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Advanced School on Quantum Monte Carlo Methods in Physics and Chemistry

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Introduction to the phaseless auxiliary field quantum Monte Carlo method

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Introduction to the phaseless auxiliary field quantum Monte Carlo method

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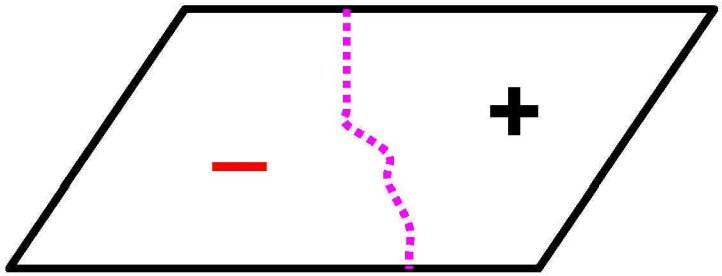
Outline

- Why auxiliary-field QMC?
  - A new approach: stochastic mean-field theory
  - Motivation: reduce QMC error & increase predictive power;
    more "black-box" like LDA or HF?
- Random walks in Slater determinant space
  - Understanding the sign (phase!) problem in this framework
  - How to control it? (approximate)
- What applications are possible?
  - Molecules and solids: T=0K
    plane-wave+Psps or Gaussians
  - Models for strongly correlated systems: T=0 and T>0K

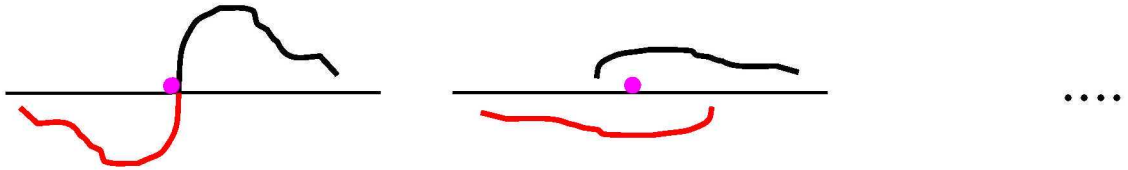
Introduction: why auxiliary-field methods?

Recall sign problem:

1 particle, first excited state:



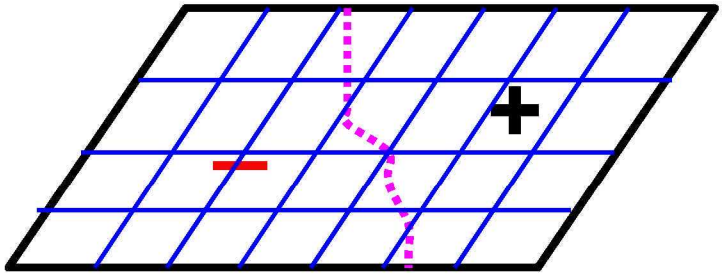
In real-space QMC, we need + and – walkers to cancel



Why auxiliary-field methods?

Recall sign problem:

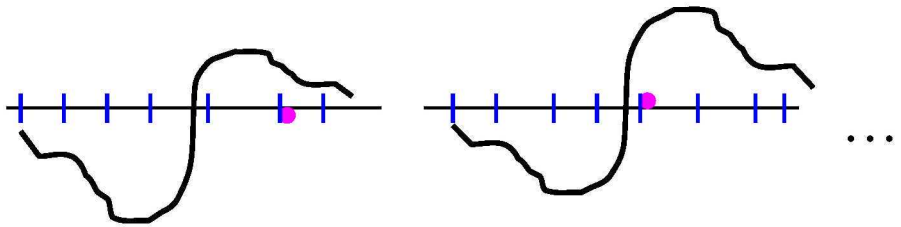
1 particle, first excited state:



Solid state or quantum chemistry?

➔ **basis**

$$e^{-\tau H} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \cdot \\ \cdot \\ \psi_N \end{pmatrix}$$



Explicit --- matrix x vec

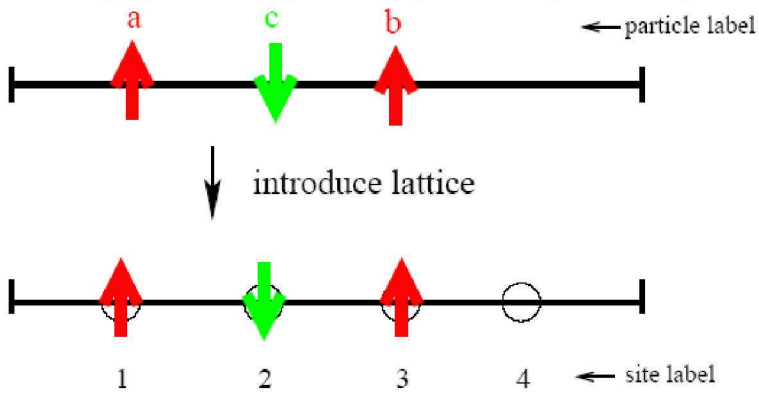
No sign problem

Why auxiliary-field methods?

Many particles?

A toy problem – trapped fermion atoms:

- 3 fermions in a box, two with $\uparrow$ spin and one with $\downarrow$ spin;
  contact interaction $V(R) = a_s\delta(r_{\rm a} - r_{\rm c}) + a_s\delta(r_{\rm b} - r_{\rm c})$ (no s-wave bt. a & b)



- Use a crude lattice basis with $i = 1, 2, 3, 4$ sites (circles). In second quantized form:

$$H = K + V = -t \sum_{\langle ij\rangle\sigma} (c^\dagger_{i\sigma} c_{j\sigma} + c^\dagger_{j\sigma} c_{i\sigma}) + U \sum_i n_{i\uparrow} n_{i\downarrow}$$

($\langle ij\rangle\sigma$: near-neighbor)

- Parameters: t; $U \propto a_s$

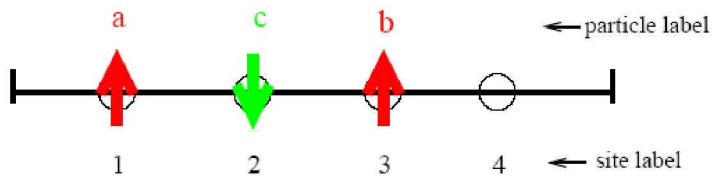
Toy problem – trapped fermions

What is the ground state when *U=0* ?

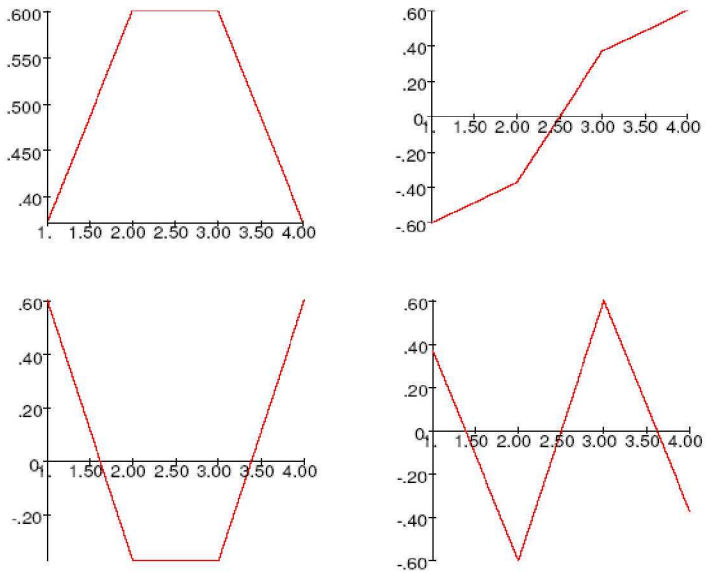
- Diagonalize *H* directly:

Single-particle Hamiltonian

$$H := \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$



Diagonalize H to find single-particle energies and w.f's
Plot wf in order of 1 , 2, 3, 4



Put fermions in lowest levels:

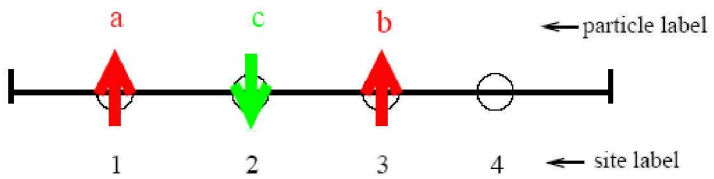
➔ many-body wf:

$$\begin{bmatrix} .3717480339 & -.6015009557 \\ .6015009541 & -.3717480349 \\ .6015009553 & .3717480339 \\ .3717480350 & .6015009543 \end{bmatrix} \cdot \begin{bmatrix} .3717480339 \\ .6015009541 \\ .6015009553 \\ .3717480350 \end{bmatrix}$$

Toy problem – trapped fermions

What is the ground state when *U=0* ?

- Diagonalize *H* directly
- Alternatively, power method:



$$e^{-\tau H}: \quad \left(\begin{array}{c} 4\times 4 \end{array} \right) \otimes \left(\begin{array}{c} 4\times 4 \end{array} \right) \equiv B_K \quad \text{operate on any } |\Psi^{(0)}\rangle \text{ repeatedly} \quad \Rightarrow |\Psi_0\rangle$$

Theorem: For any $\hat{v} = \sum_{ij} v_{ij} c_i^\dagger c_j$,
$e^{\hat{v}} |\phi\rangle = |\phi'\rangle$ where $\Phi' \equiv e^{v}\Phi$ in matrix form

Toy problem – trapped fermions

Define projection operator exp(-tau*H):

```
> P := tau -> convert(evalf(exponential((H+1.6),-tau)),Matrix);
```

For example exp(-0.1*H) looks like: (tau=0.1)

```
> P(0.1);
```

$$\begin{bmatrix} .8564116151 & .08549878210 & .004271380206 & .0001422371517 \\ .08549878209 & .8606829955 & .08564101925 & .004271380206 \\ .004271380206 & .08564101925 & .8606829955 & .08549878210 \\ .0001422371517 & .004271380206 & .08549878210 & .8564116153 \end{bmatrix}$$

>

Pick an arbitrary initial wf to project from:

```
>      --- note we're only writing out the up component
```

$$PsiT := \begin{bmatrix} 1. & -1. \\ 1. & -1. \\ 1. & 1. \\ 1. & 1. \end{bmatrix}$$

Project for a beta of 10, i.e. exp(-n*tau*H)|Psi_T>, with n*tau=10:

```
> (v0, v1) = Multiply(P(10.), PsiT)
```

$$\begin{bmatrix} .866609121199999999 & -.0000636598000000043740 \\ 1.40220301329999986 & -.0000393430999999777598 \\ 1.40220301359999988 & .0000393434000000025819 \\ .866609121099999991 & .0000636596999999961000 \end{bmatrix}$$

```
> GramSchmidt({v0,v1},normalized);
{[-.6015041283, -.3717422466, .3717450812, .6015031834],
   [.3717488488, .6015014581, .6015004522, .3717472200]}
```

Same as from direct diag.:

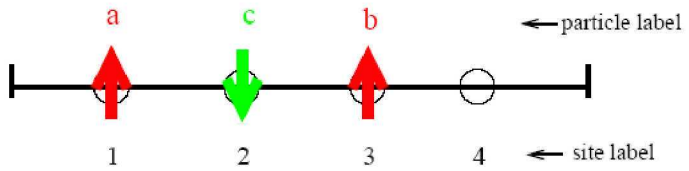
```
ground-state wf:
```

$$\begin{bmatrix} .3717480339 & -.6015009557 \\ .6015009541 & -.3717480349 \\ .6015009553 & .3717480339 \\ .3717480350 & .6015009543 \end{bmatrix} \cdot \begin{bmatrix} .3717480339 \\ .6015009541 \\ .6015009553 \\ .3717480350 \end{bmatrix}$$

Toy problem – trapped fermions

What is the ground state when *U=0* ?

