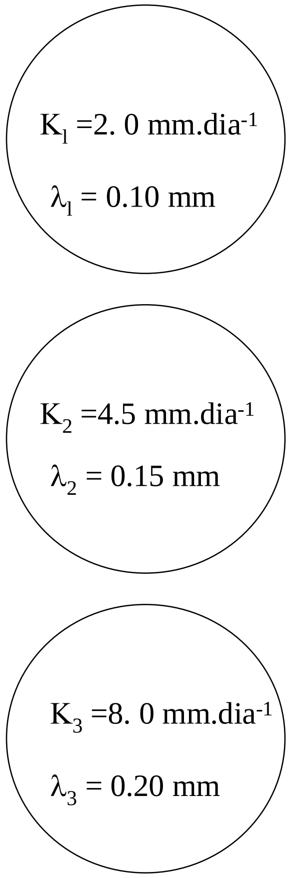
Figure 2 – Similar capillaries in water.

The hydraulic conductivity K is proportional to the area (λ^2) available for water flow (k = intrinsic permeability, L^2), and using the known relation $K = k\rho g/\eta$ or $K/k = \rho g/\eta$ = constant, we have for the I soils:

$$\frac{K_1}{k_1} = \frac{K_2}{k_2} = \ldots\ldots\ldots\ldots \frac{K_i}{k_i} = \text{constant}$$

$$K^* = \frac{\eta K_1}{\lambda_1^2 \rho g} = \frac{\eta K_2}{\lambda_2^2 \rho g} = \ldots\ldots\ldots \frac{\eta K_i}{\lambda_i^2 \rho g} \qquad (7)$$

where K* is the hydraulic conductivity of the standard soil, assuming $\lambda^* = r^* = k^* = 1$ (Figure 3).



$$\frac{K_1}{\lambda_1^2} = \frac{K_2}{\lambda_2^2} = \frac{K_3}{\lambda_3^2} = \text{constant}$$

$$\frac{2}{(0.10)^2} = \frac{4{,}5}{(0.15)^2} = \frac{8}{(0.20)^2} = 200$$

Figure 3 – Cross-sections of soil columns with their respective conductivities.

Through the definition of soil water diffusivity D = K.dh/dθ, it is possible to verify that the soil water difusivity D* is given by:

$$D^* = \frac{\eta\, D_1}{\lambda_1\, \sigma} = \frac{\eta\, D_2}{\lambda_2\, \sigma} = ... = \frac{\eta\, D_i}{\lambda_i\, \sigma} \qquad (8)$$

To make equation 3 dimensionless it is now needed to make the time t dimensionless. In accordance to all other variables, we can have a time t* for the standard soil, as follows:

$$t^* = \frac{\lambda_1\, \sigma\, t_1}{\eta \left(x_{1\max}\right)^2} = \frac{\lambda_2\, \sigma\, t_2}{\eta \left(x_{2\max}\right)^2} = ... = \frac{\lambda_i\, \sigma\, t_i}{\eta \left(x_{i\max}\right)^2} \qquad (9)$$

It can now be seen that if we substitute θ by Θ, x by X, t by t_i and D by D_i in equation 3, we obtain the differential equation for the standard soil, which differs from the equations of all other soils by factors λ_i, not seen in equation 10, but built-in the definitions of t* and D*:

$$\frac{\partial \Theta}{\partial t^*} = \frac{\partial}{\partial X}\left[D^*(\Theta) \frac{\partial \Theta}{\partial X} \right] \qquad (10)$$

subject to conditions:

$$\Theta = 0 \ , \ X \geq 0 \ , \ t^* = 0 \tag{11}$$

$$\Theta = 1 \ , \ X = 0 \ , \ t^* > 0 \tag{12}$$

the solution of which is:

$$X = \phi^*(\Theta).\left(t^*\right)^{1/2} \tag{13}$$

It is interesting to analyze equation 9, of the non dimensional infiltration time, in light of the physical similarity of the kingdoms of Liliput and Brobdingnag, which shows that to compare different soils (considered similar media), their times have to be different and dependent of λ which is a length ! We could even suggest that this fact contributes to explain how time is considered the forth coordinate, together with x, y and z, in Modern Physics.

