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Preparatory School to the Winter College on Micro and Nano Photonics for Life Sciences

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Diffraction theory, coherence, and geometrical optics
Part I and II
Introduction to optical imaging modalities

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Review of Fourier transforms

1D)

$$\tilde{f}(K_x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iK_x x}\, dx = \hat{\mathfrak{f}}_x f(x)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(K_x)\, e^{iK_x x}\, dK_x = \hat{\mathfrak{f}}_x^{-1} \tilde{f}(K_x)$$

Some Properties

- Shift

$$\hat{\mathfrak{f}}_x f(x-a) = \frac{1}{\sqrt{2\pi}} \int f(\underbrace{x-a}_{x'})\, e^{-iK_x x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x')\, e^{-iK_x(x'+a)}\, dx'$$

$$= \underline{e^{-iK_x a}\, \tilde{f}(K_x)} \Rightarrow \underline{\hat{\mathfrak{f}}_x^{-1}\left[e^{-iK_x a}\tilde{f}(K_x)\right] = f(x-a)}$$

- Scaling

$$\hat{\mathfrak{f}}_x f\left(\frac{x}{a}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\underbrace{\tfrac{x}{a}}_{x'})\, e^{-iK_x x}\, dx$$

$a>0$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x')\, e^{-iK_x x' a}\, a\, dx'$$

$$= \underline{a\, \tilde{f}(aK_x)} \Rightarrow \underline{\hat{\mathfrak{f}}_x^{-1} \tilde{f}\left(\frac{K_x}{b}\right) = b\, f(bx)}$$

- Derivative

$$\hat{\mathfrak{f}}_x f'(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x)\, e^{-iK_x x}\, dx$$

integrate by parts

$$= \frac{1}{\sqrt{2\pi}} \left. f(x)\, e^{-iK_x x} \right|_{-\infty}^{\infty} \to 0 \; - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left(-iK_x e^{-iK_x x}\right) dx$$

$$= \underline{iK_x \tilde{f}(K_x)} \Rightarrow \underline{\hat{\mathfrak{f}}_x^{-1}\left[K_x \tilde{f}(K_x)\right] = -i f'(x)}$$

Convolutions:

$$\hat{\mathcal{F}}_x^{-1}\left[\tilde{f}(K_x)\,\tilde{g}(K_x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(K_x)\,\underbrace{\tilde{g}(K_x)}_{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x')\,e^{-iK_xx'}\,dx'}\,e^{iK_xx}\,dK_x$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(K_x)\int_{-\infty}^{\infty} g(x')\,e^{-iK_xx'}\,dx'\,e^{iK_xx}\,dK_x$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x')\underbrace{\int_{-\infty}^{\infty} \tilde{f}(K_x)\,e^{iK_x(x-x')}\,dK_x}_{\sqrt{2\pi}\,f(x-x')}\,dx'$$

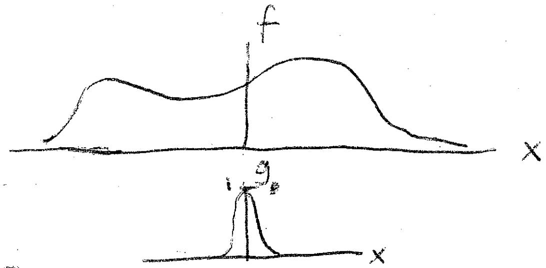
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-x')\,g(x')\,dx' = \frac{1}{\sqrt{2\pi}}\,\underbrace{f * g}_{\text{convolution}}$$

Definition of Convolution:

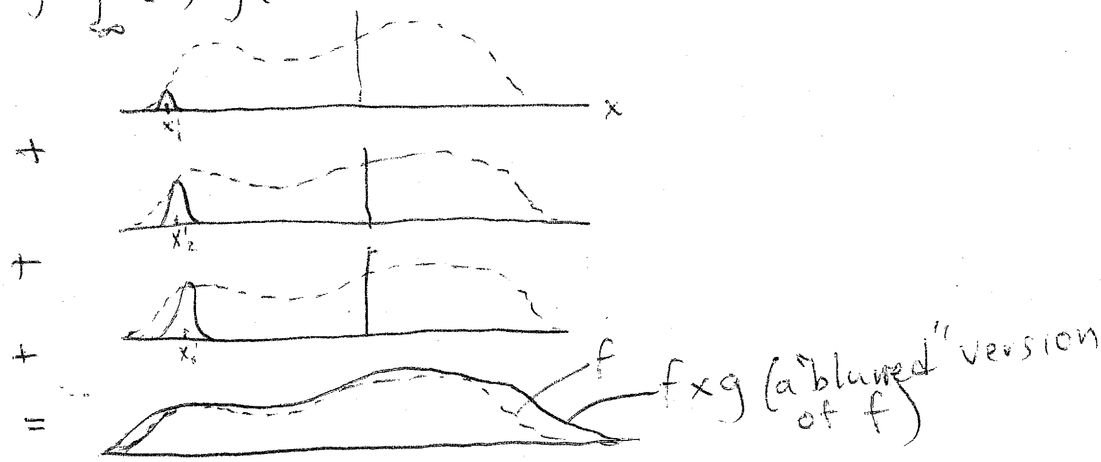
$$f(x) * g(x) = \int_{-\infty}^{\infty} f(x-x')\,g(x')\,dx' = \int_{-\infty}^{\infty} g(x-x')\,f(x')\,dx'$$

Meaning:

suppose:



then, $f \times g = \int_{-\infty}^{\infty} f(x')\,g(x-x')\,dx'$



2D) Let $\underline{x} = (x, y)$, $\underline{K} = (K_x, K_y)$.

$$f(\underline{x}) = f(x, y)$$

Then
$$\tilde{f}(\underline{K}) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} f(\underline{x})\, e^{-i\underline{K}\cdot\underline{x}}\, dx\,dy = \hat{\mathcal{F}}_{xy} f(\underline{x})$$

$$f(\underline{x}) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \tilde{f}(\underline{K})\, e^{i\underline{K}\cdot\underline{x}}\, dK_x\,dK_y = \hat{\mathcal{F}}_{xy}^{-1} \tilde{f}(\underline{K})$$

<u>Some Properties:</u>

- Shift $\quad \underline{a} = (a_x, a_y)$

$$\hat{\mathcal{F}}_{xy} f(\underline{x} - \underline{a}) = e^{-i\underline{K}\cdot\underline{a}}\, \tilde{f}(\underline{K}) \Rightarrow \hat{\mathcal{F}}_{xy}^{-1}\left[e^{-i\underline{K}\cdot\underline{a}}\, \tilde{f}(\underline{K})\right] = f(\underline{x} - \underline{a})$$

- Scaling

$$\hat{\mathcal{F}}_{xy} f(\underline{x}/a) = a^2 \tilde{f}(a\underline{K}) \Rightarrow \hat{\mathcal{F}}_{xy}^{-1}\left[\tilde{f}(\underline{K}/b)\right] = b^2 f(b\underline{x})$$

- Derivative $\quad \nabla_\perp = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$

$$\hat{\mathcal{F}}_{xy}\left[\nabla_\perp f(\underline{x})\right] = i\underline{K}\, \tilde{f}(\underline{K})$$

Second derivative (Laplacian) $\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \nabla_\perp \cdot \nabla_\perp$

$$\hat{\mathcal{F}}_{xy}\left[\nabla_\perp^2 f(\underline{x})\right] = i^2 \underline{K}\cdot\underline{K}\, \tilde{f}(\underline{K}) = -(K_x^2 + K_y^2)\, \tilde{f}(\underline{K})$$

- Convolution

$$\hat{\mathcal{F}}_{xy}^{-1}\left[\tilde{f}(\underline{K})\, \tilde{g}(\underline{K})\right] = \frac{1}{2\pi} f(\underline{x}) * g(\underline{x})$$

$$= \frac{1}{2\pi} \iint_{-\infty}^{\infty} f(\underline{x} - \underline{x}')\, g(\underline{x}')\, dx'dy'$$

$$= \frac{1}{2\pi} \iint_{-\infty}^{\infty} g(\underline{x} - \underline{x}')\, f(\underline{x}')\, dx'dy'$$

Exercises:

1) Write the 2D Fourier transform in polar coordinates, using: $\underline{X} = (\rho\cos\phi, \rho\sin\phi)$
$\underline{K} = (K\cos\varphi, K\sin\varphi)$

2) Assume that $f(\underline{x})$ depends only on $\rho = |\underline{x}|$ and not on ϕ. Simplify your result for (1). Does it depend on both K and φ?

3) Solve the examples:
a) $f(\rho) = \delta(\rho - a)$
b) $f(\rho) = \begin{cases} 1, & \rho \leq a \\ 0 & \text{otherwise} \end{cases}$
c) $f(\rho) = \begin{cases} 1 - \rho^2/a^2, & \rho \leq a \\ 0 & \text{otherwise} \end{cases}$

Formulas you might need:

Bessel functions $J_n(u)$

$$J_n(u) = \frac{1}{2\pi i^n}\int_0^{2\pi} e^{i(u\cos\alpha + n\alpha)}\,d\alpha \quad , \quad n = \text{integer}$$

$$\int_0^u u' J_0(u')\,du' = u J_1(u)$$

$$\int_0^u u'^3 J_0(u')\,du' = 2u^2 J_2(u) - u^3 J_3(u)$$

$$J_{n+1} + J_{n-1} = \frac{2n}{u} J_n(u) .$$

From Maxwell's equations to the Helmholtz Equation

Free space:

$$\left.\begin{aligned}
\nabla\cdot\vec{E} &= 0 && \text{(i)}\\
\nabla\cdot\vec{B} &= 0 && \text{(ii)}\\
\nabla\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} && \text{(iii)}\\
\nabla\times\vec{B} &= \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} && \text{(iv)}
\end{aligned}\right\}\ \text{Maxwell's equations}$$

Take the curl of (iii):

$$\underbrace{\nabla\times(\nabla\times\vec{E})}= \underbrace{-\nabla\times\left(\frac{\partial\vec{B}}{\partial t}\right)}$$

$$\nabla(\underbrace{\nabla\cdot\vec{E}}_{\text{use (i)}})-\nabla^2\vec{E} = -\frac{\partial}{\partial t}(\underbrace{\nabla\times\vec{B}}_{\text{use (iv)}})$$

$$-\nabla^2\vec{E} = -\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2} \Longrightarrow \boxed{\nabla^2\vec{E}-\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}=\vec{0}}$$

wave equation

(same for $\vec{B}$)

$\mu_0\varepsilon_0 = \frac{1}{c^2}$, c = speed of light in vacuum.

Let $\vec{E}(\vec{r},t)$ be expressed as an inverse FT in time:

$$\underbrace{\vec{E}(\vec{r},t)}_{\text{real}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\underbrace{\tilde{\vec{E}}(\vec{r},\omega)}_{\text{complex}}\,e^{-i\omega t}\,d\omega$$

notice that a different sign convention is used for time FT.