- Diagonalize *H* directly
- Alternatively, power method:



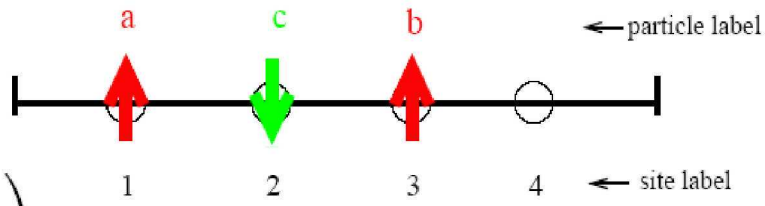
$$e^{-\tau H}: \quad \left(\begin{array}{c} 4\times 4 \end{array} \right) \otimes \left(\begin{array}{c} 4\times 4 \end{array} \right) \equiv B_K \ \text{operate on any } |\Psi^{(0)}\rangle \text{ repeatedly} \quad \Rightarrow |\Psi_0\rangle$$

- Applies to any non-interacting system
- Re-orthogonalizing the orbitals prevents fermions from collapsing to the bosonic state

→ Eliminates 'sign problem' in non-interacting systems

Toy problem – trapped fermions

Properties of Slater determinants:



$$|\phi\rangle : \quad \Phi = \begin{pmatrix} 0.37 & -0.60 \\ 0.60 & -0.37 \\ 0.60 & 0.37 \\ 0.37 & 0.60 \end{pmatrix} \otimes \begin{pmatrix} 0.37 \\ 0.60 \\ 0.60 \\ 0.37 \end{pmatrix} \quad \leftarrow \text{matrix rep.}$$

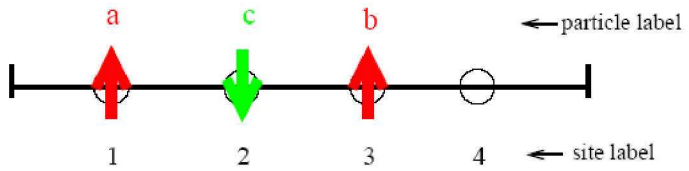
- What is the probability to find the electron configuration shown in the picture? That is, how to calculate $\langle R|\phi\rangle$?
- How to calculate $E_0 = \langle\phi|H|\phi\rangle$ from the wave function?
- How to calculate the density matrix? The spin-spin correlation function?

A: Simple matrix manipulations (See Lab exercises)

Toy problem – trapped fermions

What is the ground state when *U=0* ?

- Diagonalize *H* directly
- Alternatively, power method:



$$e^{-\tau H}: \quad \left(\; 4 \times 4 \; \right) \otimes \left(\; 4 \times 4 \; \right) \equiv B_K \;\; \text{operate on any } |\Psi^{(0)}\rangle \text{ repeatedly} \quad \Rightarrow |\Psi_0\rangle$$

What is the ground state, if we turn on *U* ?

- Lanczos (scaling !)
- Can we still write $e^{-\tau H}$ in one-body form?

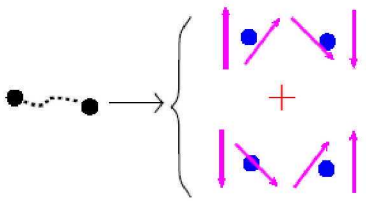
  Yes, with Hubbard-Stratonivich transformation

Introduction – why auxiliary-field methods?

Hubbard-stratonivich transformation

- Interacting two-body problem can be turned into a linear combination of non-interacting probems living in fluctuating external fields ('completion of square'):

$$e^{\tau \hat{v}^2} \xrightarrow{\text{Hubbard}-\text{Strotonivich transformation}} \int e^{-\sigma^2/2}\, e^{\sigma \sqrt{\tau}\, \hat{v}}\, d\sigma \qquad \sigma : \text{auxiliary field}$$



$$\hat{v} = \sum v_{ij} c_i^\dagger c_j : \text{one-body operator}$$

- Illustration of HS transformation — Hubbard-like interaction:

$$e^{-\tau U n_{i\uparrow} n_{i\downarrow}} \to e^{\tau U (n_{i\uparrow} - n_{i\downarrow})^2/2} = \text{factor} \times \int e^{-\frac{1}{2}x^2}\, e^{\sqrt{\tau U}\, x\, (n_{i\uparrow} - n_{i\downarrow})}\, dx$$

$$e^{-\tau U n_{i\uparrow} n_{i\downarrow}} \to e^{-\tau U (n_{i\uparrow} + n_{i\downarrow})^2/2} = \text{factor} \times \int e^{-\frac{1}{2}x^2}\, e^{\sqrt{\tau U}\, \mathrm{i}\, x\, (n_{i\uparrow} + n_{i\downarrow})}\, dx$$

Or trick by Hirsch:

$$e^{-\tau U n_{i\uparrow} n_{i\downarrow}} = e^{-\tau U (n_{i\uparrow} + n_{i\downarrow})/2} \cdot \sum\nolimits_{x=\pm 1} \frac{1}{2}\, e^{\gamma\, x\, (n_{i\uparrow} - n_{i\downarrow})} \qquad \cosh\gamma = e^{\tau U/2}$$

Back to toy problem

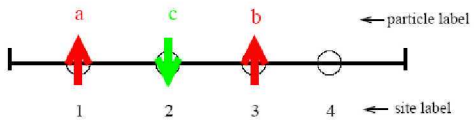
What is the ground state, if we turn on *U* ?

$$e^{-\tau U n_{i\uparrow} n_{i\downarrow}} = \text{factor} \times \sum_{x=\pm 1} \frac{1}{2}\, e^{\gamma\, x\, n_{i\uparrow}}\; e^{-\gamma\, x\, n_{i\downarrow}} \qquad \cosh\gamma = e^{\tau U/2}$$

$$e^{-\tau H} = \int d\mathbf{x}\; p(\mathbf{x}) \begin{pmatrix} e^{\gamma x_1} & 0 & 0 & 0 \\ 0 & e^{\gamma x_2} & 0 & 0 \\ 0 & 0 & e^{\gamma x_3} & 0 \\ 0 & 0 & 0 & e^{\gamma x_4} \end{pmatrix} \cdot B_{K,\uparrow}$$

$$\otimes \begin{pmatrix} e^{-\gamma x_1} & 0 & 0 & 0 \\ 0 & e^{-\gamma x_2} & 0 & 0 \\ 0 & 0 & e^{-\gamma x_3} & 0 \\ 0 & 0 & 0 & e^{-\gamma x_4} \end{pmatrix} \cdot B_{K,\downarrow}$$

$B(\mathbf{x})$ 1-particle propagator



$$\boxed{e^{-\tau H} = \int p(\mathbf{x}) B(\mathbf{x}) d\mathbf{x}} \qquad \mathbf{x} \equiv \{x_1, x_2, x_3, x_4\}$$

- With *U*, same as *U=0*, except for **integral** over **x** → Monte Carlo

Introduction to AF QMC

Standard ground-state AF QMC *Sugiyama & Koonin '86*

$$\langle \hat{O} \rangle = \frac{\langle \Psi^{(0)} | e^{-\tau H} \cdots e^{-\tau H} \, \hat{O} \, e^{-\tau H} \cdots e^{-\tau H} | \Psi^{(0)} \rangle}{\langle \Psi^{(0)} | e^{-\tau H} \cdots e^{-\tau H} \;\; e^{-\tau H} \cdots e^{-\tau H} | \Psi^{(0)} \rangle}$$

$$\Downarrow \qquad \boxed{e^{-\tau H} = \int p(\mathbf{x}) B(\mathbf{x}) d\mathbf{x}}$$

$$\frac{\int p(\mathbf{x}^{(1)}) \cdots p(\mathbf{x}^{(2L)}) \, \langle \Psi^{(0)} | B(\mathbf{x}^{(2L)}) \cdots B(\mathbf{x}^{(L+1)}) \, \hat{O} \, B(\mathbf{x}^{(L)}) \cdots B(\mathbf{x}^{(1)}) | \Psi^{(0)} \rangle \; d\mathbf{x}^{(1)} \cdots d\mathbf{x}^{(2L)}}{\int p(\mathbf{x}^{(1)}) \cdots p(\mathbf{x}^{(2L)}) \, \langle \Psi^{(0)} | B(\mathbf{x}^{(2L)}) \cdots B(\mathbf{x}^{(L+1)}) \;\; B(\mathbf{x}^{(L)}) \cdots B(\mathbf{x}^{(1)}) | \Psi^{(0)} \rangle \; d\mathbf{x}^{(1)} \cdots d\mathbf{x}^{(2L)}}$$

Choose $|\Psi^{(0)}\rangle$ as a Slater determinant $\boxed{B(\mathbf{x})|\phi\rangle = |\phi'\rangle}$

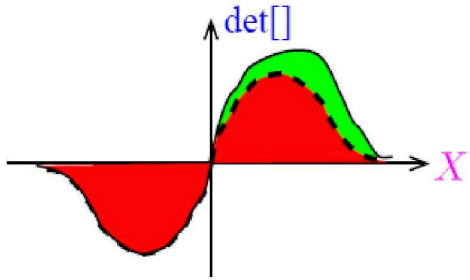
Many-dim integral can be done by Monte Carlo: $\dfrac{\int O_{\mathrm{Gr}}(X) p(X) \det[X] dX}{\int p(X) \det[X] dX}$ $\qquad X \equiv \{\mathbf{x}^{(l)}\}$

Applications mostly to "simple models":

- Hubbard model, impurity models in condensed matter
- nuclear shell model
- lattice QCD

Introduction to AF QMC

Sign problem in standard AF QMC:



As system size grows, average sign of det[ ] $\rightarrow 0$ exponentially.

$\Rightarrow$ exponential scaling

- Sign problem is often most severe where the physics is most interesting, for example, in 2-D Hubbard model when number of electrons $\sim 85\%$ number of lattice sites, where it is thought to model the CuO planes of high-T_c cuprates
- In fact, a phase (not just sign) problem appears for general 2-body interactions.

Some "lingo" from mean field

- Electronic Hamiltonian: (Born-Oppenheimer)

$$H = H_{1-\text{body}} + H_{2-\text{body}} = -\frac{\hbar^2}{2m}\sum_{i=1}^{M}\nabla_i^2 + \sum_{i=1}^{M} V_{\text{ext}}(\mathbf{r}_i) + \sum_{i<j}^{M} V_{\text{int}}(|\mathbf{r}_i - \mathbf{r}_j|)$$

can choose any single-particle basis $\{|\chi_i\rangle\}$

$$\hat{H} = \sum_{i,j}^{N} T_{ij} c_i^\dagger c_j + \sum_{i,j,k,l}^{N} V_{ijlk} c_i^\dagger c_j^\dagger c_k c_l$$

$$\int \chi_i^\star(\mathbf{r}_1)\chi_j^\star(\mathbf{r}_2)\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}\chi_k(\mathbf{r}_2)\chi_l(\mathbf{r}_1)\, d\mathbf{r}_1 d\mathbf{r}_2$$

- An orbital:

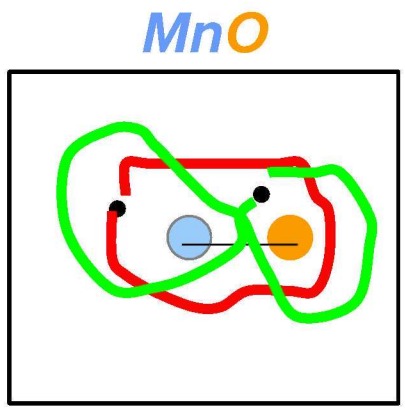
$$|\varphi_m\rangle = \sum_{i=1}^{N} \varphi_{i,m}|\chi_i\rangle$$

- A Slater determinant:

$$\begin{pmatrix} \varphi_{1,1} & \varphi_{1,2} & \cdots & \varphi_{1,M} \\ \varphi_{2,1} & \varphi_{2,2} & \cdots & \varphi_{2,M} \\ \vdots & \vdots & & \vdots \\ \varphi_{N,1} & \varphi_{N,2} & \cdots & \varphi_{N,M} \end{pmatrix}$$

N : basis

M : electrons



Summary: basic formalism of AF methods

To obtain **ground state**, use projection in imaginary-time:

$$|\Psi^{(n+1)}\rangle = e^{-\tau \hat{H}} \; |\Psi^{(n)}\rangle \quad \xrightarrow{n\to\infty} \quad |\Psi_0\rangle$$

τ: cnst, small $|\Psi^{(0)}\rangle$: arbitrary initial state

Electronic Hamiltonian: (2nd quantization, given any 1-particle basis)

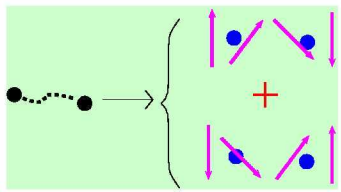
$$\hat{H} = \hat{H}_1 + \hat{H}_2 = \sum_{i,j}^{M} T_{ij} c_i^\dagger c_j + \sum_{i,j,k,l}^{M} V_{ijlk} c_i^\dagger c_j^\dagger c_k c_l$$

M : basis size

$\hat{H}_2 \to -\sum \hat{v}^2$ with $\hat{v}$ = 1-body

Hubbard-Strotonivich transf.

$$e^{-\tau \hat{H}} \quad \to \quad e^{-\tau \hat{H}_1} \int e^{-\sigma^2/2} \, e^{\sigma \sqrt{\tau}\, \hat{v}} \, d\sigma$$



interacting system $\to \sum$ (non-interacting system in auxiliary fields)

next →

AF methods: some background

- Applied in models in condensed matter, nuclear physics, (lattice QCD), ….