By analogy with what was made with h and K, we can write:

$$t_1\lambda_1 = t_2\lambda_2 = ... = t_i\lambda_i = \frac{t^*\eta\left(x_{max}\right)^2}{\sigma} = \text{ constant}$$

Once the theory was established, Reichardt et al. (1972) looked for ways to measure λ for the different soils. The "Columbus Egg" was found when they realized that if the linear regressions of x_i versus $t_i^{1/2}$ for the position of the wetting front for each soil, should overlap to one single curve for the standard soil (X versus $t^{*1/2}$), and that the factors used to rotate the line of each soil to the position of the line of the standard soil, could be used as characteristic lengths λ_i. We know that straight lines passing through the origin: $y = a_i x$ can be rotated over each other using the relation a_i/a_j of their slopes. Since in our case the lines involve a square root, the relation to be used is:

$$\frac{\lambda_i}{\lambda^*} = \left(\frac{a_i}{a^*}\right)^2 \qquad (14)$$

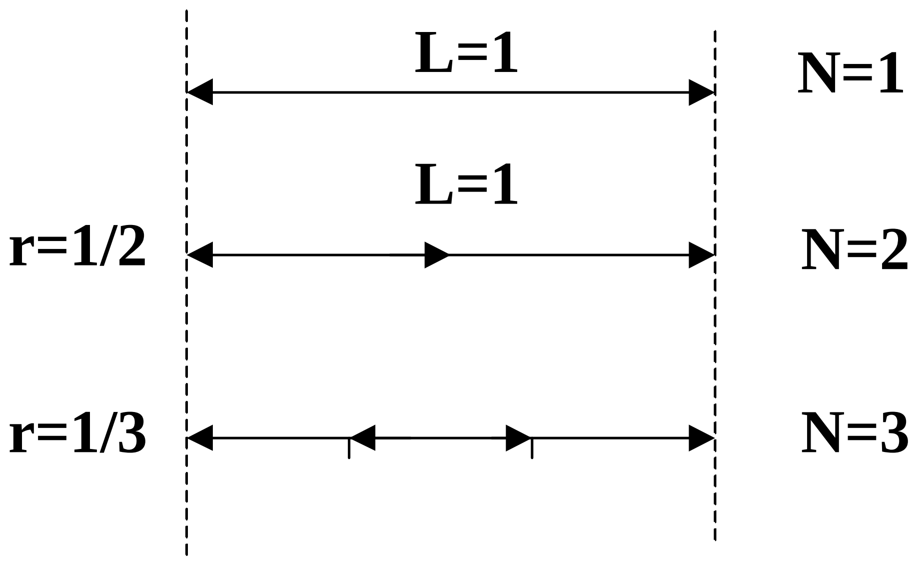
With this relation Reichardt et al. (1972) found the values λ_i for each soil, taking arbitrarily as a standard the soil of fastest infiltration, for which they postulated $\lambda^* = 1$. In this way, the slower the infiltration rate of soil i, the slower its λ_i. This way of determining λ as a scaling factor and not as a physical soil characteristic like the microscopic characteristic length of Miller & Miller (1956), facilitated the experimental part of the study and, more than that, opened the door for a much wider concept of scaling applied in other areas of Soil Physics. Reichardt et al. (1972) had only success in scaling $D(\theta)$ and a partial success in scaling $h(\theta)$ and $K(\theta)$, the reason for this being the fact that soils are not true similar media. The success of scaling $D(\theta)$ lead Reichardt & Libardi (1973) to establish a general equation to estimate $D(\theta)$ of a given soil, by measuring only the slope a_i of the wetting front advance x versus $t^{1/2}$:

$$D(\Theta) = 1{,}462 \text{ x } 10^{-5} a_i^2 \exp(8{,}087.\Theta) \qquad (15)$$

Reichardt et al. (1975) also presented a method to estimate $K(\Theta)$ through the coefficient a_i of equation (15); Bacchi & Reichardt (1988) used scaling techniques to evaluate $K(\theta)$ measurement methods, and more recently Shukla et al. (2002) used scaling to analyze miscible displacement experiments. Scaling has also widely been used in studies of soil spatial distribution, assuming characteristic values of λ for each point of a transect, making particular curves to coalesce into a single one. An excellent review of scaling techniques was made by Tillotson & Nielsen (1984), and, more recently, by Kutilek & Nielsen (1994) and Nielsen et al. (1998).