Because $\vec{E}$ is real: $\vec{E}=\vec{E}^*$, that is,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\vec{E}}(\vec{r},\omega)e^{-i\omega t}d\omega = \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\vec{E}}(\vec{r},\omega)e^{-i\omega t}d\omega\right]^* = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\vec{E}}^*(\vec{r},\omega)e^{i\omega t}d\omega$$

Change variables $\omega'=-\omega$, $d\omega'=-d\omega$

$$= -\frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty}\tilde{\vec{E}}^*(\vec{r},-\omega')e^{-i\omega' t}d\omega' = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\vec{E}}^*(\vec{r},-\omega')e^{-i\omega' t}d\omega',$$

so $$\boxed{\tilde{\vec{E}}^*(\vec{r},-\omega)=\tilde{\vec{E}}(\vec{r},\omega)}$$

This allows us to simplify:

$$\vec{E}(\vec{r},t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\vec{E}}(\vec{r},\omega')\, e^{-i\omega' t}\, d\omega'$$

$$= \frac{1}{\sqrt{2\pi}} \left[\underset{\text{use } \omega = -\omega'}{\int_{-\infty}^{0} \tilde{\vec{E}}(\vec{r},\omega')\, e^{-i\omega' t}\, d\omega'} + \underset{\text{use } \omega = \omega'}{\int_{0}^{\infty} \tilde{\vec{E}}(\vec{r},\omega')\, e^{-i\omega' t}\, d\omega'} \right]$$

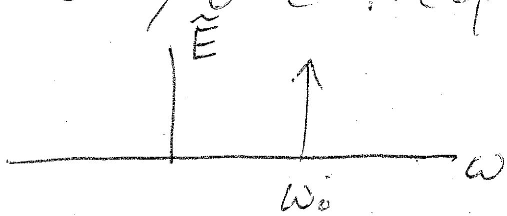
$$= \frac{1}{\sqrt{2\pi}} \left[+ \int_0^{\infty} \underbrace{\tilde{\vec{E}}(\vec{r},-\omega)}_{\tilde{\vec{E}}^*(\vec{r},\omega)}\, e^{i\omega t}\, d\omega + \int_0^{\infty} \tilde{\vec{E}}(\vec{r},\omega)\, e^{-i\omega t}\, d\omega \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \left[\tilde{\vec{E}}^*(\vec{r},\omega)\, e^{i\omega t} + \tilde{\vec{E}}(\vec{r},\omega)\, e^{-i\omega t} \right] d\omega$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} \mathrm{Re}\left\{ \tilde{\vec{E}}(\vec{r},\omega)\, e^{-i\omega t} \right\} d\omega$$

Monochromatic field

there is only one frequency:



so $$\vec{E}(\vec{r},t) = \frac{2}{\sqrt{2\pi}} \mathrm{Re}\left\{ \tilde{\vec{E}}(\vec{r},\omega_0)\, e^{-i\omega_0 t} \right\}$$

Let $$\vec{U}(\vec{r}) = \frac{\tilde{\vec{E}}(\vec{r},\omega_0)}{\sqrt{2\pi}}$$

then $$\vec{E}(\vec{r},t) = 2\,\mathrm{Re}\left\{ \vec{U}(\vec{r})\, e^{-i\omega_0 t} \right\}$$

In practice, true monochromatic fields do not exist, but lasers can come close

$\Delta\omega \ll \omega_0$

True monochromatic fields would in principle exist for all time!

Time-dependent intensity:

$$\mathcal{I}(\vec{r},t) \propto \underbrace{\vec{E}\cdot\vec{E}}_{\rightarrow 2 \text{ for convenience}} = 2\,\mathrm{Re}\{\vec{U}e^{-i\omega_0 t}\}\cdot \mathrm{Re}\{\vec{U}e^{-i\omega_0 t}\}$$

$$= 2\left[\left(\mathrm{Re}\{U_x e^{-i\omega_0 t}\}\right)^2 + \left(\mathrm{Re}\{U_y e^{-i\omega_0 t}\}\right)^2 + \left(\mathrm{Re}\{U_z e^{-i\omega_0 t}\}\right)^2\right]$$

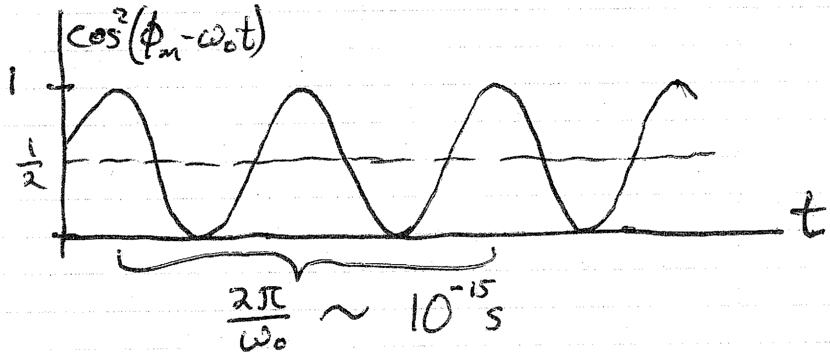
Let $U_m = |U_m|\, e^{i\phi_m}$, $m = x, y, z$. Then

$$\mathcal{I}(\vec{r},t) \propto 2\left[|U_x|^2\cos^2(\phi_x - \omega_0 t) + |U_y|^2\cos^2(\phi_y - \omega_0 t) + |U_z|^2\cos^2(\phi_z - \omega_0 t)\right]$$

In practice, the oscillations are so fast that the eye or a detector only sees an average:

$$I(\vec{r}) = \langle \mathcal{I}(r,t)\rangle_t$$

Notice: $\langle \cos^2(\phi_m - \omega_0 t)\rangle_t = \frac{1}{2}$



Then, $I(\vec{r}) \propto 2\left[|U_x|^2\frac{1}{2} + |U_y|^2\frac{1}{2} + |U_z|^2\frac{1}{2}\right] = \vec{U}^*(\vec{r})\cdot\vec{U}(\vec{r}) = |\vec{U}(\vec{r})|^2$

Substitute now $\vec{E} = 2\mathrm{Re}\{\vec{U}(\vec{r})\, e^{-i\omega_0 t}\}$
into wave eq.

$$\nabla^2 \vec{E} = 2\,\mathrm{Re}\{\nabla^2 \vec{U}(\vec{r})\; e^{-i\omega_0 t}\}$$

$$\frac{\partial^2 \vec{E}}{\partial t^2} = 2\,\mathrm{Re}\{-\omega_0^2 \vec{U}(\vec{r}) e^{-i\omega_0 t}\}$$

so

$$\mathrm{Re}\left\{\left[\nabla^2 \vec{U} + \underbrace{\frac{\omega_0^2}{c^2}}_{k_0^2} \vec{U}\right] e^{-i\omega_0 t}\right\} = 0 \quad \text{for all } t$$

therefore

$$\boxed{(\nabla^2 + k_0^2)\,\vec{U}(\vec{r}) = 0.}$$ Free-space vector Helmholtz Eq.

For a monochromatic field in a linear, isotropic dielectric: $\vec{D} = \varepsilon(\vec{r}, \omega_0)\vec{E}$,

so

$$\left(\nabla^2 + \underbrace{\omega_0^2 \mu_0 \varepsilon}_{\underbrace{\omega_0^2 \mu_0 \varepsilon_0}_{k^2} \underbrace{\frac{\varepsilon}{\varepsilon_0}}_{n^2(\vec{r},\omega_0)}}\right)\vec{U}(\vec{r}) = 0$$

$$\boxed{\left[\nabla^2 + k_0^2 n^2(\vec{r},\omega_0)\right]\vec{U}(\vec{r}) = 0}$$ Vector Helmholtz Eq. for inhomogeneous media.

This is the basis of most of what follows.

For simplicity use ω, k instead of ω_0, k_0.

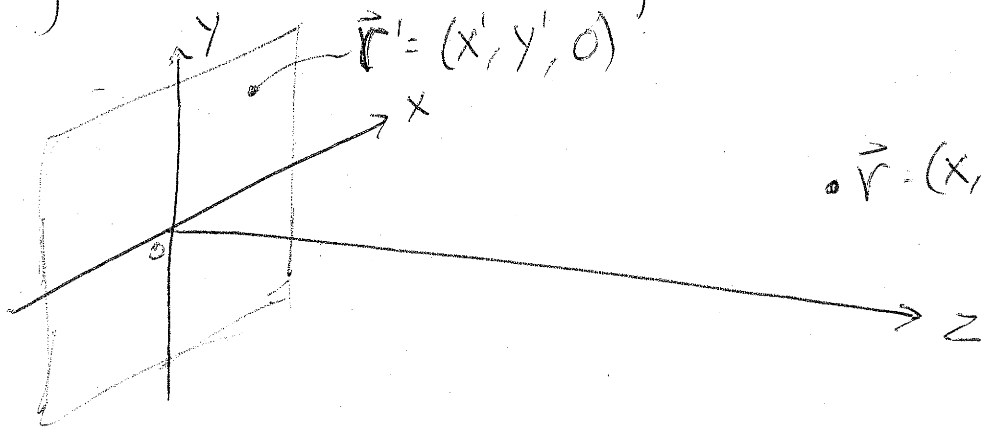
Let $U(\vec{r})$ represent a component of $\vec{U}$:

$$\boxed{\left(\nabla^2 + k^2 n^2\right) U(\vec{r}) = 0}$$ Scalar Helmholtz eq.

Diffraction theory

Solution of Helmholtz given:

- Value of U at some initial plane $(z=0)$
- knowledge that all sources are to the left of initial plane $(z<0)$, so for $z>0$ all light goes "from left to right".



We know $U(x', y', 0)$ for all x', y'.

To Solve Helmholtz, use 2D $\hat{f}$ in x&y:

$$\hat{f}\left[\left(\nabla^2 + k^2 n^2\right) U(\vec{r})\right] = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \left(\nabla^2 + k^2 n^2\right) U(\vec{r})\, e^{-i \underline{K} \cdot \underline{x}}\, dx\, dy = 0$$

Consider only the case $n=1$ for now.

Use $\nabla^2 U = \nabla_\perp^2 U + \frac{\partial^2}{\partial z^2} U$, so

$$\hat{\mathcal{F}}_{xy} \nabla^2 U = -\underline{K}\cdot\underline{K}\tilde{U} + \frac{\partial^2}{\partial z^2}\tilde{U},$$

and the $\hat{\mathcal{F}}_{xy}$ of Helmholtz Eq. gives

$$\left[-\underbrace{\underline{K}\cdot\underline{K}}_{|\underline{K}|^2} + \frac{\partial^2}{\partial z^2} + k^2\right]\tilde{U}(\underline{K},z) = 0.$$

We can rewrite this as

$$\frac{\partial^2}{\partial z^2}\tilde{U} = -\left(k^2 - |\underline{K}|^2\right)\tilde{U},$$

which has the solution:

$$\tilde{U}(\underline{K},z) = \begin{cases} A_I(\underline{K})\, e^{iz\sqrt{k^2-|\underline{K}|^2}} + B_I(\underline{K})\, e^{-iz\sqrt{k^2-|\underline{K}|^2}}, & |\underline{K}| \le k, \\ A_{II}(\underline{K})\, e^{-z\sqrt{|\underline{K}|^2-k^2}} + B_{II}(\underline{K})\, e^{z\sqrt{|\underline{K}|^2-k^2}}, & |\underline{K}| > k. \end{cases}$$

Therefore

$$U(\vec{r}) = \hat{\mathcal{F}}^{-1}_{xy}\,\tilde{U}(\underline{K},z)$$

$$= \frac{1}{2\pi}\iint_{|\underline{K}|\le k} A_I(\underline{K})\, e^{i\left(K_x x + K_y y + \sqrt{k^2-|\underline{K}|^2}\, z\right)} dK_x\, dK_y \qquad (1)$$

$$+ \frac{1}{2\pi}\iint_{|\underline{K}|\le k} B_I(\underline{K})\, e^{i\left(K_x x + K_y y - \sqrt{k^2-|\underline{K}|^2}\, z\right)} dK_x\, dK_y \qquad (2)$$

$$+ \frac{1}{2\pi}\iint_{|\underline{K}|> k} A_{II}(\underline{K})\, e^{i(K_x x + K_y y) - \sqrt{|\underline{K}|^2-k^2}\, z} dK_x\, dK_y \qquad (3)$$

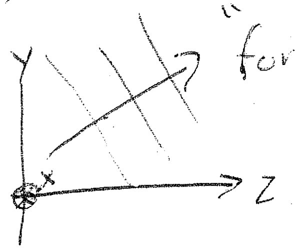
$$+ \frac{1}{2\pi}\iint_{|\underline{K}|> k} B_{II}(\underline{K})\, e^{i(K_x x + K_y y) + \sqrt{|\underline{K}|^2-k^2}\, z} dK_x\, dK_y. \qquad (4)$$

(1) $e^{i(K_x x + K_y y + \sqrt{k^2-|\underline{K}|^2}\, z)} = e^{i\vec{K}\cdot\vec{r}}$ plane wave!

where $\vec{K} = (K_x, K_y, \sqrt{k^2-|\underline{K}|^2})$

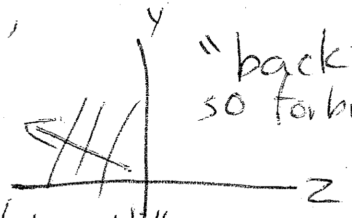
Note $|\vec{K}|^2 = k^2$.

direction of propagation



(2) $e^{i(K_x x + K_y y - \sqrt{k^2-|\underline{K}|^2}\, z)} = e^{i\vec{K}\cdot\vec{r}}$,

where $\vec{K} = (K_x, K_y, -\sqrt{k^2-|\underline{K}|^2})$



"backward" so forbidden.

since all light must go from "left to right",
we set $B_I = 0$.