  Scalapino, Sugar, Hirsch, White *et al.*; Koonin; Sorella, ….

  $$\text{interacting} \rightarrow \sum (\text{non-interacting in fields})$$

  basic idea: Monte Carlo to do **sum** (path integral)

- However,
  - sign problem for "simple" interactions (Hubbard)
  - phase problem for realistic interaction

    Fahy & Hamann; Baroni & Car; Wilson & Gyorffy; Baer *et. al.*; ….

- Reformulate ---

Slater determinant random walk (preliminary I)

- In general, we can choose any single-particle basis $\{|\chi_i\rangle\}$, with $i = 1, 2, \cdots, N$
- A single-particle orbital (labeled by m) is given by $\hat{\varphi}_m{}^\dagger|0\rangle \equiv \sum_{i=1}^{N} \varphi_{i,m}|\chi_i\rangle$
- If we have M identical fermions ($M \leq N$), a Slater determinant $|\phi\rangle$ is given by:

$$|\phi\rangle \equiv \hat{\varphi}_1{}^\dagger \hat{\varphi}_2{}^\dagger \cdots \hat{\varphi}_M{}^\dagger |0\rangle$$

- $|\phi\rangle$ is represented by an $N \times M$ matrix:

$$\Phi \equiv \begin{pmatrix} \varphi_{1,1} & \varphi_{1,2} & \cdots & \varphi_{1,M} \\ \varphi_{2,1} & \varphi_{2,2} & \cdots & \varphi_{2,M} \\ \vdots & \vdots & & \vdots \\ \varphi_{N,1} & \varphi_{N,2} & \cdots & \varphi_{N,M} \end{pmatrix}$$

- E.g., $\langle\phi|\phi'\rangle = \det(\Phi^{\mathrm{T}}\Phi')$; $\quad G_{ij} \equiv \frac{\langle\phi|c_i^\dagger c_j|\phi'\rangle}{\langle\phi|\phi'\rangle} = [\Phi'(\Phi^{\mathrm{T}}\Phi')^{-1}\Phi^{\mathrm{T}}]_{ij}$;
  any 2-body correlation $\leftarrow \{G_{ij}\}$

Slater determinant random walk (preliminary II)

HS transformation:

For example in electronic systems:

$$H = K + V_{\mathrm{e-I}} + V_{\mathrm{e-e}} + V_{\mathrm{I-I}}$$

In plane-wave one-particle basis $|k\rangle \equiv \frac{1}{\sqrt{\Omega}} e^{i\mathbf{G}_k \cdot \mathbf{r}}$ **:**

$$V_{\mathrm{e-I}} = \sum_{i \neq j} V_{\mathrm{local}}(\mathbf{G}_i - \mathbf{G}_j) c_i^\dagger c_j + \sum_{i,j} V_{\mathrm{NL}}(\mathbf{G}_i, \mathbf{G}_j) c_i^\dagger c_j$$

$$V_{\mathrm{e-e}} = \frac{1}{2\Omega} \sum_{i,j,\mathbf{Q}\neq 0} \frac{4\pi}{|\mathbf{Q}|^2} c^\dagger_{\mathbf{G}_i+\mathbf{Q}} c^\dagger_{\mathbf{G}_j-\mathbf{Q}} c_{\mathbf{G}_j} c_{\mathbf{G}_i}$$

$$\rightarrow -\frac{1}{2\Omega} \sum_{\mathbf{Q}\neq 0} \frac{4\pi}{|\mathbf{Q}|^2} \rho^\dagger(\mathbf{Q})\, \underline{\rho(\mathbf{Q})}$$

$$\underline{\rho(\mathbf{Q})} \nwarrow \sum_i c^\dagger_{\mathbf{G}_i+\mathbf{Q}} c_{\mathbf{G}_i}$$

$$\rightarrow \sum_{\mathbf{Q}\neq 0} \sqrt{\frac{4\pi}{|\mathbf{Q}|^2}} \left([\, \underbrace{\rho^\dagger(\mathbf{Q}) + \rho(\mathbf{Q})}_{i\,\hat{v}} \,]^2 - [\, \underbrace{\rho^\dagger(\mathbf{Q}) - \rho(\mathbf{Q})}_{\hat{v}'} \,]^2 \right)$$

'density' decomposition

New AF QMC approach

Random walks in Slater determinant space:

Recall $|\Psi^{(n+1)}\rangle = e^{-\tau \hat{H}} \; |\Psi^{(n)}\rangle \xrightarrow{n\to\infty} |\Psi_0\rangle$

SZ, Carlson, Gubernatis

SZ, Krakauer

H-S transformation

$\int e^{-\sigma^2/2} \; e^{\hat{\mathbf{v}}(\sigma)} d\sigma$

1-body: $\sum_{i,j} v_{ij}(\sigma) c_i^\dagger c_j$

Schematically:

$|\Psi^{(0)}\rangle \xrightarrow{e^{-\tau \hat{H}}} |\Psi^{(1)}\rangle \; .... \; \rightarrow |\Psi_0\rangle$

sample σ from $e^{-\frac{\sigma^2}{2}}$;
apply 1-body propag.

$|\phi^{(0)}\rangle \rightarrow |\phi^{(1)}(\sigma)\rangle \rightarrow |\phi\rangle$

$|\Psi_0\rangle \doteq \sum_\phi |\phi\rangle$

Exact so far

next →

Connection with DMC

Many-dim. electronic configuration space: $R = \{ \mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_M \}$

$$\hat{H} = \sum_i^M \frac{\hat{\mathbf{p}}_i^2}{2m} + \hat{V}$$

$$|\Psi^{(n+1)}\rangle = e^{-\tau \hat{H}} \; |\Psi^{(n)}\rangle \; \rightarrow \; |\Psi_0\rangle$$

$$e^{-\tau \hat{\mathbf{p}}_i^2/2m} = \int e^{-\sigma^2/2} \; e^{i \hat{\mathbf{p}}_i \cdot (\gamma \, \sigma)} \; d\sigma \qquad \gamma = \sqrt{\tfrac{\tau}{m}}$$

$$e^{-\tau \hat{H}} = \int e^{-\vec{\sigma}^2/2} \; e^{i \hat{P} \cdot (\gamma \, \vec{\sigma})} \; d\vec{\sigma} \; e^{-\tau \hat{V}} \qquad \vec{\sigma}\text{: } 3M\text{-dim vector}$$

translation op.

Random walk realization of $\boxed{\cdots}$: basic idea (importance sampling can also be derived)

$$|\Psi^{(0)}\rangle \xrightarrow{\quad e^{-\tau H} \quad} |\Psi^{(1)}\rangle \; .... \; \rightarrow \; |\Psi_0\rangle$$

$$|R^{(0)}\rangle \xrightarrow[\text{sample } \vec{\sigma} \text{ from Gaussian;}]{\text{multiply weight by } e^{-\tau V(R^{(0)})}} |R^{(1)}\rangle \; \rightarrow \; |R\rangle \quad \text{diffusion + branching}$$

$$\text{translate } R^{(0)} \text{ by } (-\gamma \vec{\sigma})$$

$$\begin{array}{ccc} \vdots & \vdots & \vdots \end{array}$$

Random walks in Slater determinant space

Standard DMC

$|R\rangle = |\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_M\rangle$

$|\Psi_0\rangle = \sum_R \Psi_0(R)|R\rangle$

$\Downarrow$

$|\Psi_0\rangle \doteq \sum_{\mathrm{MC}} |R\rangle$

Slater determinant RW

$|\phi\rangle = |\psi_1, \psi_2, \cdots, \psi_M\rangle$

$\sum_k c_{k,i}|\chi_k\rangle$ basis

$|\Psi_0\rangle = \sum_\phi \Psi_\phi|\phi\rangle$

$\Downarrow$

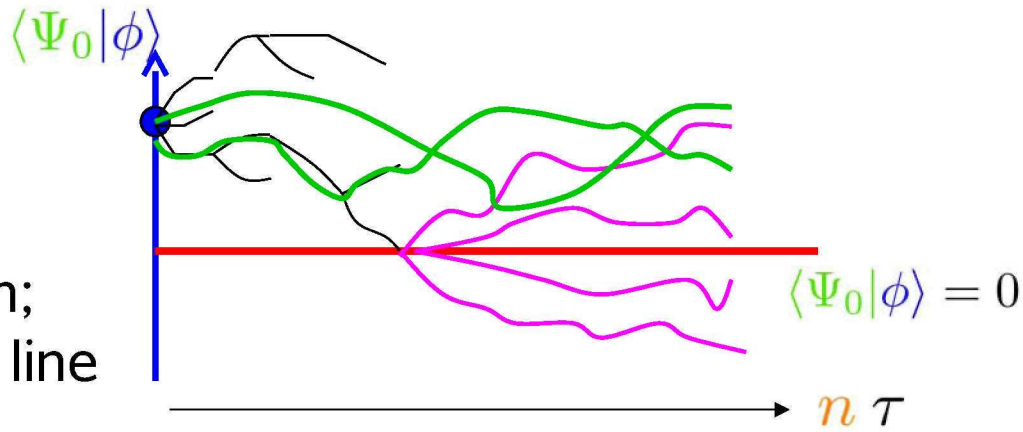
$|\Psi_0\rangle \doteq \sum_{\mathrm{MC}} |\phi\rangle$

- The formalism is appealing — each random walker is a full Slater determinant
- Close formal relation to mean-field approaches. The QMC thus shares the same machinery as DFT or Hartree-Fock, using *any* one-particle basis
  - Second-quantization, antisymmetry automatically imposed
  - The single-particle problem ($\hat{H}_1$) is solved exactly, with no statistical error
  - Correlation effects are obtained by building stochastic ensembles of independent-particle solutions
- Core-electron problem: non-local pseudopotential can be implemented straightforwardly — *locality approximation* eliminated

But … sign problem

E.g., in Hubbard:

- $e^{-\tau \hat{H}}$ → paths in Slater determinant space



- Suppose $|\Psi_0\rangle$ is known; consider "**hyper-node**" line

- If path reaches **hyper-node**

$$\langle \Psi_0 | \phi \rangle = 0$$
$$\Rightarrow \langle \Psi_0 | e^{-n\tau \hat{H}} | \phi \rangle = 0$$

then its descendent paths collectively contribute 0

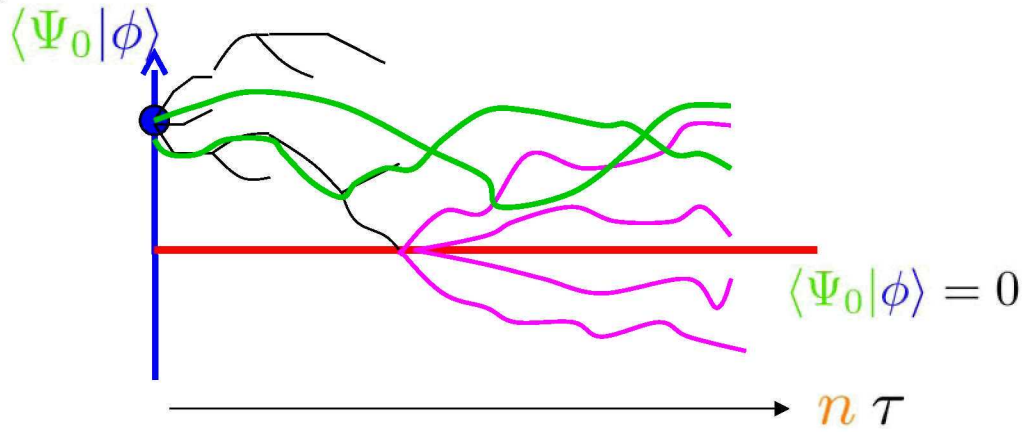
- MC signal is exponentially small compared to noise

In special cases (1/2 filling, or U<0), symmetry keeps paths to one side → no sign problem

next →

How to control the sign problem?