18.8. FRACTAL GEOMETRY AND FRACTAL DIMENSION

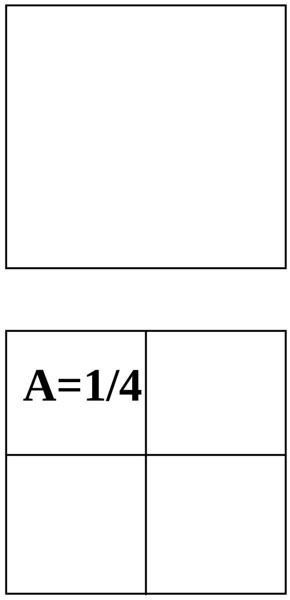
The fractal geometry in contrast to the euclidian admits fractional dimensions. The term **fractal** is defined in Mandelbrot (1982) as comming from the latin ***fractus***, signifying broken. According to Mandelbrot, **fractals** are non topologic objects, i.e., objects whose dimension is a real non integer number. For topologic objects or geometric Euclidian forms, the dimensão is na integer (0 for a point, 1 for a curve, 2 for a surface and 3 for volumes). The "**fractal dimension**", is a measure of the degree of the irregularity of the observed object in all scales of observation. An object that normally is considered unidimensional , like a straight line segment, can be divided in N identical parts, so that each part is a new segment represented in a scale $r=1/N$ of the original segment, so that $Nr^1 = 1$.



$$\boldsymbol{D_L} = \frac{\log \boldsymbol{N}}{\log(1/\boldsymbol{r})} = \frac{\log 2}{\log 2} = \frac{\log 3}{\log 3} = 1$$

Figura 18.4 – Generalization of the relation $N.r^D = 1$, for the case D = 1, i.e., $N.r^1 = 1$.

In a similar way a bidimensional object can be divided in N similar square areasdradas in a scale $r = 1/\sqrt{N}$ of the original área, so that $Nr^2 = 1$.



N=1
A=1

r(linear) =1/2
r(área) =1/4

N=4
A=1/4

r(linear) =1/4
r(área) =1/16

N=16
A=1/16

$$D_A = \frac{\log N}{\log(1/r)} = \frac{\log 4}{\log 2} = \frac{\log 16}{\log 4} = 2$$