(3) $\underbrace{e^{i(K_x x + K_y y)}}_{\substack{\text{fast oscillation} \\ \text{in xy plane } (|\underline{K}|>k)}} \; \underbrace{e^{-\sqrt{|\underline{K}|^2-k^2}\, z}}_{\text{decay in z}}$ = <u>evanescent wave</u>

decays quickly in z (within a few wavelengths)

(4) $e^{i(K_x x + K_y y)} \; \underbrace{e^{\sqrt{|\underline{K}|^2-k^2}\, z}}_{\substack{\text{exponential growth in z} \\ \text{which is unphysical.}}}$

we set $B_{II} = 0$.

Let $A(\underline{K}) = \begin{cases} A_I(\underline{K}), & |\underline{K}| \le k \\ A_{II}(\underline{K}), & |\underline{K}| > k \end{cases}$

Then:

$$\boxed{U(\vec{r}) = \frac{1}{2\pi}\left[\underbrace{\iint_{|\underline{K}|\le k} A(\underline{K})\, e^{i(\underline{K}\cdot\underline{X}+\sqrt{k^2-|\underline{K}|^2}\,z)}\, dK_x dK_y}_{\text{Homogeneous plane waves}} + \underbrace{\iint_{|\underline{K}|>k} A(\underline{K})\, e^{i\underline{K}\cdot\underline{X}-\sqrt{|\underline{K}|^2-k^2}\,z}\, dK_x dK_y}_{\text{Evanescent waves.}}\right]}$$

Boundary condition

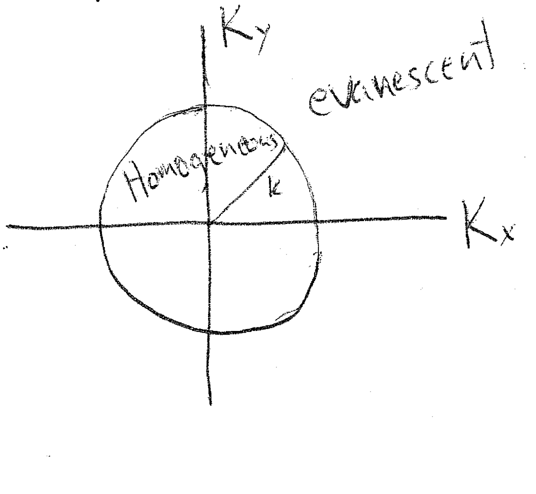
At $z=0$,

$$U(x,y,0) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} A(\underline{K})\, e^{i\underline{K}\cdot\underline{X}}\, dK_x dK_y$$

so

$$\boxed{A(\underline{K}) = \hat{f}_{xy}\, U(x,y,0) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} U(x,y,0)\, e^{-i\underline{K}\cdot\underline{X}}\, dx\,dy}$$

A is called the "angular spectrum."



Upon propagation, the high spatial frequency information ($|\underline{K}|>k$) is lost.

We can describe propagation as

$$U(x,y,z) = \hat{f}^{-1}_{xy}\left[e^{iK_z(\underline{K})z}\, \hat{f}_{xy}\{U(x,y,0)\}\right],$$

where $K_z(\underline{K}) = \begin{cases} \sqrt{k^2 - |\underline{K}|^2}, & |\underline{K}| \leq k \\ i\sqrt{|\underline{K}|^2 - k^2}, & |\underline{K}| > k. \end{cases}$

Notice that, due to the convolution theorem:

$$U(x,y,z) = U(x,y,0) * \hat{f}^{-1}_{xy}\left(e^{iK_z(\underline{K})z}\right)$$

Can show (not easy) that

$$\hat{f}^{-1}\left(e^{iK_z(\underline{K})z}\right) = \left(-ik + \frac{1}{\sqrt{x^2+y^2+z^2}}\right)\frac{z}{\sqrt{x^2+y^2+z^2}}\,\frac{e^{ik\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}$$

so

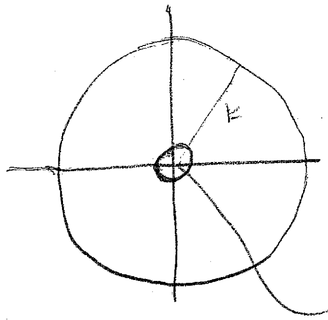
$$\boxed{U(\vec{r}) = \iint_{-\infty}^{\infty} U(x',y',0)\left(\frac{-ik}{2\pi} + \frac{1}{2\pi|\vec{r}-\vec{r}'|}\right)\frac{z}{|\vec{r}-\vec{r}'|}\,\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\,dx'dy'}$$

Rayleigh-Sommerfeld formula I

Paraxial approximation

Assume $A(\underline{K}) \neq 0$ only for $|\underline{K}| \ll k$

$$\sin\theta = \frac{|\underline{K}|}{k} \ll 1$$



No evanescence.

$A \neq 0.$

Then $\quad K_z(\underline{K}) = \sqrt{k^2 - |\underline{K}|^2}$

$$\approx k - \frac{|\underline{K}|^2}{2k},$$

$$U(\vec{r}) = \hat{f}_{xy}^{-1}\left[e^{ikz} e^{-i\frac{|\underline{K}|^2}{2k}z} \hat{f}_{xy}\{U(x,y,0)\}\right]$$

$$= e^{ikz}\, \hat{f}_{xy}^{-1}\left[e^{-i\frac{|\underline{K}|^2 z}{2k}} \hat{f}_{xy}\{U(x,y,0)\}\right]$$

$$= \frac{e^{ikz}}{2\pi} U(x,y,0) * \hat{f}_{xy}^{-1}\left(e^{-i|\underline{K}|^2\frac{z}{2k}}\right)$$

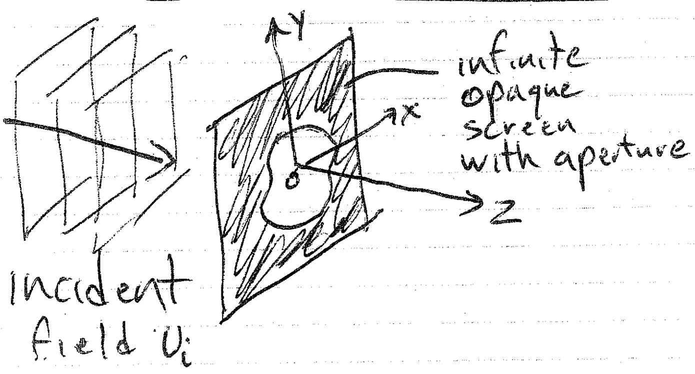
but $\hat{f}_{xy}^{-1} e^{-i|\underline{K}|^2\frac{z}{2k}} = \frac{-ik}{z} e^{-ik\frac{(x^2+y^2)}{2z}}$

so

$$\boxed{U(\vec{r}) = \frac{-ik}{2\pi}\frac{e^{ikz}}{z}\iint_{-\infty}^{\infty} U(x',y',0)\; e^{-ik\frac{[(x'-x)^2+(y'-y)^2]}{2z}}\, dx'dy'}$$

Fresnel diffraction Formula

Diffraction by flat obstacles (apertures)

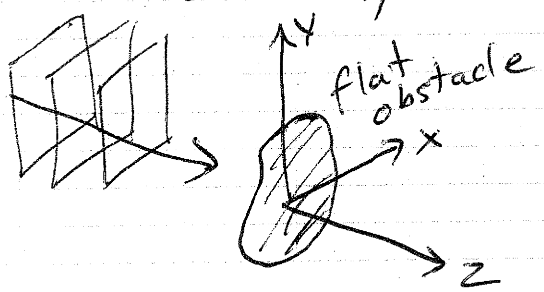


Kirchhoff approximate boundary conditions

The field right after the screen is:

$$U_t(x,y,0) = \begin{cases} U_i(x,y,0), & (x,y) \text{ at aperture} \\ 0, & \text{otherwise.} \end{cases}$$

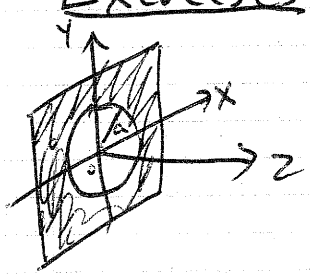
Similarly for an obstacle:



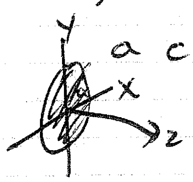
$$U_t(x,y,0) = \begin{cases} U_i(x,y,0), & (x,y) \text{ away from obstacle} \\ 0, & (x,y) \text{ at obstacle.} \end{cases}$$

<u>Babinet principle</u>: the sum of the field transmitted through an aperture of some shape, and the field transmitted around a flat obstacle of the same shape equals the unperturbed incident field. (This is an approximation).

<u>Exercises</u>: 1) Consider a circular aperture of radius a. There is no closed form for the diffracted field everywhere, except at the z axis. Try to find these expressions using a) RSI, and b) Fresnel (paraxial).



2) Use Babinet's principle to find the axial field for a circular obstacle of radius a.

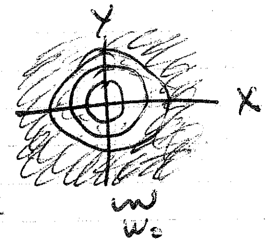


Focused Fields

Let us start with the paraxial case.

Excercise: Gaussian beam

Let $U(x,y,0) = \underset{\uparrow \text{ constant}}{U_0}\, e^{-\frac{x^2+y^2}{2w_0^2}}$ $\quad w_0$ = width at the waist.