Constrained path appr.



keep only **paths that never reach the node**

require $\langle \Psi_{\mathbf{T}} | \phi \rangle > 0$

Trial wave function

Zhang, Carlson, Gubernatis, '97

Zhang, '00

next →

Introduction to T>0 method

Standard finite-T method *Blankenbecler, Scalapino, and Sugar, '81*

Partition function for Hamiltonian H is: $(\beta = 1/k\,T)$

$$\mathrm{Tr}(e^{-\beta H}) = \mathrm{Tr}(\, e^{-\tau H}\, e^{-\tau H}\, \cdots\, e^{-\tau H}\,)$$

Need: $e^{-\tau H} = \sum_{\mathbf{x}} B(\mathbf{x})$

$$\langle O \rangle = \frac{\mathrm{Tr}(O\, e^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})} = \frac{\sum_{\{\mathbf{x}_l\}} \mathrm{Tr}(O B(\mathbf{x}_L) B(\mathbf{x}_{L-1}) \cdots B(\mathbf{x}_1))}{\sum_{\{\mathbf{x}_l\}} \mathrm{Tr}(B(\mathbf{x}_L) B(\mathbf{x}_{L-1}) \cdots B(\mathbf{x}_1))}$$

Analytically evaluate trace: $\mathrm{Tr}(e^{-\beta H}) = \sum_{\{\mathbf{x}_l\}} \det[\, I + B(\mathbf{x}_L)\, B(\mathbf{x}_{L-1}) \cdots B(\mathbf{x}_1)\,]$

Sample fields $\{\mathbf{x}_l\}$ by Metropolis Monte Carlo to compute sum.

Sign Problem in standard finite-T AF QMC:

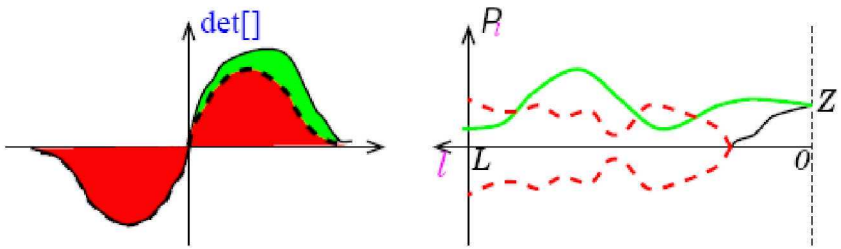
- As T lowers, average sign of $\det[\,]$ $\to 0$ exponentially.
- We need to control the sign problem — focus on real auxiliary fields, i.e., real $\hat{v}$

The sign problem at finite-T

Imagine introducing path integrals one time slice at a time: *Zhang, '99*

$$
\begin{aligned}
Z &= \mathrm{Tr}\left(e^{-\tau H} e^{-\tau H} \cdots e^{-\tau H} e^{-\tau H} \right) && P_0 \\
&= \sum_{\{\mathbf{x}_1\}} \mathrm{Tr}\left(e^{-\tau H} e^{-\tau H} \cdots e^{-\tau H} B(\mathbf{x}_1) \right) && P_1(\{\mathbf{x}_1\}) \quad \leftarrow \text{integrand} \\
&= \sum_{\{\mathbf{x}_1,\mathbf{x}_2\}} \mathrm{Tr}\left(e^{-\tau H} e^{-\tau H} \cdots B(\mathbf{x}_2)\, B(\mathbf{x}_1) \right) && P_2(\{\mathbf{x}_1,\mathbf{x}_2\}) \\
&= \cdots \\
&= \sum_{\{\mathbf{x}_l\}} \det[\, I + B(\mathbf{x}_L)\, B(\mathbf{x}_{L-1}) \cdots B(\mathbf{x}_1) \,] && P_L(\{\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_L\})
\end{aligned}
$$

Suppose we know $e^{-\tau H}$. Consider P_l:



- If $P_l = 0$, all future paths $\{\mathbf{x}_{l+1}, \mathbf{x}_{l+2}, \cdots, \mathbf{x}_L\}$ collectively contribute 0 in Z.
- A complete path $\{\mathbf{x}_l\}$ contributes to Z **iff** $P_l > 0$ for all l.

Constrained path method at finite-T

Constraint to control the sign problem

Require: $P_1(\{\mathbf{x}_1\}) > 0$; $P_2(\{\mathbf{x}_1, \mathbf{x}_2\}) > 0$; ....; $P_L(\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_L\}) > 0$.

- Constraint eliminates all noise paths ('dashed lines').
- In practice, we use trial B_T for $e^{-\tau H}$ — approximate.

(HF propagator)

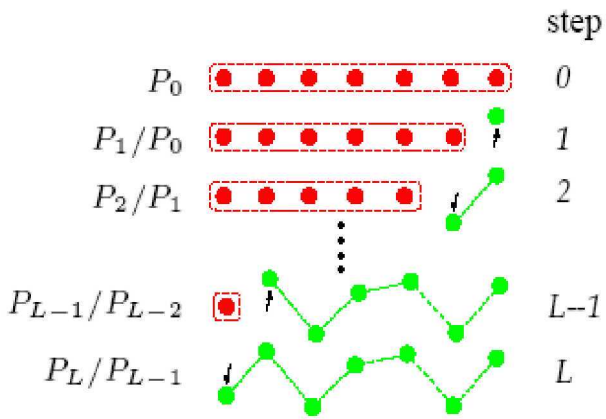
Monte Carlo sampling algorithm to incorporate constraint

If B_T is $\sum$(mean-field), then Tr $\rightarrow$ det[ ] in P_l.

Sampling — random walk of L steps:

Note:

$$P_L = \frac{P_L}{P_{L-1}} \frac{P_{L-1}}{P_{L-2}} \cdots \frac{P_2}{P_1} \frac{P_1}{P_0} P_0$$



Recovery from wrong trial w.f.

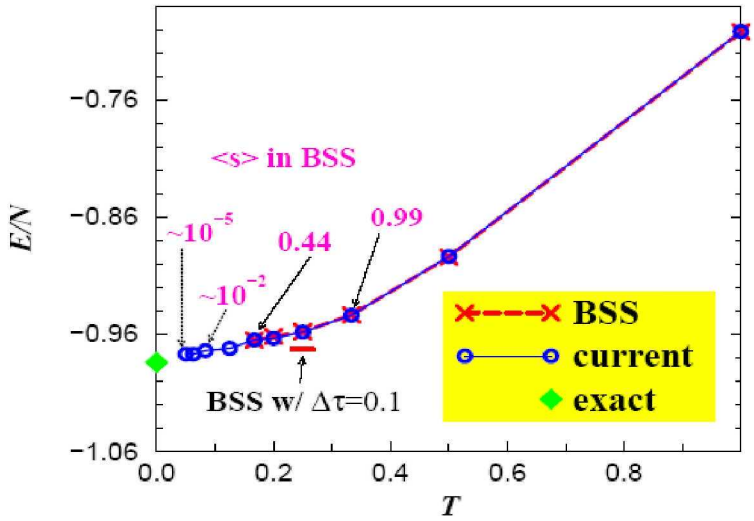
More predictive QMC: requires reducing reliance on trial wf

2-D Hubbard model: **finite-*T***

- *U*>0; 12% doping, 4x4
- Sign problem severe <s>~10^-5

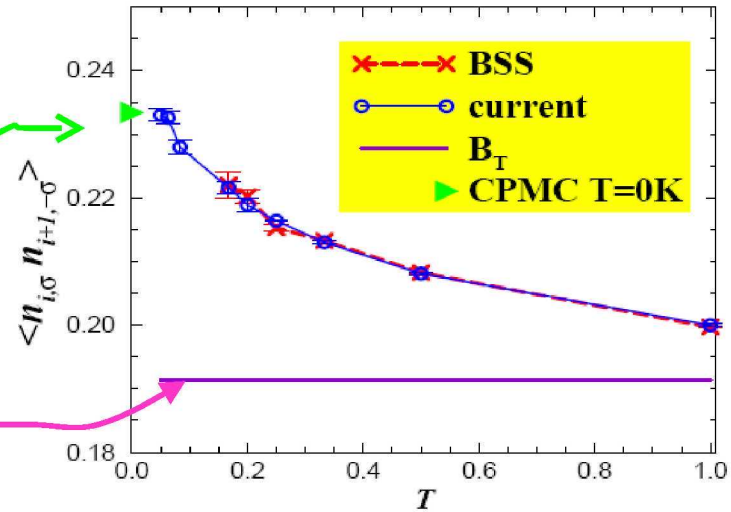
Compare with:

- high T: exact calculation with sigr problem
- T=0K: exact diag.



AFM order

wrong trial



New AF QMC approach

Random walks in Slater determinant space:

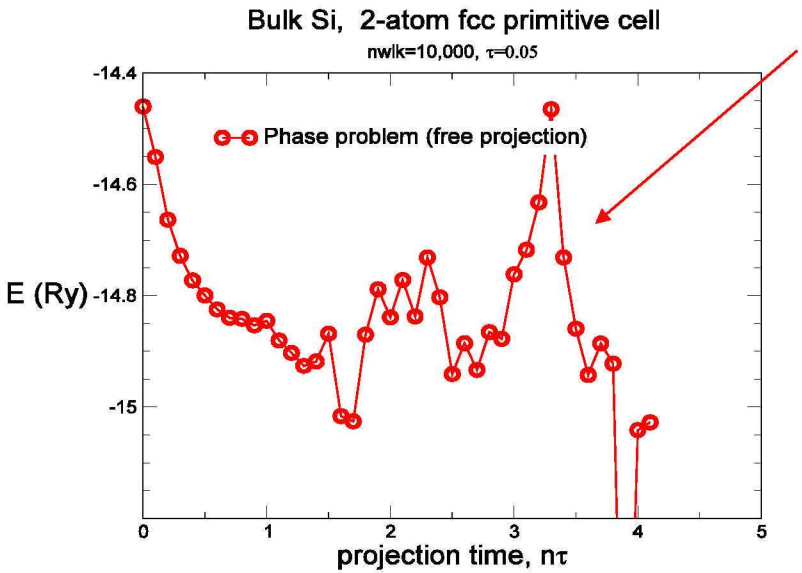
$$\text{Recall } |\Psi^{(n+1)}\rangle = e^{-\tau \hat{H}} \; |\Psi^{(n)}\rangle \xrightarrow{n\to\infty} |\Psi_0\rangle$$

H-S transformation

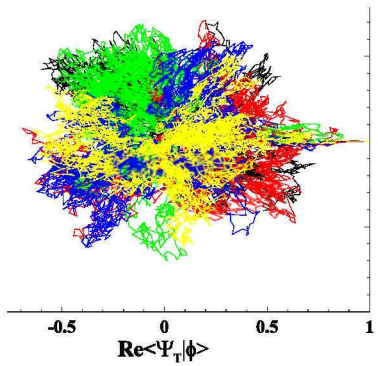
$$\int e^{-\sigma^2/2} \, e^{\hat{\mathbf{v}}(\sigma)} d\sigma$$

1-body: $\sum_{i,j} v_{ij}(\sigma) c_i^\dagger c_j$

For general interaction, phase problem:



Exponential noise



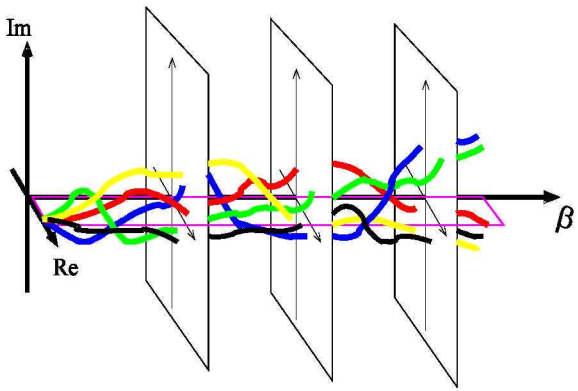
problem!