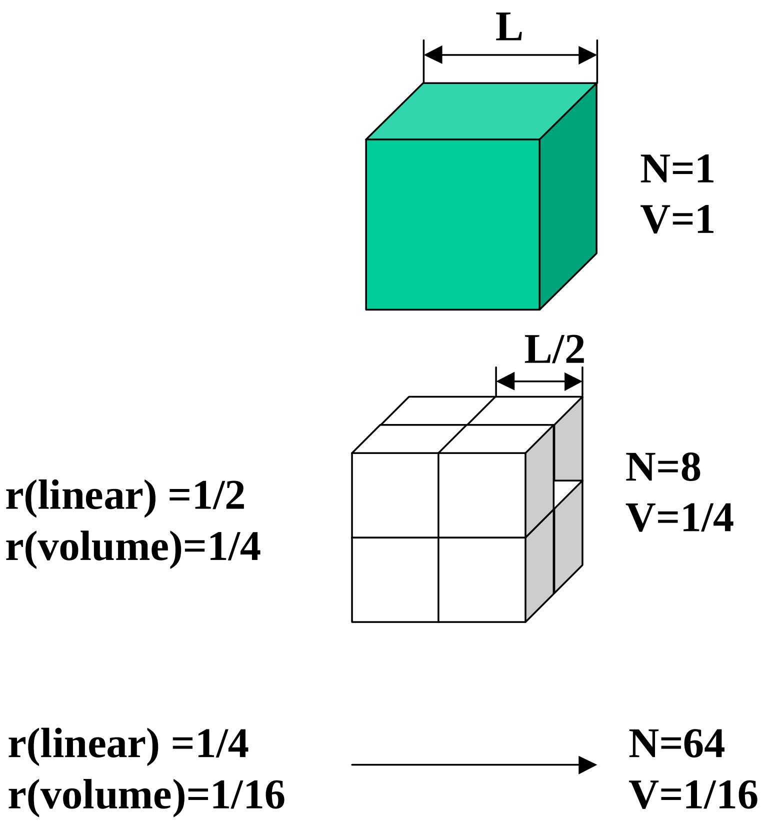
N.r²=1

$$r = \frac{1}{\sqrt{N}}$$

$$D_L = D_A - 1$$

Figura 18.5 – Bidimensional objects.

This scaling can be extended to tridimensional objects and the relation between the number of similar segments (N) and its scale in relation to the original object (r) can be generalized by $Nr^{D}=1$, where D defines the fractal dimension.



$$D_v = \frac{\log N}{\log(1/r)} = \frac{\log 8}{\log 2} = \frac{\log 64}{\log 4} = 3$$

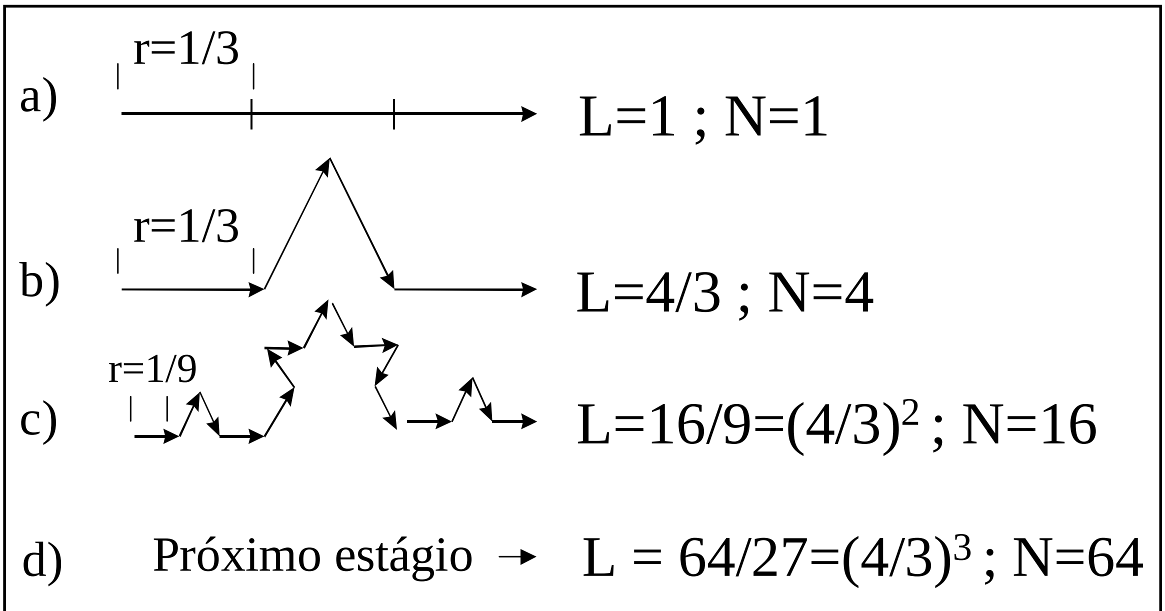
$$N.r^3=1$$

$$r = \frac{1}{\sqrt[3]{N}}$$

$$D_L = D_V - 2$$

Figura 18.6 – Tridimensional objects.