1) Find $U(x,y,z)$ in closed form

Hints:
$$\hat{f}_x\, e^{-\frac{x^2}{2a^2}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2a^2}} e^{-ik_x x}\,dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2a^2}(x^2+2ia^2k_x x \underbrace{-a^4k_x^2}_{\text{add to complete square}})}dx\; \underbrace{e^{-\frac{a^4k_x^2}{2a^2}}}_{\text{add to compensate}}$$
$$= \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2a^2}(\underbrace{x+ia^2k_x}_{\text{call } x'})^2}\underbrace{dx}_{dx=dx'}}_{\int_{-\infty}^{\infty} e^{-\frac{x'^2}{2a^2}}dx' = \sqrt{2\pi}\,a}\; e^{-\frac{a^2k_x^2}{2}} = \underline{a\, e^{-\frac{a^2k_x^2}{2}}}$$

Also, use the form of Fresnel propagation given by
$$U(x,y,z) = e^{ikz}\,\hat{f}_{xy}^{-1}\left[e^{i\frac{z}{2k}|\mathbf{k}|^2}\,\hat{f}_{xy}\{U(x,y,0)\}\right]$$

2) What is the spacing of the wavefronts? (Hint: evaluate the on-axis field)

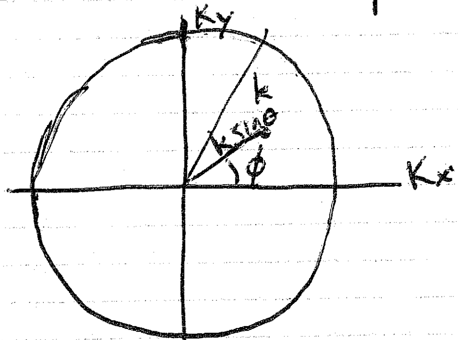
3) A spherical wave centered at $(0,0,z_0)$ is:
$$U(\vec{r}) = \underset{\nearrow \text{ constant}}{u_0}\,\frac{e^{ik\sqrt{x^2+y^2+(z-z_0)^2}}}{\sqrt{x^2+y^2+(z-z_0)^2}}.$$
Let now $z_0 = iq$, for $kq \gg kr$. Expand the $\sqrt{\ }$ for $|z-iq|^2 \gg x^2+y^2$ to 2^{nd} order in the exponent & 0^{th} order in the amplitude. Find q & u_0 so that this matches the result of 1).

Nonparaxial focused fields

$$U(\vec{r}) = \frac{1}{2\pi}\iint A(K_x,K_y)\, e^{i\vec{K}\cdot\vec{r}}\, dK_x dK_y$$

Suppose now that A is not found from $U(x,y,0)$ but that it is dictated by the focusing instrument.

Since we are away from sources, we can ignore evanescent components. Let us change variables to



$$K_x = k\sin\theta \quad \cos\phi$$
$$K_y = k\sin\theta \quad \sin\phi$$

Note that $\frac{|K|}{k} = \sin\theta$

then $$\frac{\partial(K_x,K_y)}{\partial(\theta,\phi)} = \left|\frac{\partial K_x}{\partial\theta}\frac{\partial K_y}{\partial\phi} - \frac{\partial K_x}{\partial\phi}\frac{\partial K_y}{\partial\theta}\right|$$
$$= k^2\left|\cos\theta\cos^2\phi\sin\theta + \sin\theta\sin^2\phi\cos\theta\right|$$
$$= k^2 \sin\theta\cos\theta \qquad \left(0 \le \theta \le \tfrac{\pi}{2}\right)$$

$$U(\vec{r}) = \frac{k^2}{2\pi}\int_0^{2\pi}\int_0^{\pi/2} A(k\sin\theta\cos\phi, k\sin\theta\sin\phi)\, e^{ik[\sin\theta(x\cos\phi + y\sin\phi) + z\cos\theta]} \cdot \sin\theta\cos\theta\, d\theta\, d\phi$$

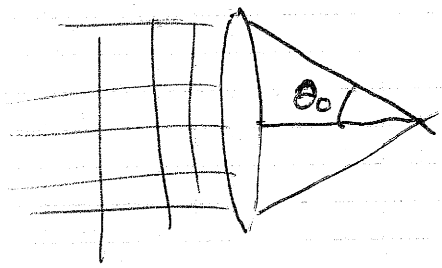
Rotationally symmetric fields: A is independent of ϕ.
Let us write A as $A = \frac{P(\theta)}{k^2\cos\theta}$. Then, if we use cylindrical coordinates: $x = \rho\cos\varphi$, $y = \rho\sin\varphi$:

$$U(\vec{r}) = \int_0^{\pi/2} P(\theta)\, \underbrace{\frac{1}{2\pi}\int_0^{2\pi} e^{ik\rho\underbrace{[\cos\phi\cos\varphi + \sin\phi\sin\varphi]}_{\cos(\phi-\varphi)}\sin\theta}\, d\phi}_{J_0(k\rho\sin\theta)}\; e^{ikz\cos\theta}\,\sin\theta\, d\theta$$

$$= \int_0^{\pi/2} P(\theta)\, J_0(k\rho\sin\theta)\, e^{ikz\cos\theta}\,\sin\theta\, d\theta$$

In practice P is $\neq 0$ only for $\theta \leq \theta_0$, so

$$\boxed{U(\vec{r}) = \int_0^{\theta_0} P(\theta)\, J_0(k\rho \sin\theta)\, e^{ikz\cos\theta} \sin\theta\, d\theta}$$



Numerical aperture

$\boxed{NA = n \sin\theta_0}$

refractive index.

Axial field:

$$U(0,0,z) = \int_0^{\theta_0} P(\theta)\, e^{ikz\underbrace{\cos\theta}_{u}} \underbrace{\sin\theta\, d\theta}_{-du}$$

$$= \int_{\cos\theta_0}^{1} P(\arccos u)\, e^{ikzu}\, du$$

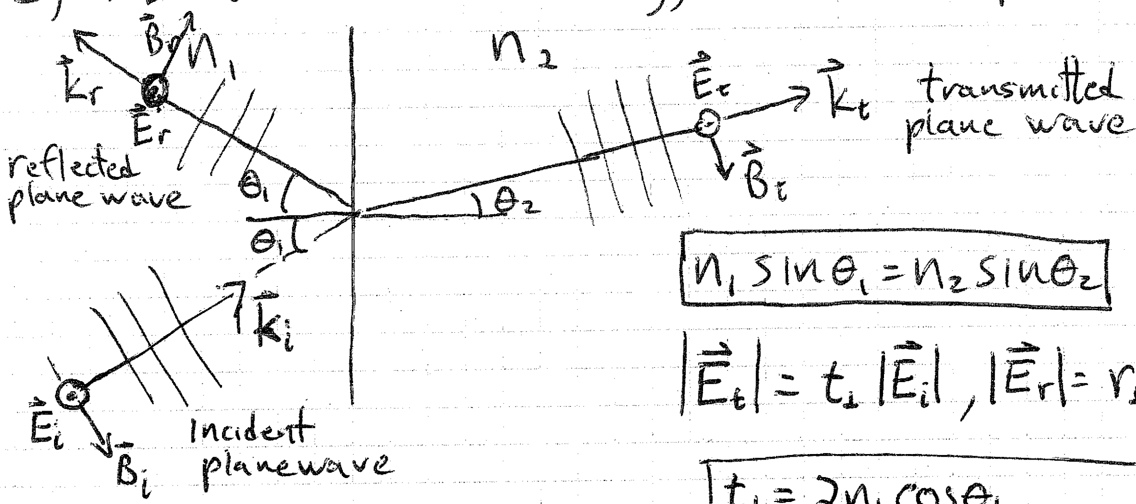
Similar to a Fourier transform!

Excercises: a) $P(\theta) = 1$ b) $P(\theta) = \cos\theta$

Can you find the focal $(z=0)$ and axial $(x=y=0)$ fields?

Fresnel transmission & reflection coefficients

1) TE (transverse electric), also called s-polarization



$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}$$ Snell's law

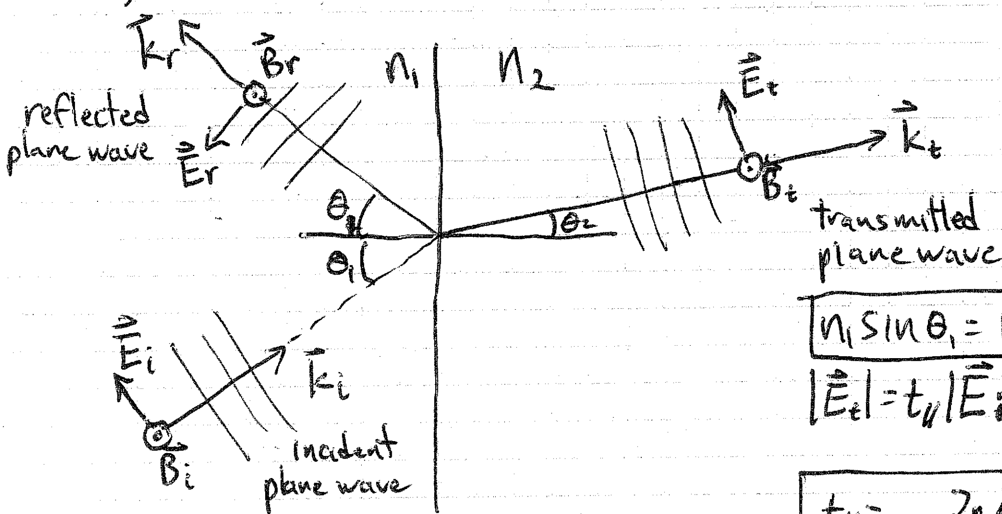
$$|\vec{E}_t| = t_\perp |\vec{E}_i| \,, \quad |\vec{E}_r| = r_\perp |\vec{E}_i|$$

$$t_\perp = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + n_2\cos\theta_2}$$

$$r_\perp = \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2}$$

Fresnel TE coefficients

2) TM (transverse magnetic), also called p-polarization



$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}$$ same Snell's law

$$|\vec{E}_t| = t_\parallel |\vec{E}_i| \,, \quad |\vec{E}_r| = r_\parallel |\vec{E}_i|$$

$$t_\parallel = \frac{2n_1\cos\theta_1}{n_1\cos\theta_2 + n_2\cos\theta_1}$$

$$r_\parallel = \frac{n_2\cos\theta_1 - n_1\cos\theta_2}{n_1\cos\theta_2 + n_2\cos\theta_1}$$

Fresnel TM coefficients

Recall, for $n_1 > n_2$ and $|\theta_1| > \theta_c$, where $\theta_c = \arcsin\left(\frac{n_2}{n_1}\right)$, the transmitted wave is evanescent.