$\sum_{\text{MC}} |\phi\rangle \to 0$

next →

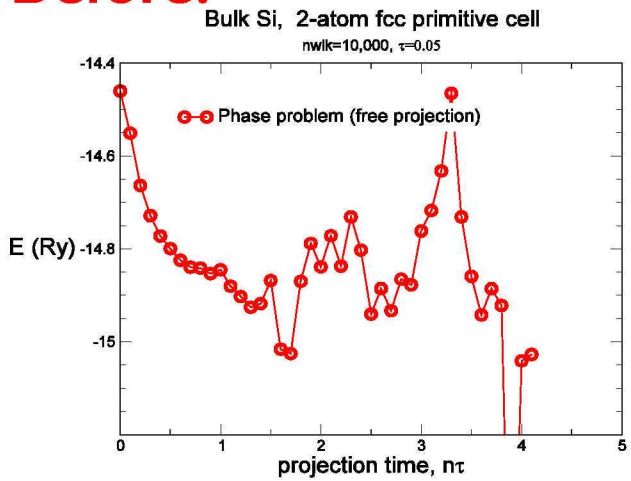
Controlling the phase problem

Sketch of approximate **solution**:

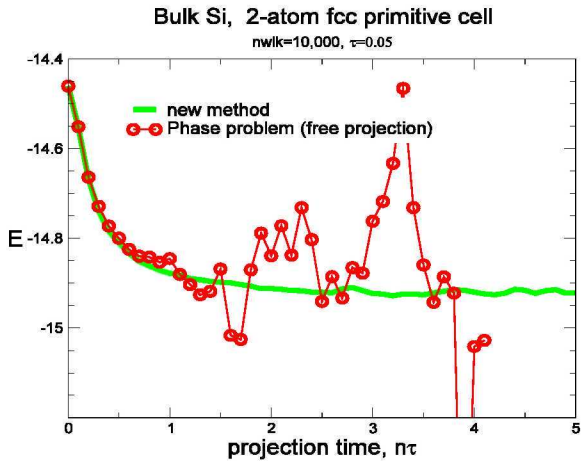


- Modify propagator by "importance sampling":
  phase → degeneracy (use trial wf)
- Project **to one overall phase**: $\sum_\phi \frac{|\phi\rangle}{\langle \Psi_T | \phi \rangle}$
  break symmetry (+/- → rotation)

Before:



After:



Controlling the phase problem --- more details

(a) Phaseless formalism *SZ & Krakauer*

- Seek MC representation of $|\Psi_0\rangle$ in the form: $|\Psi_0\rangle \doteq \sum_\phi \frac{|\phi\rangle}{\langle\Psi_T|\phi\rangle}$

  i.e., the contribution of each $|\phi\rangle$ is independent of its phase (if $|\psi_T\rangle$ is exact)

- This is accomplished by an "importance-sampling" transformation to modify the propagator:

$$\int \langle\Psi_T|\phi'(\sigma)\rangle\, e^{-\frac{1}{2}\sigma^2} B(\sigma)\, d\sigma\, \frac{1}{\langle\Psi_T|\phi\rangle} = e^{-\tau\hat{H}_1} \int e^{-\sigma^2/2}\, e^{(\sigma-\bar{\sigma})\sqrt{\tau}\,\hat{v}}\, d\sigma\; e^{-\tau Re\{E_L(\phi)\}}$$

  ⋆ Force bias: $\bar{\sigma} \equiv -\frac{\langle\Psi_T|\sqrt{\tau}\,\hat{v}|\phi\rangle}{\langle\Psi_T|\phi\rangle}$ ← complex!

  ⋆ Local energy: $E_L(\phi) \equiv \frac{\langle\Psi_T|\hat{H}|\phi\rangle}{\langle\Psi_T|\phi\rangle}$

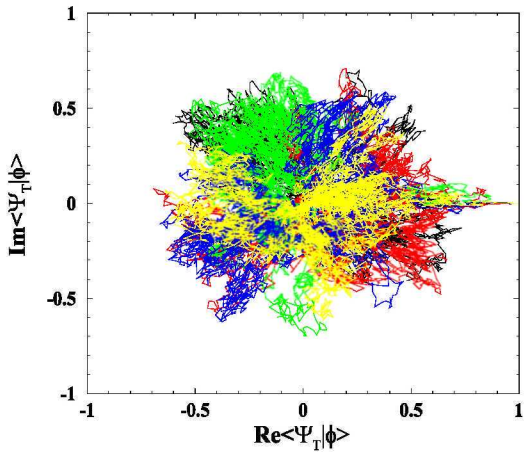
(b) Projection to break "rotational invariance"

- With **(a)**, we can confine the RW to one overall phase (e.g., 0)
- This is accomplished by projecting the RW onto 1D: reducing the weight of a walker according to its phase change, e.g., by $\cos(\Delta\theta)$

Controlling the phase problem: some comments

Subtleties:

- Constraint **before** importance sampling:
  $\mathrm{Re}\langle\Psi_T|\phi\rangle > 0,$
  then use $\mathrm{Re}\langle\Psi_T|\phi\rangle$ as importance function
  --- natural (!?), but does not work well
- Instead, project **after** "importance sampling":
  use complex importance function $\langle\Psi_T|\phi\rangle$



It helps to subtract "mean-field background" in HS:

$$\hat{v}^2 \rightarrow (\hat{v} - \langle\hat{v}\rangle)^2 + 2\hat{v}\langle\hat{v}\rangle - \langle\hat{v}\rangle^2$$

If $\hat{v}$ is real, method reduces to constrained path MC

Two-dimensionality unique -
connection and difference(!) with fixed-phase

Discussion – new AF QMC

- Pluses
  - **Sign problem** is often found to be reduced

    ← more robust and predictive methods
  - Can do down-folded Hamiltonians (realistic models)
  - **Uses a basis** --- walkers are Slater determinants

    formal connection to DFT --- k-pts, non-loc psp's, PAW's, ….
- Minuses
  - **Uses a basis** --- finite basis-size error
  - Mixed-estimator of total energy is not variational
  - Not straightforward to include a Jastrow factor in trial w.f. **(….)**

Application: Hubbard model

- Simplest model combining band structure and interaction:

$$H = K + V = -t \sum_{\langle ij \rangle \sigma} (c^\dagger_{i\sigma} c_{j\sigma} + c^\dagger_{j\sigma} c_{i\sigma}) + U \sum_i n_{i\uparrow} n_{i\downarrow}$$

electrons on a 2-D lattice

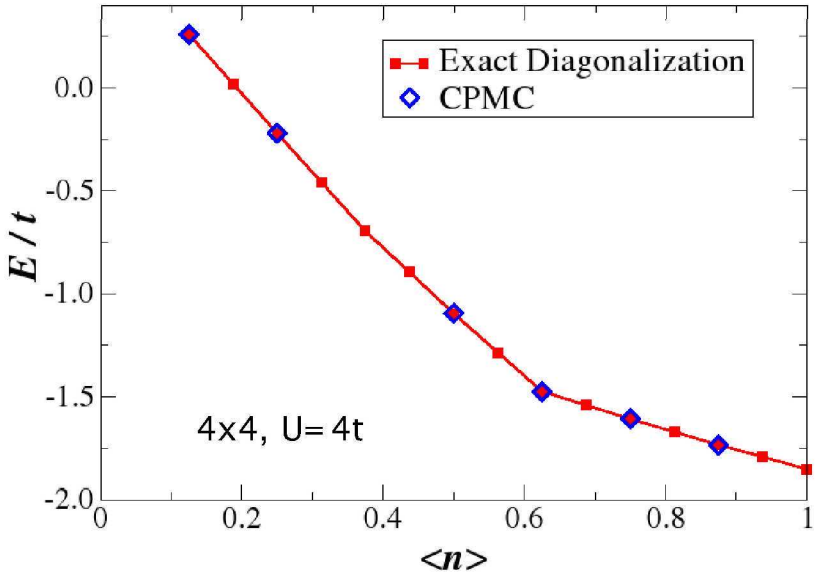
- near-neighbor hopping
- on-site repulsion

Size $N = L \times L$

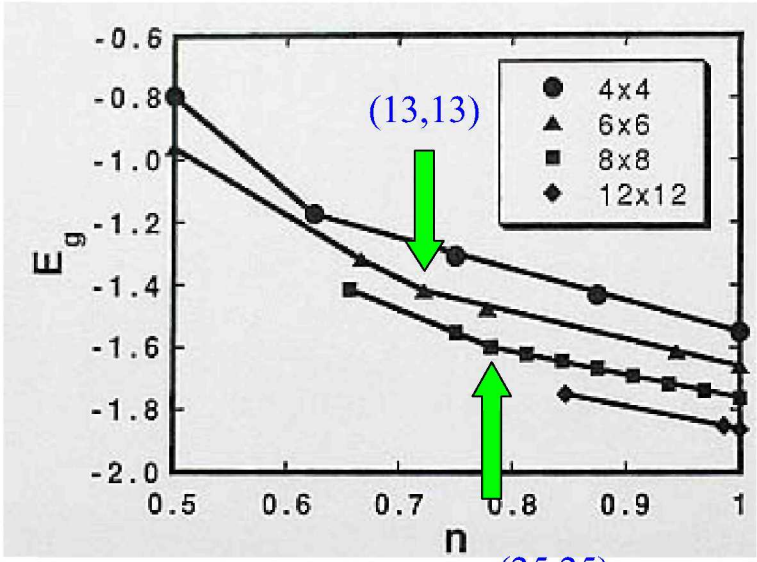
Filling $\langle n \rangle = \frac{N_\uparrow + N_\downarrow}{N}$

- Renewed interest due to many experimental opportunities:
  - optical lattices
  - trapped Fermi gas (unitarity: QMC key)
- Long-standing: connection to cuprates? phase separation?
- We look at *ground-state* energy vs. filling