Therefore the geometric euclidian dimensions 0, 1, 2 e 3, with which we are familiar, can be seen as particular cases of an infinite number of dimensions that occur in nature. Figure 18.7, adapted from Barnsley et al. (1988), known as von Koch's curve, is constructed in na iterative or recursive way, starting from a straight line segment (a) divided in 3 equal parts and the central segment is substituted by 2 equal segments so as to take part of an equilaterous triangle (b). In the following stage each of these 4 segments is again divided in 3 parts and each of them is substituída by new 4 segments of length equal to 1/3 of the original one, disposed according to standard (b), and so sucessivelly. After stage b, in each stage change the total length L of the figure increases by a fator 4/3, the number N of similar element of those of the previous stage increses by a factor 4 and its dimensions are in scale r=1/3 of the previous stage. At each stage the figure can be divided into N similar elements , so that $N.r^D=1$, D being the fractal dimension of the object. Essa this curve represents the fractal dimension of D=1,26 , which is greater than 1 and less than 2, meaning that this curve "fills"more space than a simple (D=1) line, and less than na euclidian area (D=2).



$N.r^D=1$

$$D = \frac{\log N}{\log(1/r)} = \frac{\log 4}{\log 3} = \frac{\log 16}{\log 9} = \frac{\log 64}{\log 27} = 1{,}26..$$

Figura 18.7 – Curva de von Koch.

Forms and strututures highly complex and irregular, very comon in Nature, can be reproduced with detail richness through similar procedures, indicating that behind an apparent disorder of these forms, strututures and dynamic processes that occur in nature , there is som regularity able to be better understood. A new approach appeared, called “Teory of the Caos” which, matematically defines the casuality gernerated by simple dynamic derterministic systems. This approach allows the description of a given order in dynamis processes, before defined as compleately aleatory.

With the help of computers the fractal geometry is growing in different áreas of knowledge, including arts, as a new tool to understand nature. Agronomy treats basically all processes ocurring in the soil (movement of water, gases and solutes), soil structure, plant growth and development, drainage processes in hydrographic basins, etc., so that figura 18.8, taken from Barnsley et al. (1988), shows the image of a plant generated by graphic computation **.**

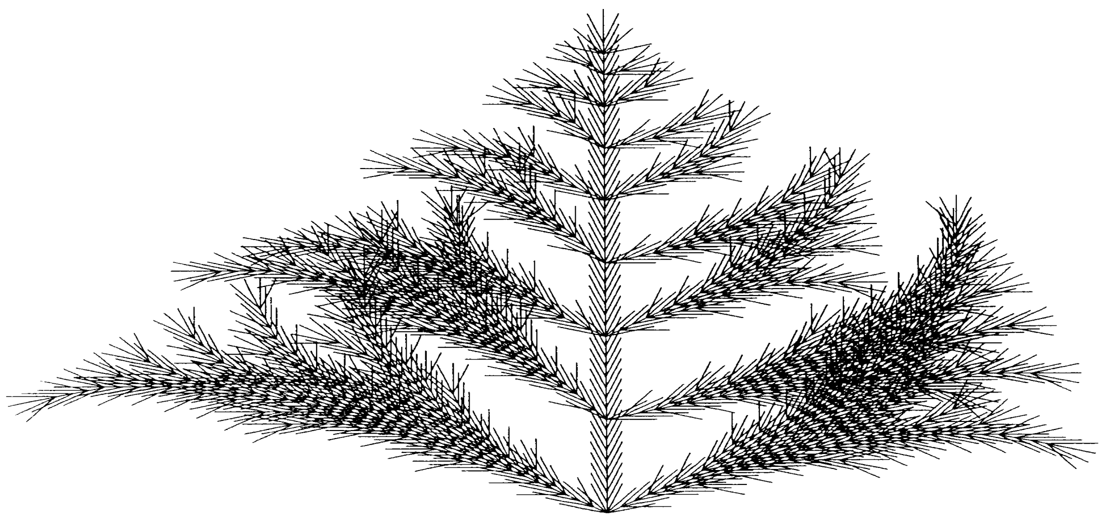


Figura 18.8 – Simulation of a plant generated by graphic computation, Barnsley et al. (1988).