Ray optics (or geometrical optics)

Valid for k "large"

Let: $U(\vec{r}) = \underbrace{A(\vec{r})}_{\text{real}} e^{ik\underbrace{\Phi(\vec{r})}_{\text{real}}}$

In free space:

$$(\nabla^2 + k^2) U(\vec{r}) = 0$$

$$(\nabla^2 + k^2)(A e^{ik\Phi}) = 0$$

Notice: $\nabla(A e^{ik\Phi}) = (\nabla A + A ik \nabla\Phi) e^{ik\Phi}$,

$$\nabla^2 (A e^{ik\Phi}) = (\nabla^2 A + 2ik \nabla A \cdot \nabla\Phi + A ik \nabla^2 \Phi - k^2 A |\nabla\Phi|^2) \cdot e^{ik\Phi}$$

so:

$$0 = e^{-ik\Phi}\left[(\nabla^2 + k^2)(A e^{ik\Phi})\right]$$

$$= \underbrace{\nabla^2 A - k^2 A |\nabla\Phi|^2 + k^2 A}_{\text{real}} + \underbrace{ik(2\nabla A \cdot \nabla\Phi + A\nabla^2\Phi)}_{\text{imaginary}}$$

Both parts must $= 0$ independently:

real: $\boxed{|\nabla\Phi|^2 = 1 + \frac{\nabla^2 A}{k^2 A}}$

imaginary times A: $2A\nabla A \cdot \nabla\Phi + A^2 \nabla^2 \Phi$

$$= \nabla A^2 \cdot \nabla\Phi + A^2 \nabla^2\Phi$$

$$= \boxed{\nabla \cdot (A^2 \nabla\Phi) = 0}$$

Transport Equation

If $\boxed{\frac{\nabla^2 A}{A} \ll k^2}$, then

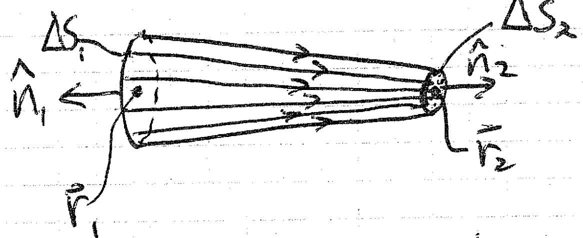
$|\nabla\Phi|^2 = 1$, and, if $n \neq 1$, $\boxed{|\nabla\Phi|^2 = n^2}$: Eikonal Equation (ray optics)

$\nabla\Phi$ = ray direction

spacing between wavefronts $=$ $\frac{2\pi}{|\nabla(k\Phi)|} = \underbrace{\frac{2\pi}{k}}_{\lambda} \frac{1}{\sqrt{1 + \frac{\nabla^2 A}{k^2 A}}} = \lambda!$

This extra factor causes Gouy shift

Consider a volume V occupied by an infinitesimal ray bundle



Integrate the transport equation over V:

$$0 = \int_V \nabla\cdot(A^2 \nabla\Phi)\, d^3r = \underset{\substack{\uparrow \\ \text{use Gauss' theorem}}}{} \int_{\text{surface of } V} A^2 \nabla\Phi \cdot d\vec{s} = \Delta S_2\, A^2(\vec{r}_2)\, \nabla\Phi(\vec{r}_2)\cdot\hat{n}_2$$

$$+ \Delta S_1\, A^2(\vec{r}_1)\, \nabla\Phi(\vec{r}_1)\cdot\hat{n}_1$$

$$+ \cancel{\int_{\text{sides of } V} A^2 \nabla\Phi\cdot d\vec{s}} \to 0$$

because $\nabla\Phi$ is tangent to the surface.

Assume $k \gg \sqrt{\frac{\nabla^2 A}{A}}$, so $|\nabla\Phi| = 1$.

Then:

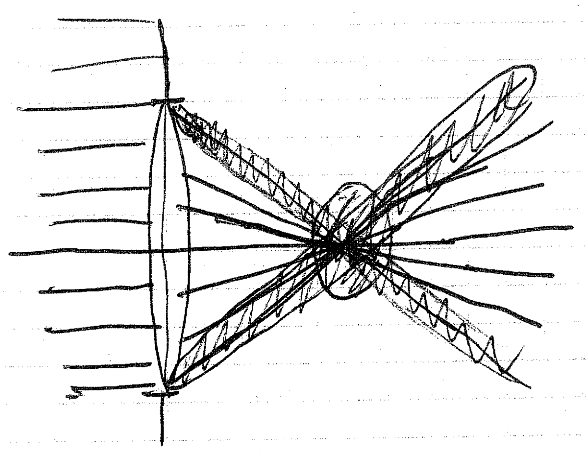
$$0 = \Delta S_2\, A^2(\vec{r}_2)\, \underbrace{\nabla\Phi(\vec{r}_2)\cdot\hat{n}_2}_{1} + \Delta S_1\, A^2(\vec{r}_1)\, \underbrace{\nabla\Phi(\vec{r}_1)\cdot\hat{n}_1}_{-1}$$

or

$$A^2(\vec{r}_2)\,\Delta S_2 = A^2(\vec{r}_1)\,\Delta S_1.$$

This implies that $A^2(\vec{r})$ scales as the density of rays.

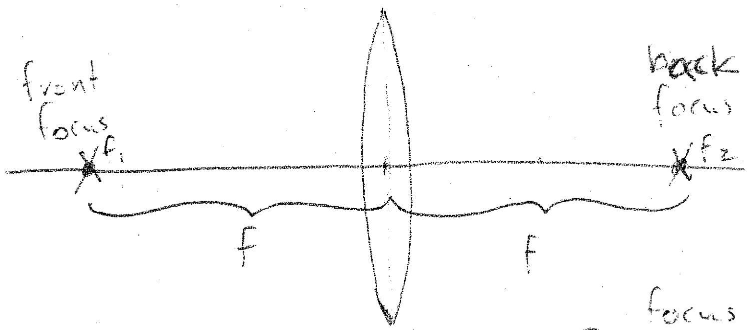
Consider a focused field:



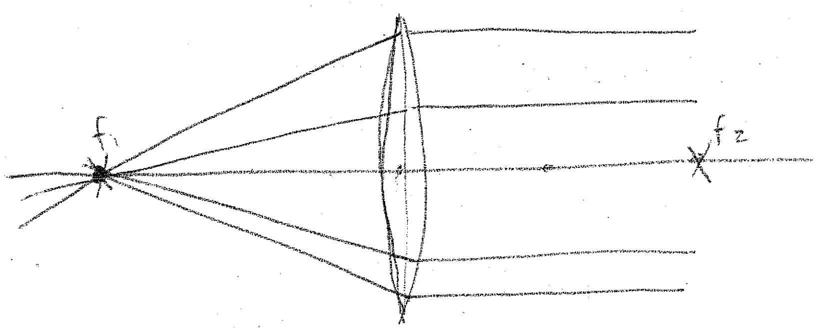
Geometrical optics does not give accurate results in the shaded areas (the surface of the beam and the focus) because A varies quickly there, so $\frac{\nabla^2 A}{A}$ is not $\ll k^2$.

Ray optics of the single convergent thin lens (1^{st} order or paraxial)

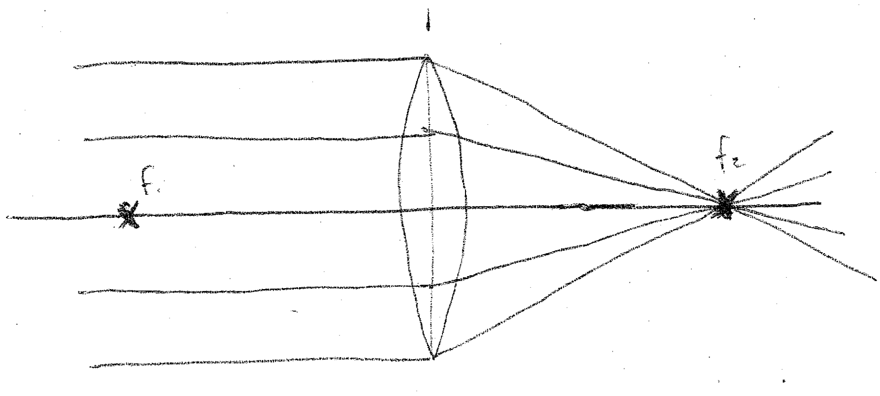
Foci



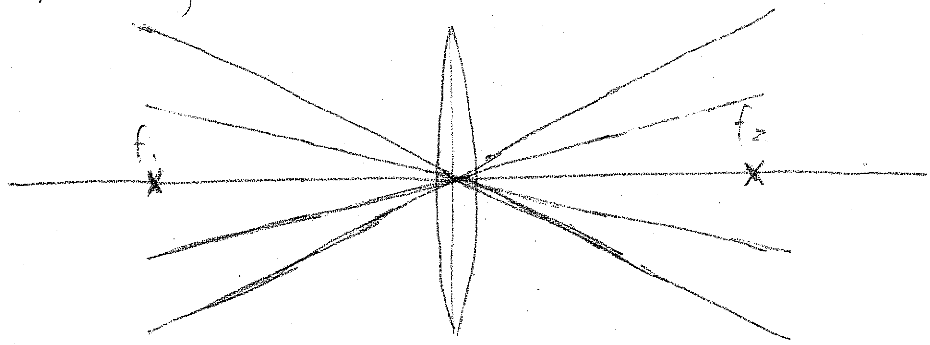
1) Any ray through the front $\overbrace{\text{focal point}}^{\text{focus}}$ emerges from the lens parallel to the axis



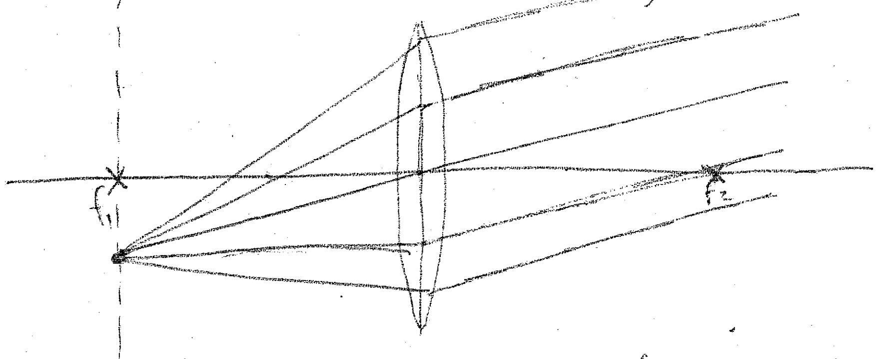
2) Any ray that arrives parallel to the axis, bends towards the back focus



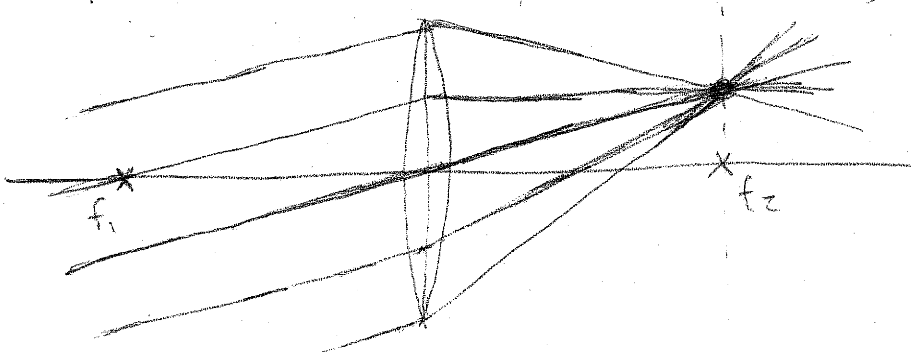
3) Any ray through the center of the lens emerges with unchanged direction



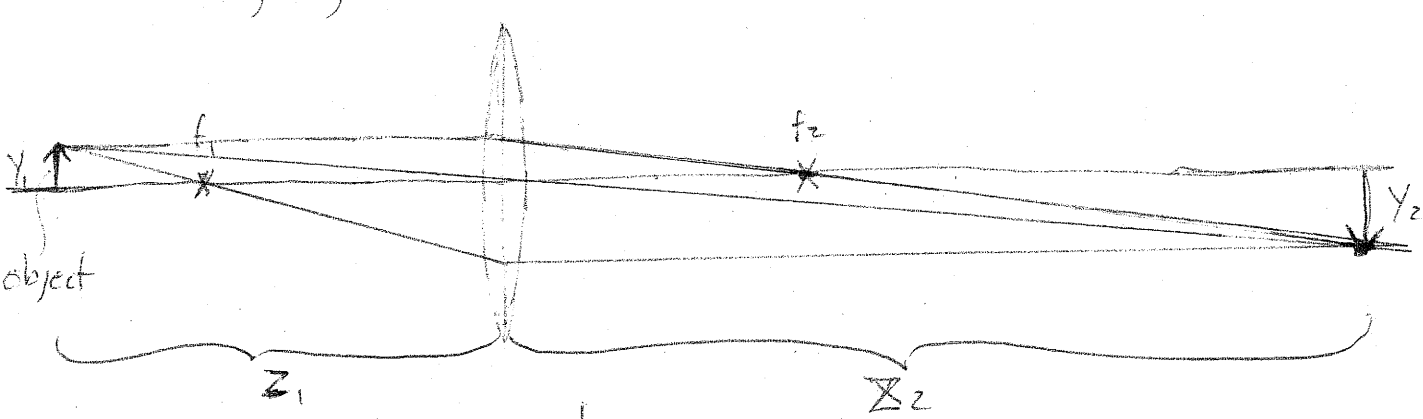
4) All rays that come from a point at the front focal plane emerge from the lens parallel to the ray that crosses through the center