Hubbard model: equation of state



Exact diagonalization: Dagotto et.al. 1992
CPMC: Zhang et.al., 1997

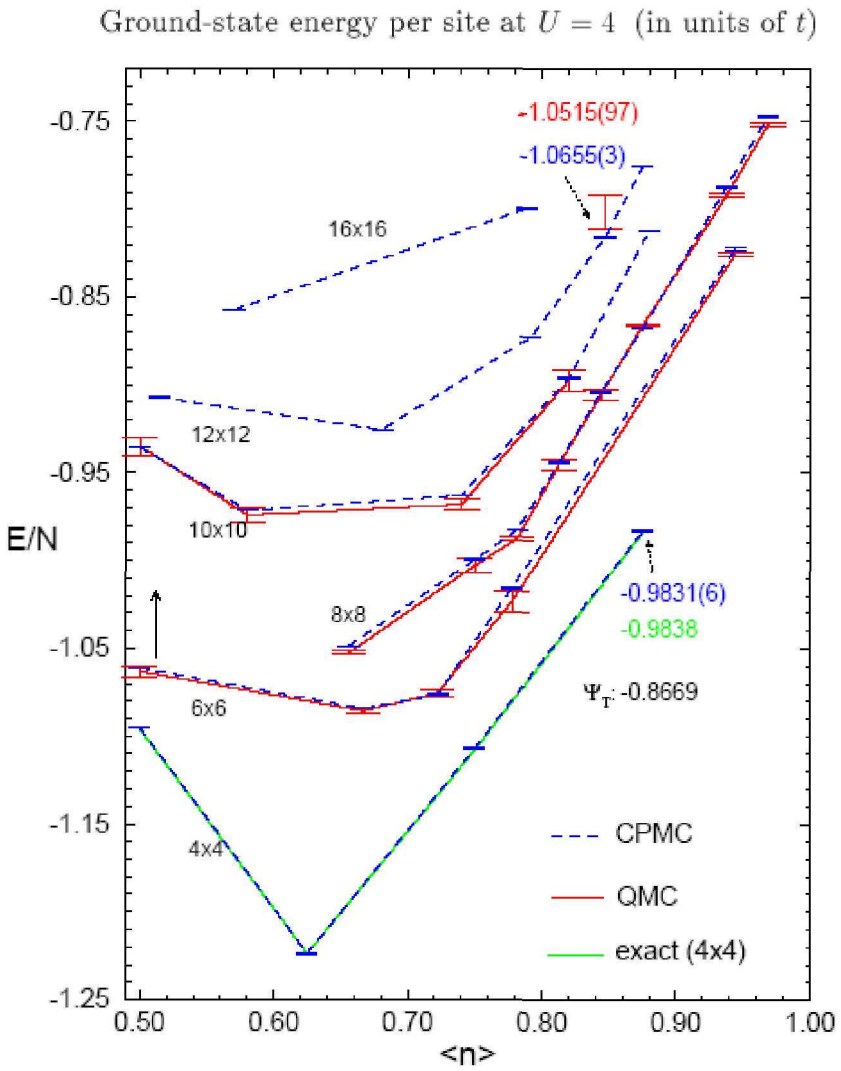


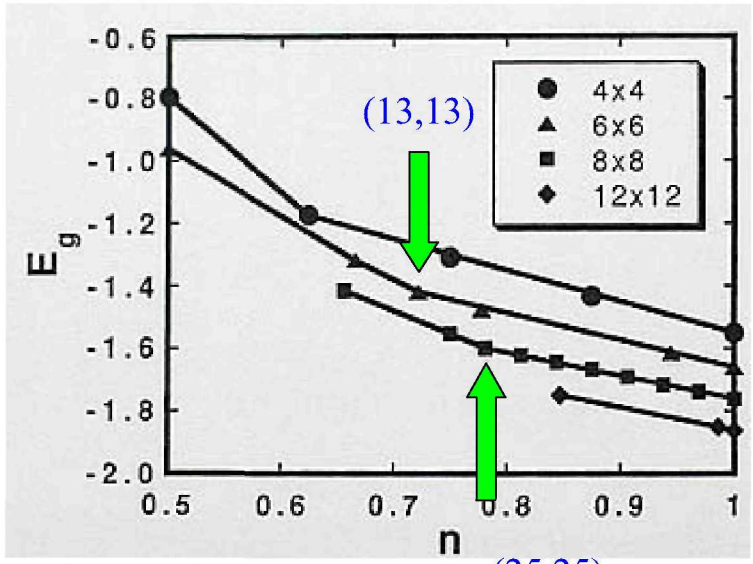
Furukawa and Imada, 1992

(25,25)

- Constrained-path auxiliary field QMC (CPMC) is accurate.
- There are kinks at closed-shell fillings => large shell effects.

Hubbard model: equation of state



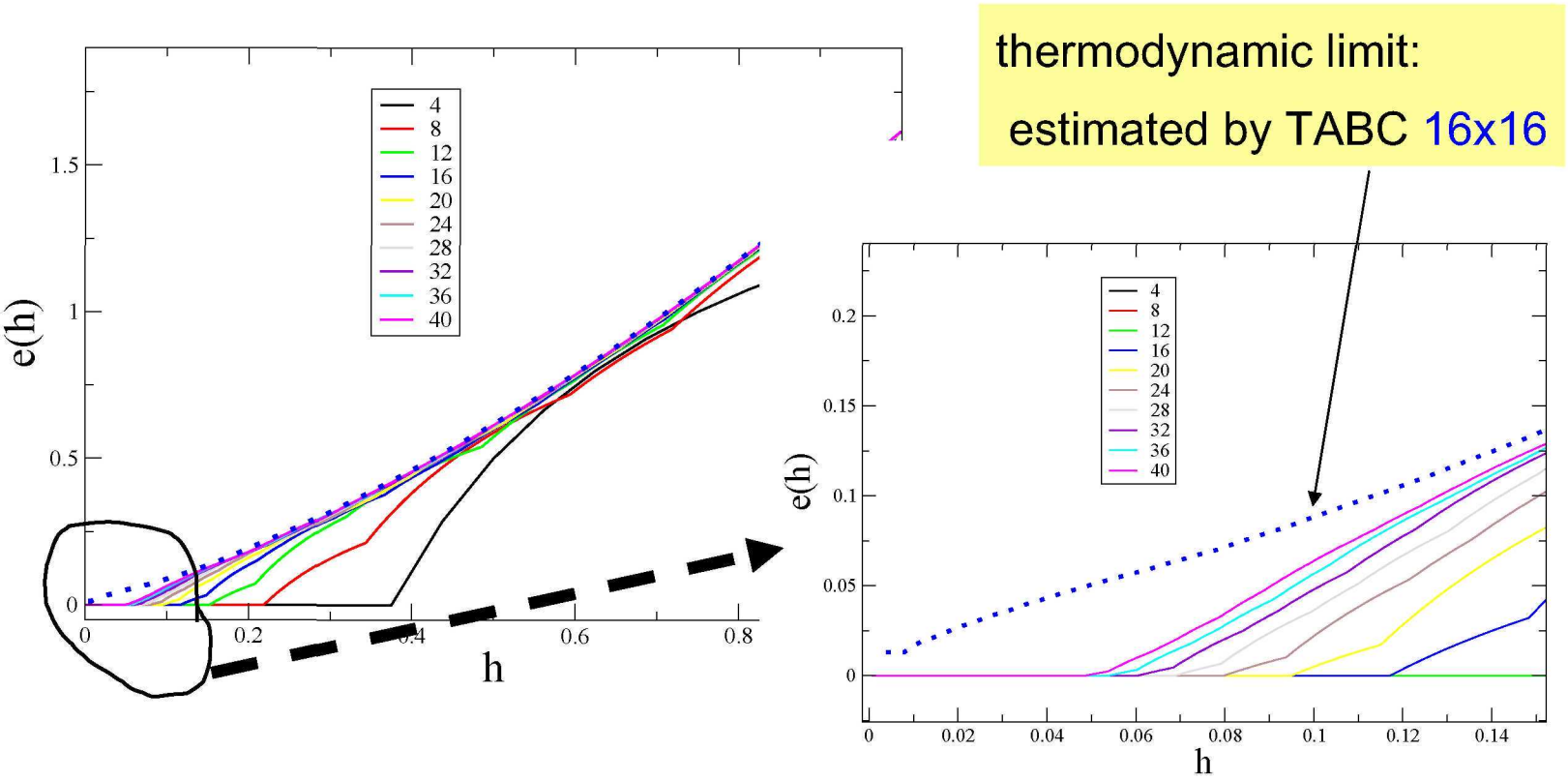


Furukawa and Imada, 1992

(25,25)

CPMC data

Hubbard model: persistent shell effects



- One signal for phase separation: does e(h) turn ?
- Shell effect persists to >*40x40,* leads to bias

Twist averaged boundary conditions (TABCs)

- TABCs have been widely used in band structure methods; some in QMC (Foulkes et.al., Lin, Zhong & Ceperley...), and exact diagonalizations (Jullien & Martin, Poilblanc, Gross...).

- E.g. in one dimension:
  - The particle picks up a phase when it goes around the lattice:

$$\Psi(x+L) = e^{i\theta_x}\Psi(x)$$

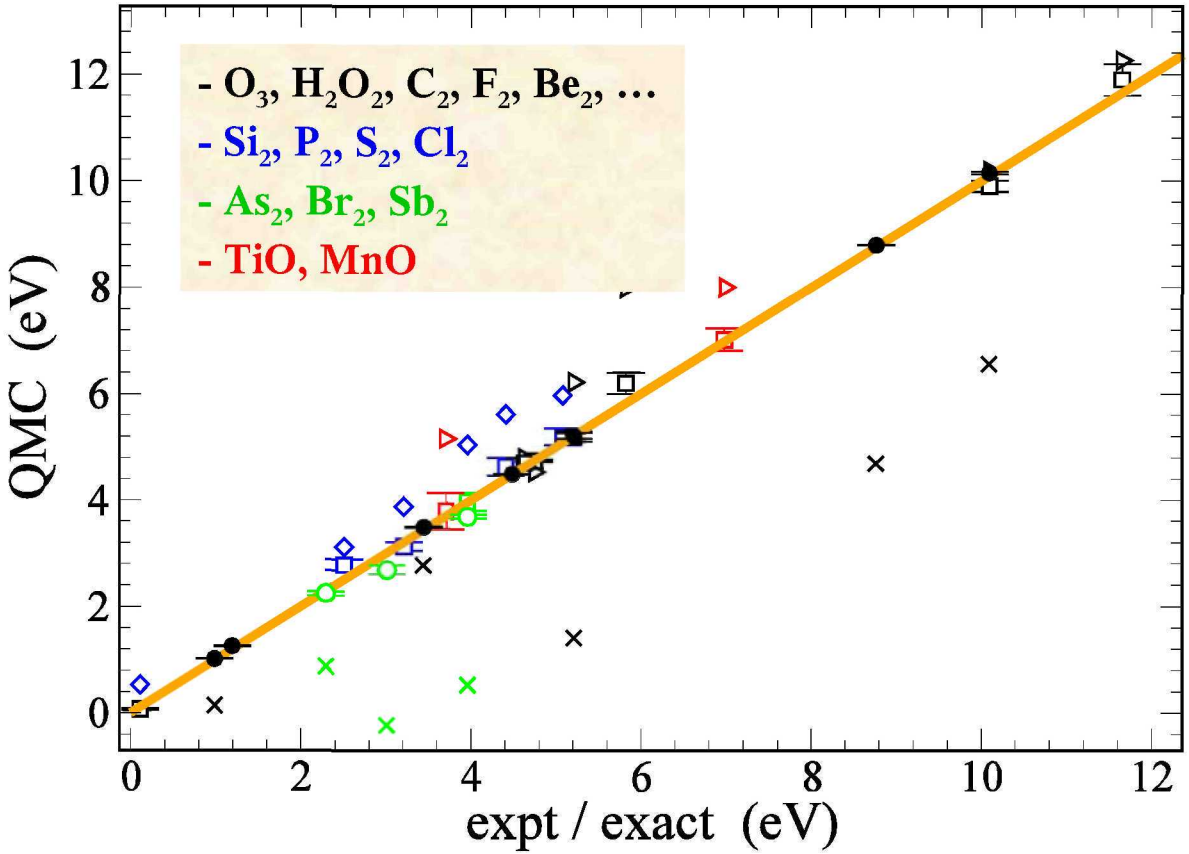
  - In the 1D Hubbard Hamltonian:

$$H = \sum_{i,\sigma}\left(-te^{i\theta_x/L}c^{\dagger}_{i+1\sigma}c_{i\sigma} - te^{-i\theta_x/L}c^{\dagger}_{i-1\sigma}c_{i\sigma}\right) + U\sum_{i} n_{i\uparrow}n_{i\downarrow}$$

$$E_{free}(k,\theta_x) = -2t\cos\left(k + \frac{\theta_x}{L}\right)$$

- Breaks degeneracy in free-particle spectrum.
  But introduces phase problem
  → use the new method