Soil structure hás also been modelled (Figure 18.9) and tested against real structures.

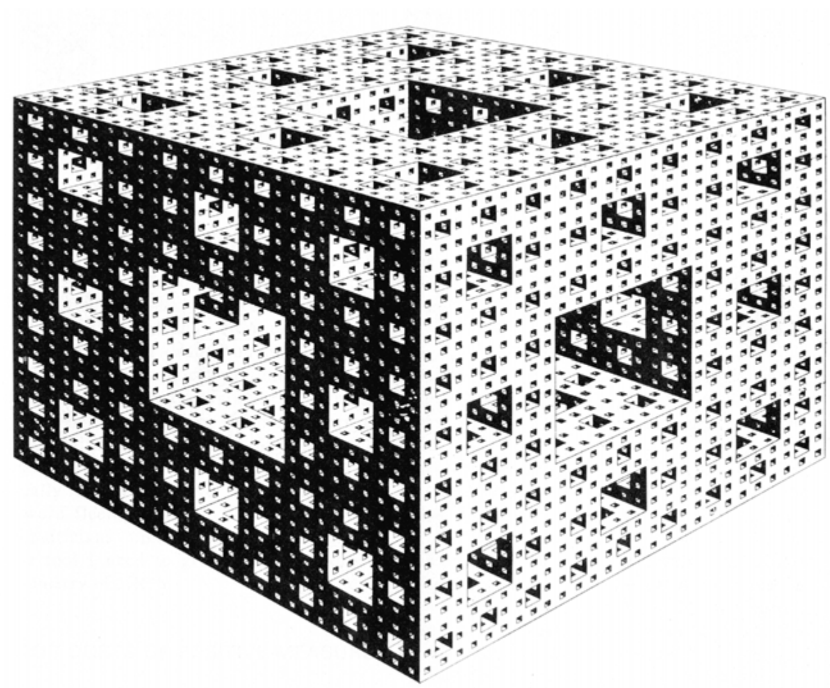


Figura 18.9 – Simulation of soil matrix.

Let us now clarify in more detail Figures 18.4 to 18.7. When we measure a length L, which can be a straight linr segment , the contour line of a country’s map, we use a an unit a linear ruler of “size” $\in$, much smaller than L. If $\in$ fits N times into L, we have:

$$L(r) = N(r)\, r, \quad \text{where} \quad r = \frac{\in}{L}$$

We write L(r) since a tortous length L, measured with a linear ruler depends on the size of the ruler since "arches" are measured as straight lines. The smaller the ruler , the better the measurement. In Figure 18.4, L is a straight line and nothing is lost in tortuosity. In the first case , L = 1, N = 1 and r = 1, that is, the ruler is L itself. It the ruler would be half of L, we would have N = 2 and r = ½. If it would be one third, N = 3 e r = 1/3.

It can be demostrated that :

$$Nr^{D} = 1 \tag{18.16}$$

D being the fractal dimension. In the euclidian geometry, D = 1 (line); D = 2 (area); D = 3 (volume). Aplying logaritm to both sides of equation 18.16, we have:

$N = r^{-D}$, or log N = -D.log r, or log N = D.log (1/r) and so:

$$D = \frac{\log N}{\log(1/r)} \tag{18.17}$$

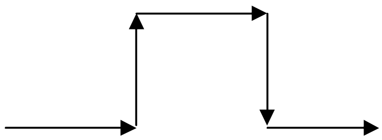
In Figure 18.4 we used the symbol D_L for the linear dimension, in which we can see through equation 18.17, that the measure is linear : $D_L = 1$, in agreement with the euclidian geometry.

In Figure 18.5 we measure bidimensional objects, i.e., areas and the euclidian dimension is $D_A = 2$, with $D_L = D_A - 1$. for tridimensional objects (volumes), we have $D_v = 3$, with $D_L = D_v - 2$ (Figure 18.6).