5) A parallel bundle of rays hitting the lens is focused at the back focal plane at the point determined by the ray through the center



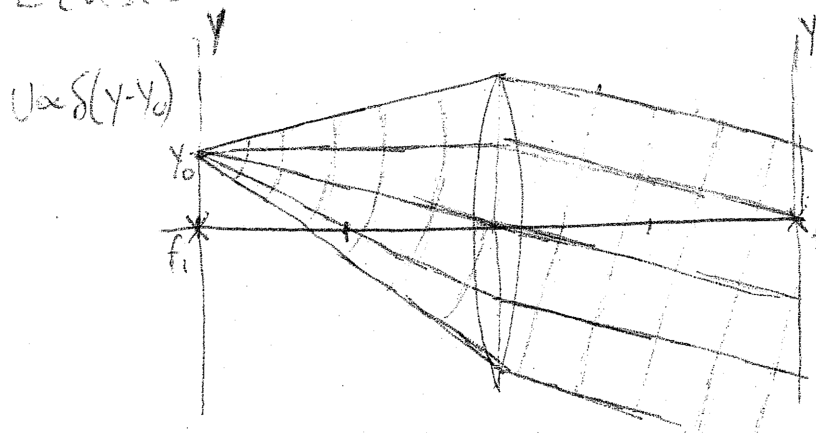
Imaging with one lens



$$\frac{1}{z_1} + \frac{1}{z_2} = \frac{1}{f}$$

$$y_2 = M_T y_1 = \left(-\frac{z_2}{z_1}\right) y_1$$

Lenses also do <u>Fourier transforms</u> (wave optics)



The field at the back focal plane is approximately the $\mathcal{F}$ of the field at the front focal plane

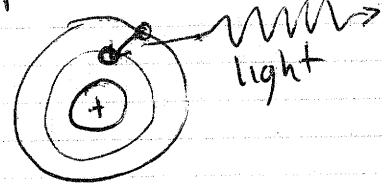
$$U \propto e^{-i\left(\frac{ky'}{f}\right) y_0}$$

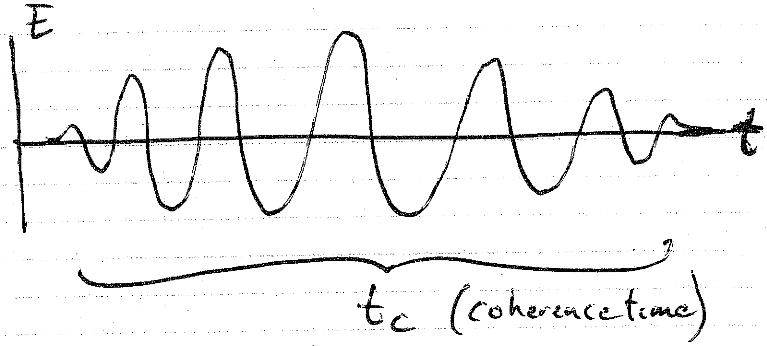
Coherence and partial coherence

Temporal coherence

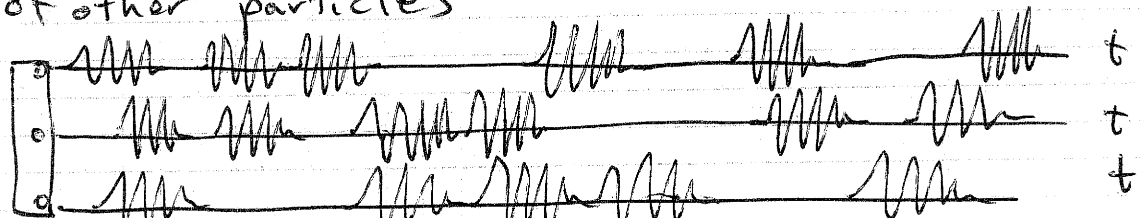
Pure monochromatic fields do not exist.

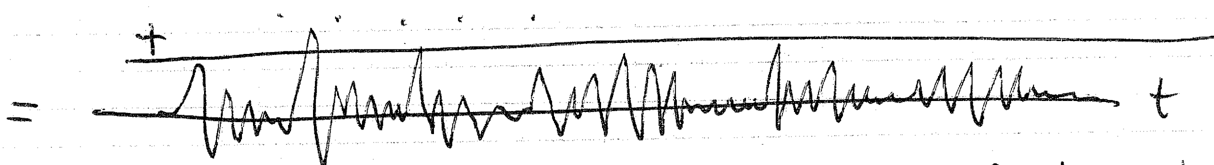
In practice, an atom or molecule emits a "photon" following a transition:





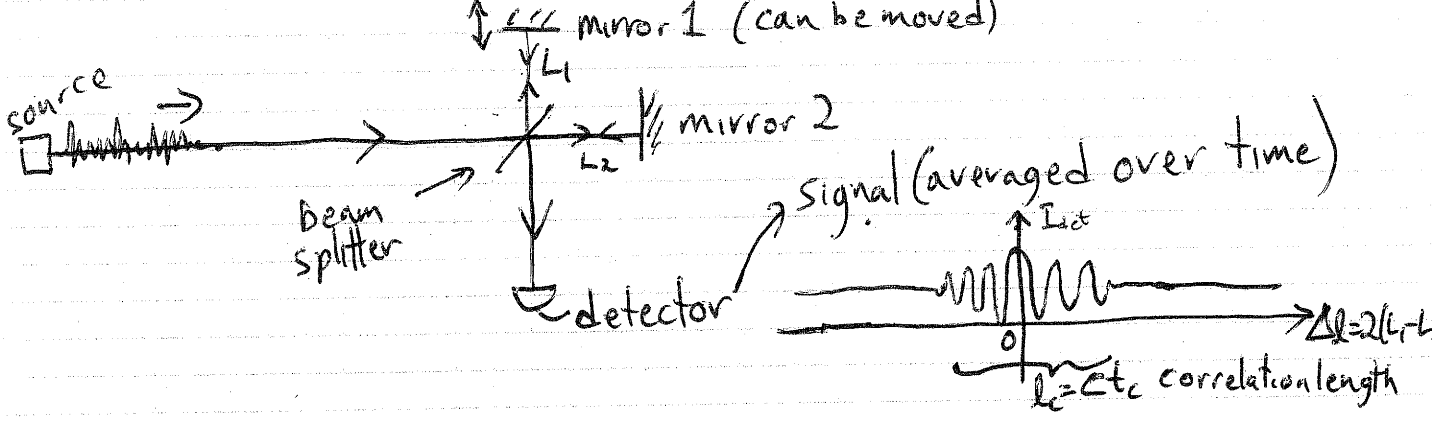
In an incoherent source, each particle emits light pulses randomly, uncorrelated from each other and from those of other particles





The oscillations are only correlated for time differences smaller than t_c.

Michelson interferometer



Where

$$I_{det} = \left\langle \left| \frac{E(t)}{2} + \frac{E(t+\overset{\frac{\Delta l}{c}=\tau}{\tau})}{2} \right|^2 \right\rangle_t = \frac{1}{4}\left\langle \left(E(t)+E(t+\tau)\right)^* \left(E(t)+E(t+\tau)\right) \right\rangle_t$$

$$= \underbrace{\frac{\langle |E(t)|^2\rangle_t}{4}}_{\frac{I_0}{4}} + \underbrace{\frac{\langle |E(t+\tau)|^2\rangle_t}{4}}_{\frac{I_0}{4}} + \frac{2}{4}\,\mathrm{Re}\underbrace{\left\langle E^*(t+\tau)\,E(t)\right\rangle_t}_{\mathcal{C}(\tau)}$$

$$= \frac{I_0 + \mathrm{Re}[\mathcal{C}(\tau)]}{2}$$

Notice: The Correlation $\mathcal{C}(\tau)$ satisfies

$$\mathcal{C}(0) = I_0 \implies I_{det}(0) = I_0$$

$$\mathcal{C}(\tau \gg t_c) = 0 \implies I_{det}(\tau \gg t_c) = \frac{I_0}{2}$$

Wiener-Khinchin theorem (roughly)

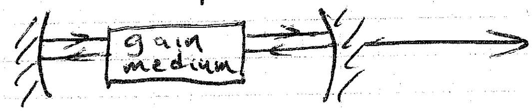
$$\mathcal{C}(\tau) = \langle E^*(t+\tau)\,E(t)\rangle_t \propto \int \underbrace{E^*(t+\tau)}_{\text{write as}\left[\frac{1}{\sqrt{2\pi}}\int \tilde{E}(\omega)e^{-i\omega(t+\tau)}d\omega\right]^*} E(t)\,dt$$

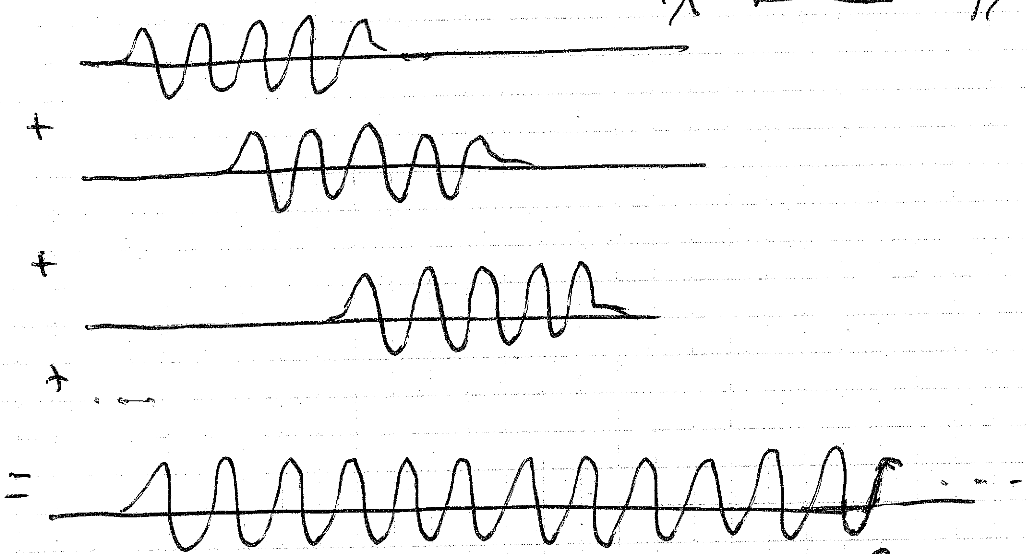
$$\mathcal{C}(\tau) \propto \frac{1}{\sqrt{2\pi}}\iint \hat{E}^*(\omega)\, e^{+i\omega(t+\tau)}\, E(t)\,dt\,d\omega$$

$$= \int \hat{E}^*(\omega)\underbrace{\frac{1}{\sqrt{2\pi}}\int E(t)\,e^{i\omega t}\,dt}_{\tilde{E}(\omega)}\; e^{i\omega\tau}\,d\omega$$

$$= \int \underbrace{|\tilde{E}(\omega)|^2}_{S(\omega)\,=\,\text{spectrum}} e^{i\omega\tau}\,d\omega, \quad \text{so } \mathcal{C}(\tau) = I_0\frac{\hat{f}^{-1}_{\omega\to\tau}\,S(\omega)}{\left[\hat{f}^{-1}_{\omega\to\tau}\,S(\omega)\right]\Big|_{\tau=0}}$$

In a laser, on the other hand, each photon triggers the coherent emission of another one (stimulated emission), so the pulses are not (entirely) random:

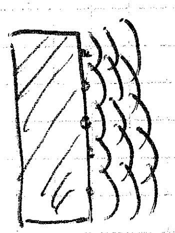




Very long correlation time & length!