Application: molecular binding energies



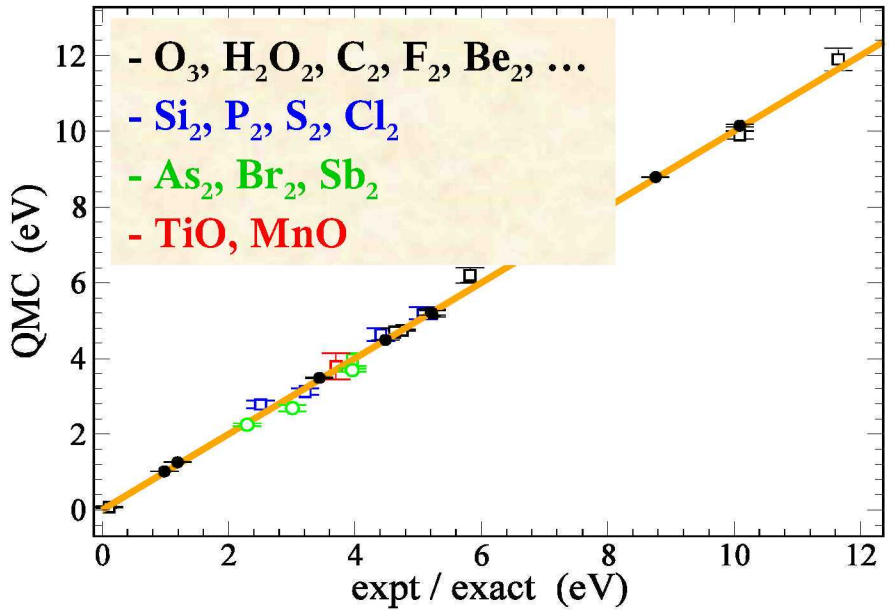
3 types of calc's:

- PW +psp:
- Gaussian/AE:
- Gaussian/sc-ECP:

N_{val} up to ~ 60

- All with single mean-field determinant as trial w.f.
- "automated" post-HF or post-DFT

Molecular binding energies



3 types of calc's:

- PW +psp:
- Gaussian/AE:
- Gaussian/sc-ECP:

N*val* up to ~ 60

- ~ 100 systems (also IP, EA, a_B, ω): eq. geom., moderate correlation
- Error < a few mHa (0.1 eV)
- Accuracy ~ CCSD(T) (gold standard in chemistry, but N^7)
- A QMC algorithm that complements DMC/GFMC
- reduced dependence on trial wf

- **Larger systems? strong correlation?**

Large extended systems

Cohesive energies: (eV/atom)

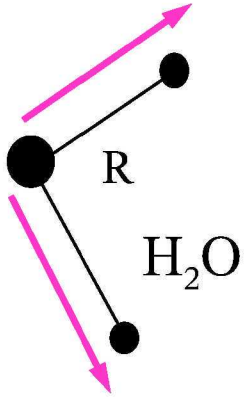
	diamond Si	bcc Na
LDA	5.086	1.21
DMC	4.63(2)	0.991(1) w/o CPP
		1.022(1) w/ CPP
present	4.59(3)	1.143(7)
expt.	4.62(8)	1.13

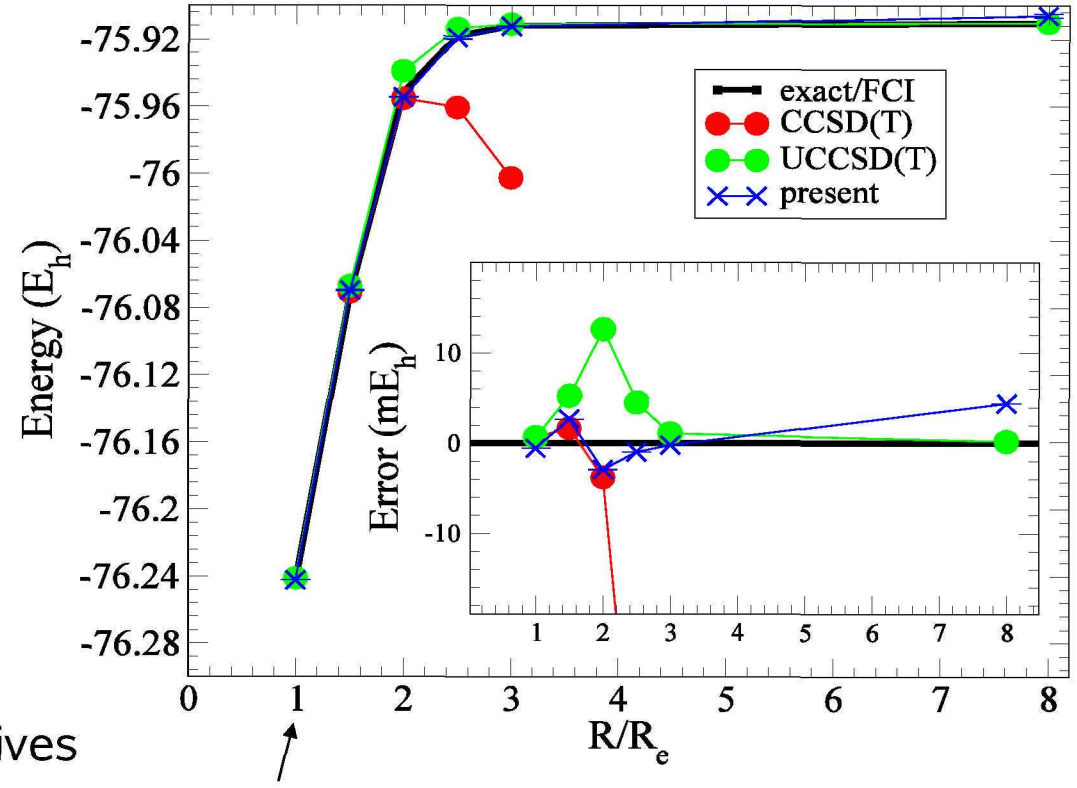
- **Na** (*preliminary*):
  - metal
  - new finite-size correction scheme
- plane-wave + pseudopotential calculations
- DMC -- Needs *et al* (Cambridge group)

Benchmark: H_2O bond breaking

Mimics increasing correlation effects:

(Quantum-chemistry-like calculation with **Gaussian basis**)





- CCSD(T) methods (excellent at eq.) have problems
- The new method gives more uniform accuracy (error < 4 mHa)

Equilibrium → **Dissoc. limit**

"bonding" ***"insulating"***

F_2 bond breaking

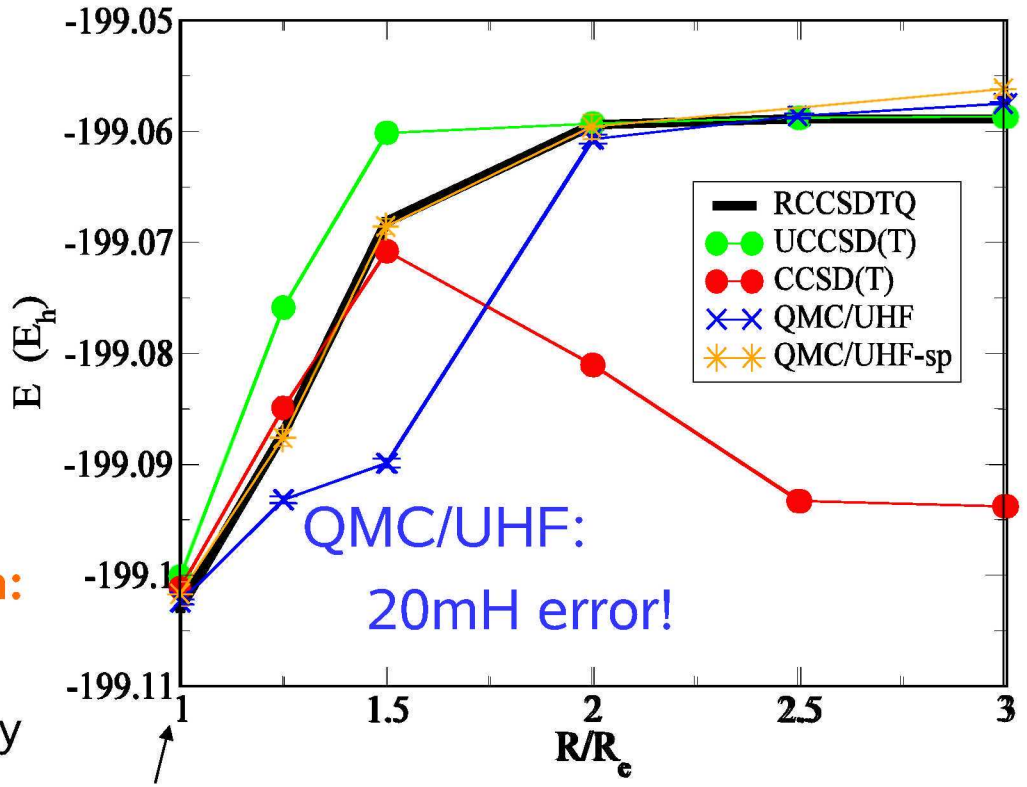
Mimics increasing correlation effects:

- UHF unbound.
  Nonetheless, large
  dependence on trial wf??

- **No. Spin-contamination:**
  - $|\Psi_{UHF}\rangle$: not eigenstate of S^2
  - low-lying triplet in F_2

- **Simple fix – spin-projection:**
  - Let $|\Psi^{(0)}\rangle=|\Psi_{RHF}\rangle$
  - HS preserves spin symmetry
  - each walker determinant:

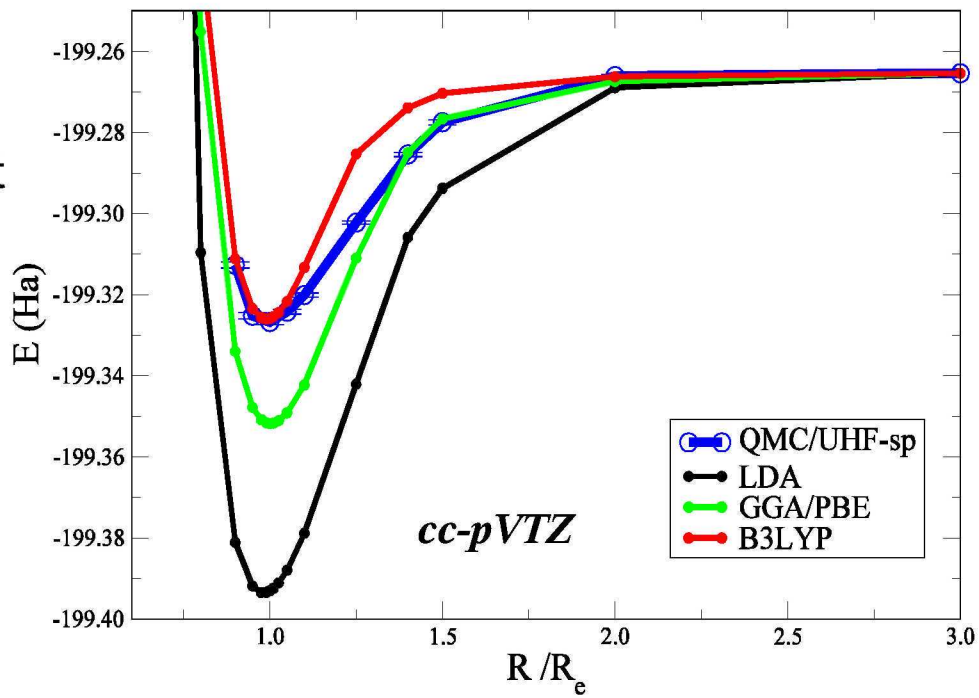
  free of contamination



F_2 bond breaking --- larger basis

How well does DFT do?