Equation 18.16 also admits fractional dimensions when we measure tortuous curves L or irregular areas A and volumes V . In Figure 18.7, the tortuosity is shown in a progressive way : in a) a basic length $L_{o\ \text{is given}}$; in b) 1/3 of L_o is added so that it fits into the same space as shown. If the ruler has length L_o, it will not measure L_1 which is 4/3 L_o. In c) for each increment along b, the same arrangement is made and a greater lenth L_2 = 16/9 L_o,is obtained which is also not recognized by the ruler L_o.

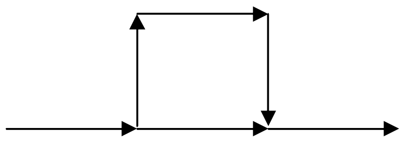
Through equation 18.17 results a dimension of D = 1.26...., greater than 1 and less than 2 of the euclidian geometry. It is not a straight line neigther an area, it is a tortuous line.

In the case of Figure 18.7, if we add 2 parts, we have:



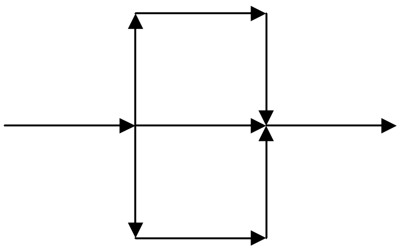
$$D = \frac{\log 6}{\log 3} = 1.63$$

and if we add 4 parts:.



$$D = \frac{\log 7}{\log 3} = 1.77$$

or still 6 parts:



$$D = \frac{\log 9}{\log 3} = 2$$

obtaining $D = D_A = 2$ which signifies that the tortuosity is so intense that the "curve" tends to an area.

In Soil Physics, since the path followed by the water , the íons os the gases, floowing the particle distribution are all tortuous, the concept of fractals could be good option for modelling. Along these lines, Tyler & Wheatcraft (1989) measured the volumetric fractal dimension of a soil using as a basis the particle distribution, measuring the slope of the realions log N versus log R, where N is the numeber of particles of radia less than R. Later , Tyler & Wheatcraft (1992) recognized the dificulty of measuring the number of particles N and used the mass os particles in a non-dimensional way $M (R < R_i)/M_t$ and the radia were also made non-dimensional R_i/R_l.

Bacchi & Reichardt (1993) used these concepts to model soil water retention curves, estimating the pore length L_i corresponding to a given textural class, employing the empiric expression of Arya & Paris (1981): $L_i = 2R_iN_i^{\alpha}$, where $2R_i$ is the diameter of the particles of class i and N_i the number of particles of this class. No success was obtained for this reseach line and it is still open for new thoughts. Bacchi et al. (1996) compared the use of the particle distribution and of the pore distribution to measure the soil fractal dimension D_v and applied their effects on soil hydraulic conductivity data.

Still among the Brazilans, Guerrini (1992, 2000) aplied the fractal geometry with success in agronomy. The basic text for fractal geometry is Mandelbrot (1982) and in addition to the already cited papers, the following should be of great interest: Puckett et al. (1985), Turcotte (1986), Tyler & Wheatcraft (1990), Guerrini & Swartzendruber (1994, 1997) e Perfect & Kay (1995).

18.9. HUMAN DIMENSIONS

I cannot finish this topic without mentioning the hunan dimensions, which are not EXACT, cannot be deterministically quantified by similar equations, calculation or indexes, Being, however, not of less importance. The subject is complex and does not fit exactly into this context, but we dare to present the dimensions proposed by Boff (1997), that of the FALK (F) and that of the CHICKEN (C), both fundamental for human existence! C is the dimension of rooting, i.e. , being fixed to standard behaviours of the daily affairs, prosaic, limited, of the "square", happy with the routinepeople which symbolizes the human behaviour that is similar to that of a chicken. F represents the dimension of the openning, of the desire ,of the poetic, of the non-limited, of the jeopardized feeling, of the hights that are charcteristic of the falcon. Boff (1997) in his book "The falk and the cicken", shows the difficulties to equilibrate ("dimensionalize" in words) these two entities.

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