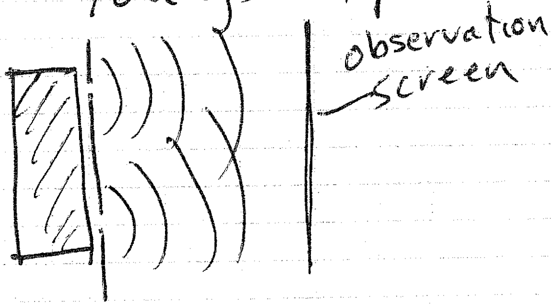
Spatial coherence

extended incoherent source



the emissions of each point are statistically uncorrelated

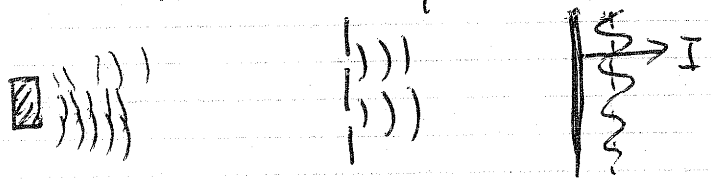
Young's experiment:



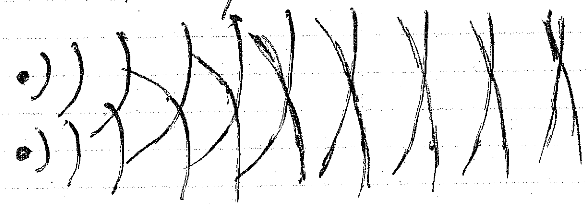
opaque screen with two pinholes

The interference between the light transmissions through the two pinholes washes out statistically over time, so the intensity at the observation screen looks uniform.

Upon propagation from the source, however, the field acquires some spatial coherence:

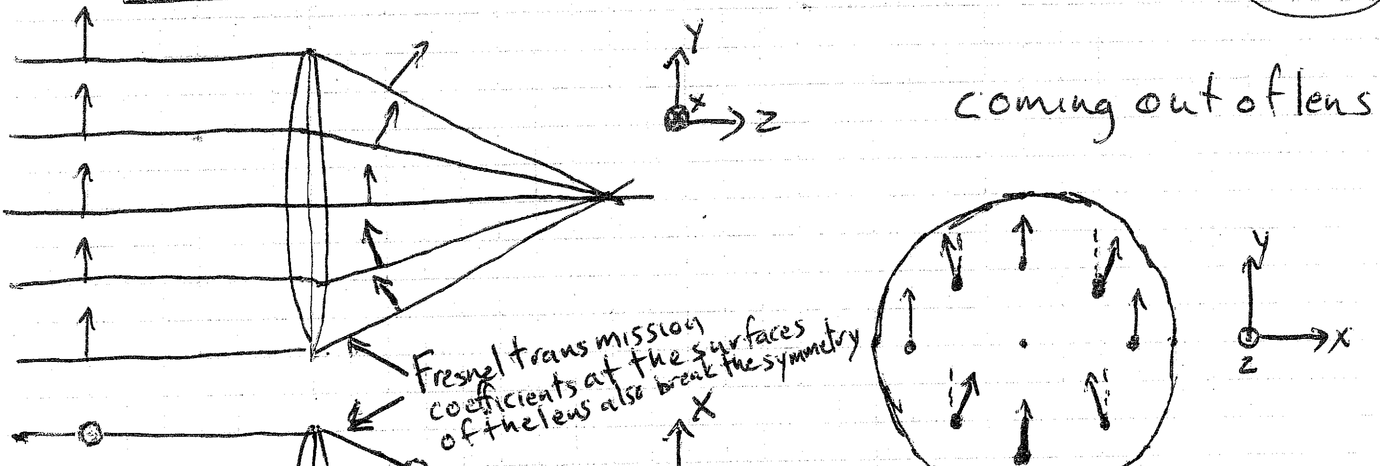


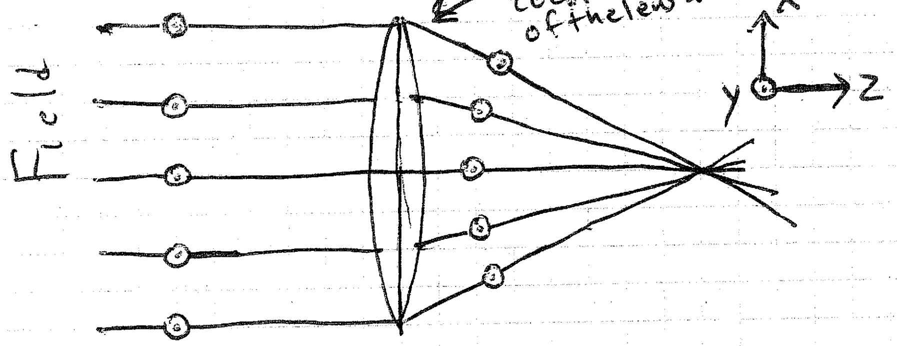
Explanation: consider a source that consists of only two incoherent emitters



as we move away, their wavefronts look more and more similar.

Effects of polarization in focused fields

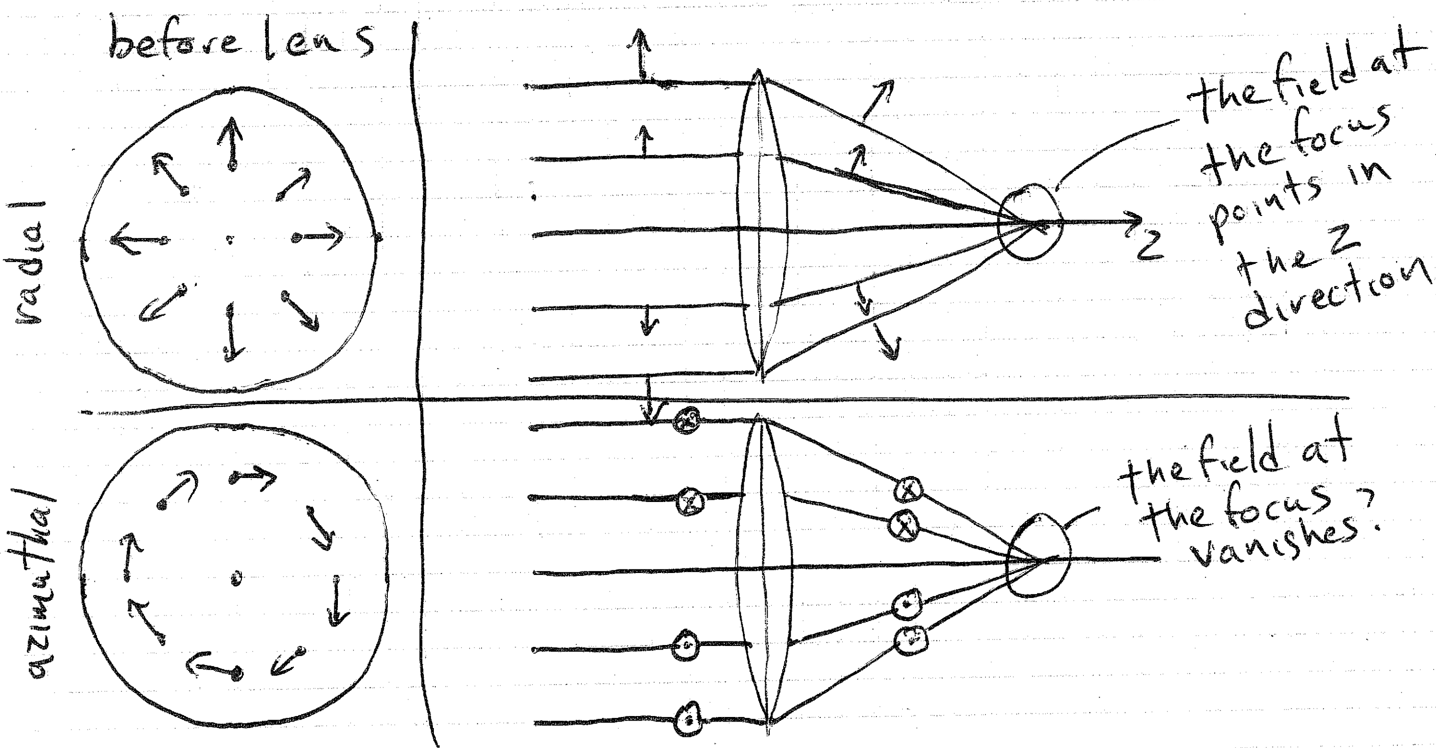




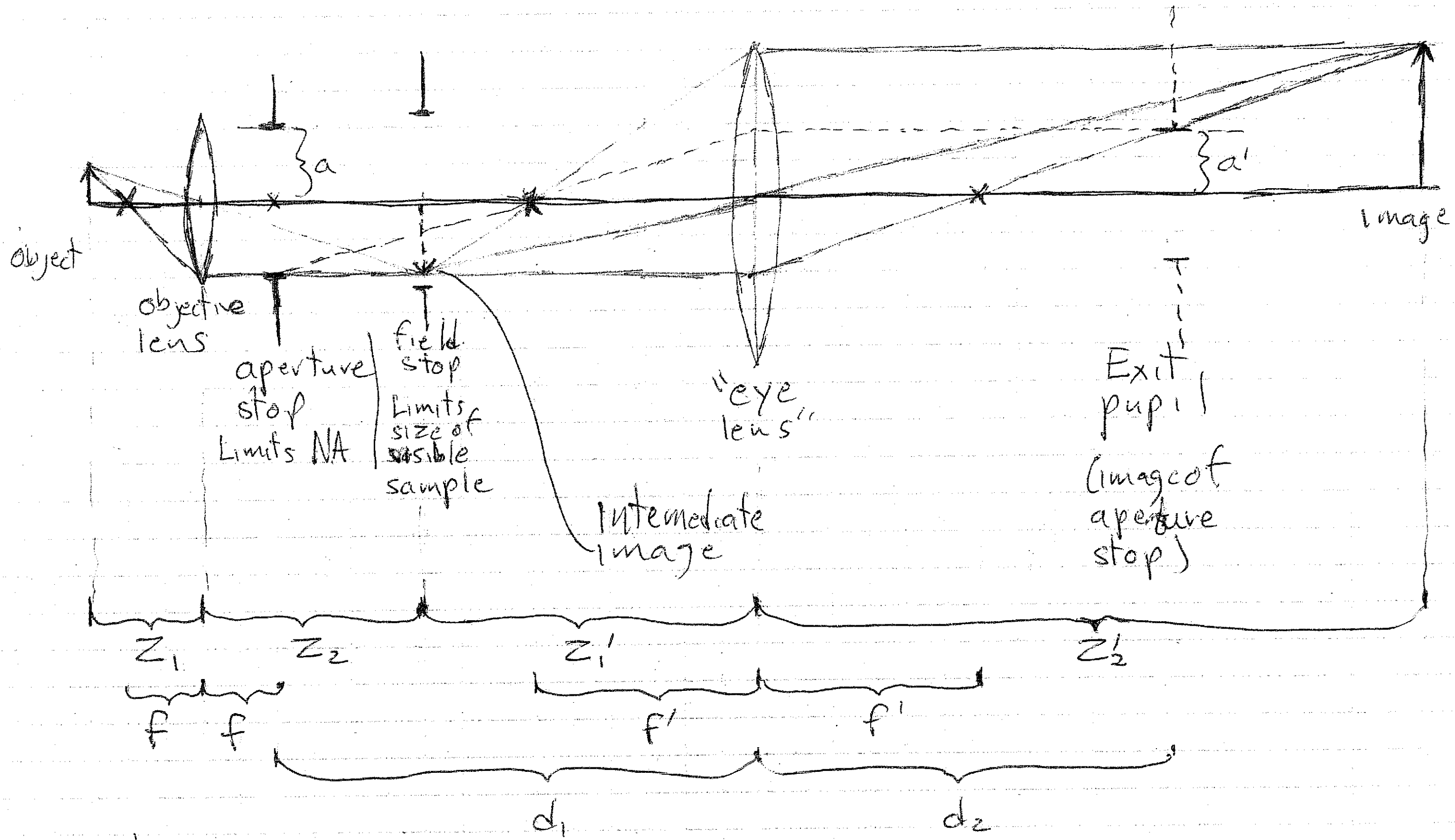
the focal spot becomes asymmetric for large NA!