- LDA and **GGA/PBE** well-depths too deep
- **B3LYP** well-depth excellent
- "Shoulder" too steep in all 3



C_2 potential energy curve

JOURNAL OF CHEMICAL PHYSICS VOLUME 121, NUMBER 19 15 NOVEMBER 2004

ARTICLES

Full configuration interaction potential energy curves for the $X^1\Sigma_g^+$, $B^1\Delta_g$, and $B'^1\Sigma_g^+$ states of C_2: A challenge for approximate methods

Micah L. Abrams and C. David Sherrill[a)]
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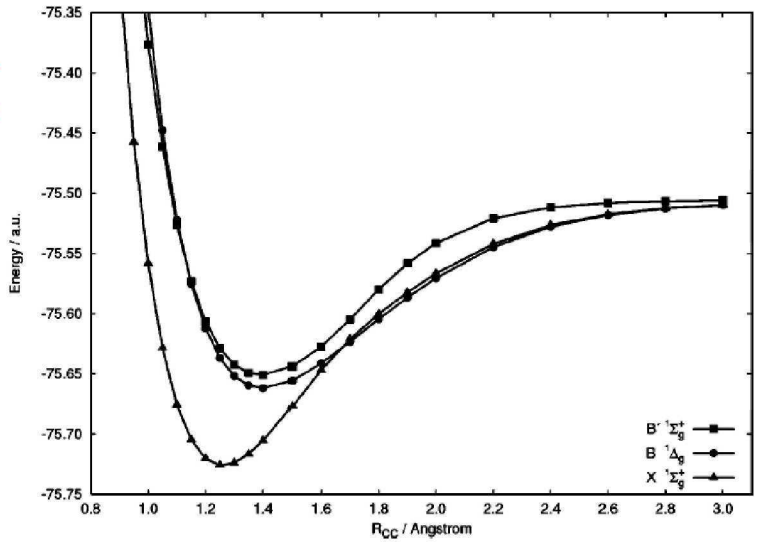
(Received 7 July 2004; accepted 17 August 2004)

The C_2 molecule exhibits unusual bonding and several low-lying excited electronic states, making the prediction of its potential energy curves a challenging test for quantum chemical methods. We

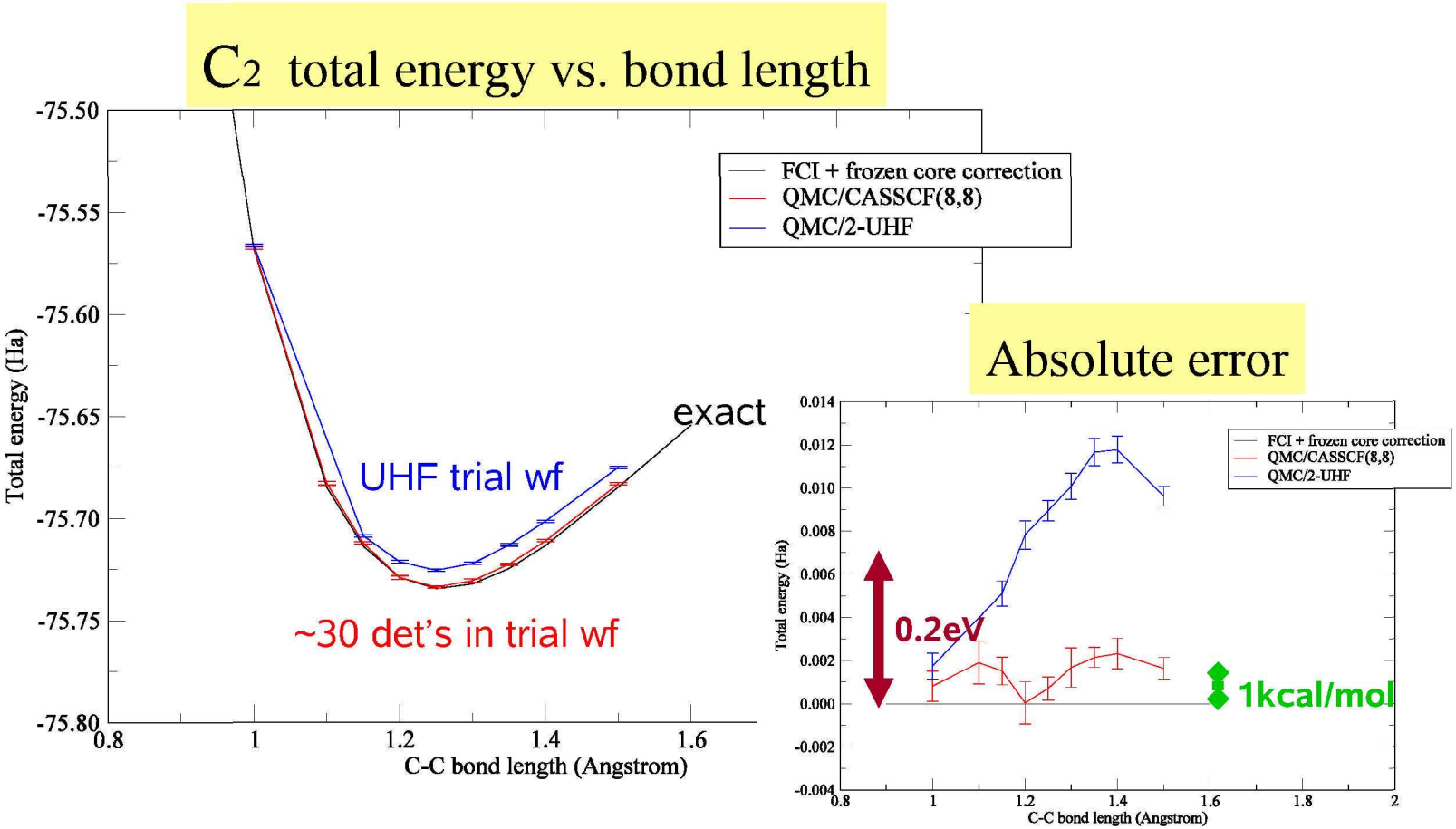
. . . .

benchmark results. Unfortunately, even couple
unrestricted Hartree–Fock reference exhibits l
ground state. The excited states are not accurat



C_2 potential energy curve

- QMC with multi-determinant MCSCF trial wf *(preliminary)*



Metal-insulator transition in H-chain

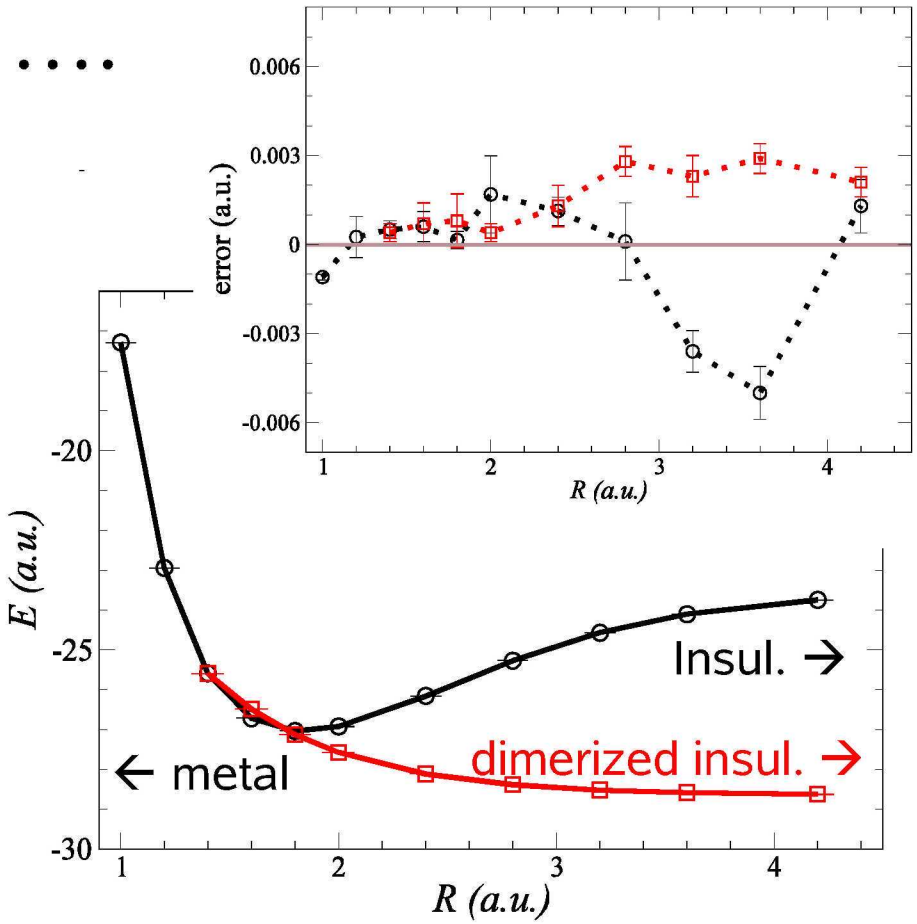
Stretching bonds in H_{50}:

. . . .

Symmetric: stretch each b

Asymmetric: stretch red bonds only

- Near-exact DMRG (solid lines)
  Chan *et. al.*, '06
- QMC agrees with DMRG to 0.002 eV/electron



Thanks:

Collaborators:

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Lecture Notes: (missing recent developments – see papers below)

- Shiwei Zhang, ``*Constrained Path Monte Carlo For Fermions*," in ``*Quantum Monte Carlo Methods in Physics and Chemistry*," Ed.M. P. Nightingale and C. J. Umrigar, NATO ASI Series (Kluwer Academic Publishers, 1998).
  (cond-mat/9909090: http://xxx.lanl.gov/abs/cond-mat/9909090v1)

- Shiwei Zhang, ``*Quantum Monte Carlo Methods for Strongly Correlated Electron Systems*," in ``*Theoretical Methods for Strongly Correlated Electrons*," Ed. by D. Senechal, A.-M. Tremblay, and C. Bourbonnais, Springer-Verlag (2003).
  (available at my website:
  http://www.physics.wm.edu/~shiwei/Preprint/Springer03.pdf)

Some references: (incomplete!)

In addition to the general QMC references from previous lectures:

1. R. Blankenbecler, D. J. Scalapino, and R. L. Sugar, Phys. Rev. D **24**, 2278 (1981)
2. G. Sugiyama and S. E. Koonin, Ann. Phys. **168**, 1 (1986)
3. S. R. White *et. al.*, Phys. Rev. B **40**, 506 (1989)
4. D. R. Hamann and S. B. Fahy, Phys. Rev. B **41**, 11352 (1990)
5. P. L. Silvestrelli and S. Baroni and R. Car, Phys. Rev. Lett. **71**, 1148 (1993)
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7. S. Zhang and J. Carlson and J. E. Gubernatis, Phys. Rev. B **55**, 7464 (1997)
8. S. Zhang, Phys. Rev. Lett. **83**, 2777 (1999)
9. S. Zhang and H. Krakauer, Phys. Rev. Lett. **90**, 136401 (2003)
10. W. Purwanto and S. Zhang, Phys. Rev. E **70**, 056702 (2004)
11. W. A. Al-Saidi, S. Zhang, and H. Krakauer, J. Chem. Phys. **124**, 224101 (2006)

What we have not covered (see references)

- Ground state method for boson systems (Ref 10))
- Back-propagation to calculate observables other than the energy (refs 7, 10)
- Finite-size correction for solids
  - Twist-averaging in solids
  - New 2-body finite-size correction scheme

    Kwee *et al* , arXiv:0711.0921
- Applications (Al-Saidi, Chang, Kwee, Purwanto, …)
  - Van der waals, post-d atoms & molecules, TM molecules, electron affinities, more bond-breaking, trapped atoms, ….

    (my website)

Summary

- New AF QMC approach: random walks in Slater det. space
  - Potentially a method to systematically go beyond independent-particle methods while using much of its machinery

    --- superposition of independent-particle calculations
  - Phaseless approximation (→ constrained path if sign problem)
  - Hybrid of real-space QMC and 'mean-field' methods
- Towards making QMC more robust, capable, black-box:
  - Electronic structure:

    Benchmarks in ~ 100 systems (w/ increased correlation effects)
  - Lattice models
  - Simple trial wfs

    QMC 'recovery' ability important for strong correlation
  - accuracy seems systematic
- Many opportunities for further development and for applications