For small N.A. (paraxial case) each of the x & y components can be treated as an independent scalar field, and the z component is negligible.

Radially and Azimuthally polarized beams



Basic Microscope



where

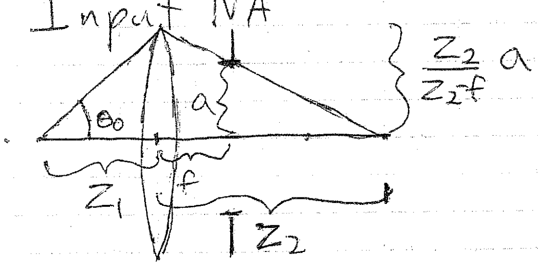
$$\frac{1}{z_1}+\frac{1}{z_2}=\frac{1}{f},\quad \frac{1}{z_1'}+\frac{1}{z_2'}=\frac{1}{f'},\quad \frac{1}{d_1}+\frac{1}{d_2}=\frac{1}{f'},\qquad z_2+z_1'=f+d_1$$

Magnification between object and intermediate image $M_T=-\dfrac{z_2}{z_1}$

Magnification between intermediate and final image $M_T'=-\dfrac{z_2'}{z_1'}$

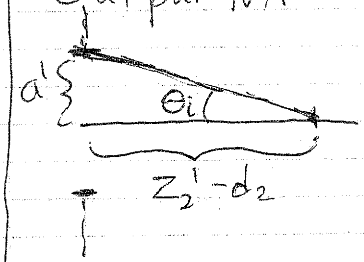
Total magnification: $\underline{M}_T=M_T M_T'=\dfrac{z_2 z_2'}{z_1 z_1'}$

Input NA



$NA_o=\sin\theta_o\approx\tan\theta_o=\dfrac{z_2}{z_2-f}\,\dfrac{a}{z_1}$

Output NA



$NA_i=\sin\theta_i\approx\tan\theta_i=\dfrac{a'}{z_2'-d_2}$

Exercise:

$NA_i=$ _____ NA_o

find this factor and write it here.

For cases where the exit pupil is sufficiently far from the image (i.e. $\gg \lambda$), we can approximate the image of the axial point source as:

$$\underbrace{U_{image}(x_i, y_i) \propto \frac{J_1(kNA_i \rho_i)}{kNA_i \rho_i}}_{\text{Airy function}}$$

The intensity is then

$$I_{image}(x_i, y_i) \propto \left| \frac{J_1(kNA_i \rho_i)}{kNA_i \rho_i} \right|^2$$

If the object is shifted to a point $(\bar{X}_o, \bar{Y}_o)$, the field at the image plane becomes:

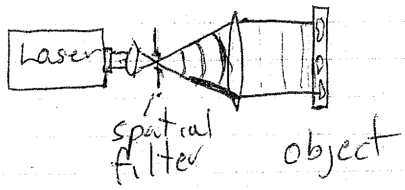
$$U_{image}(x_i, y_i) \propto \underbrace{e^{ik\frac{(\bar{X}_i x_i + \bar{Y}_i y_i)}{z_2'}}}_{\text{tilt factor}} \frac{J_1\left(kNA_i\sqrt{(x_i-\bar{X}_i)^2+(y_i-\bar{Y}_i)^2}\right)}{kNA_i\sqrt{(x_i-\bar{X}_i)^2+(y_i-\bar{Y}_i)^2}},$$

where $(\bar{X}_i, \bar{Y}_i) = \underline{M}_T(\bar{X}_o, \bar{Y}_o)$. The intensity is then

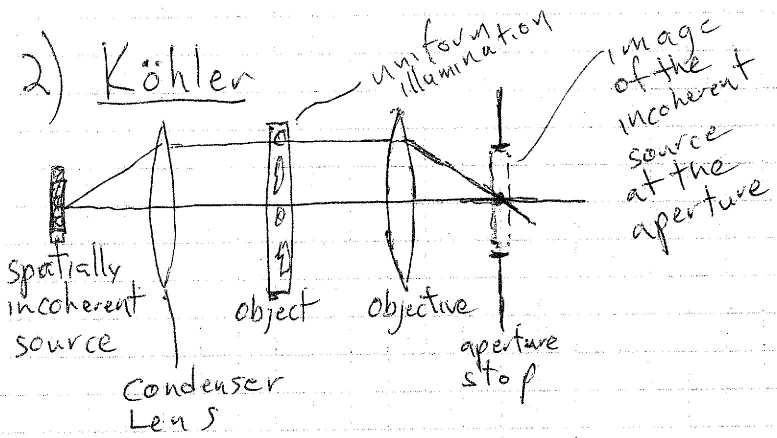
$$I_{image}(x_i, y_i) \propto \left| \frac{J_1\left(kNA_i\sqrt{(x_i-\bar{X}_i)^2+(y_i-\bar{Y}_i)^2}\right)}{kNA_i\sqrt{(x_i-\bar{X}_i)^2+(y_i-\bar{Y}_i)^2}} \right|^2$$

Object illumination

1) Coherent



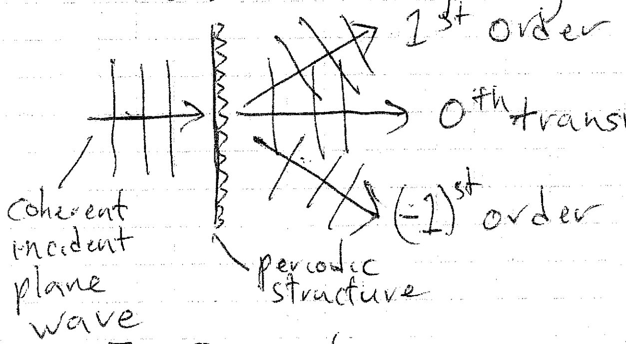
2) Köhler



If the image of the source fills the aperture, this behaves like incoherent illumination.

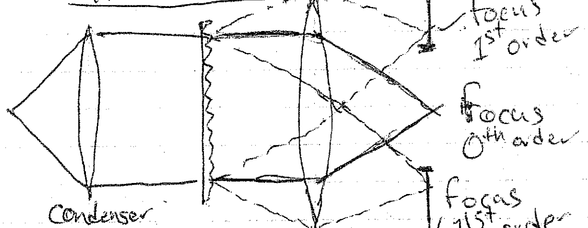
Advantages of "incoherent" (Köhler) illumination:

Consider an object given by a diffraction grating:



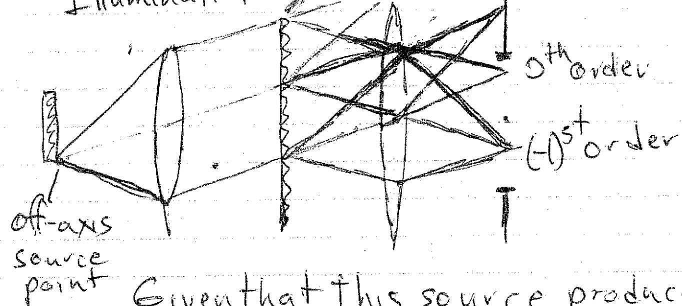
(neglect for now all others)

For Coherent Illumination:



If the grating's period is short, only the 0th order is transmitted through the system, and the grating's information is lost

For Köhler Illumination



Given that this source produces (mutually incoherent) plane waves at different angles hitting the object, some higher frequencies are admitted through the system

Resolution

Consider two object point sources, one at the axis, and one at $(\bar{X}_o, 0)$. The image intensity will then be:

a) If the two point sources are mutually coherent:

phase between sources

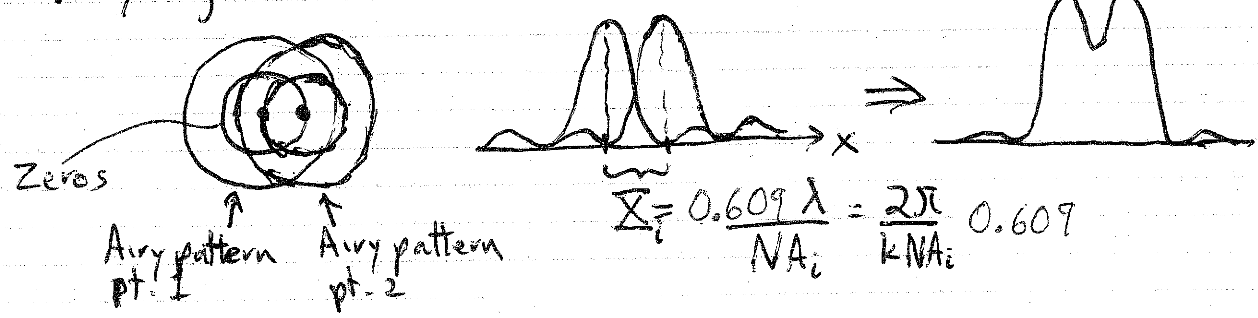
$$I_{image} \propto \left| \frac{J_1\left(k NA_i \sqrt{x_i^2+y_i^2}\right)}{k NA_i \sqrt{x_i^2+y_i^2}} + e^{i\phi} e^{\frac{ik\bar{X}_i x_i}{z_i}} \frac{J_1\left(k NA_i \sqrt{(x_i-\bar{X}_i)^2+y_i^2}\right)}{k NA_i \sqrt{(x_i-\bar{X}_i)^2+y_i^2}} \right|^2$$

The resolution depends strongly on ϕ.

b) If the two point sources are mutually incoherent:

$$I_{image} \propto \left| \frac{J_1\left(k NA_i \sqrt{x_i^2+y_i^2}\right)}{k NA_i \sqrt{x_i^2+y_i^2}} \right|^2 + \left| \frac{J_1\left(k NA_i \sqrt{(x_i-\bar{X}_i)^2+y_i^2}\right)}{k NA_i \sqrt{(x_i-\bar{X}_i)^2+y_i^2}} \right|^2$$

Rayleigh criterion for resolution:

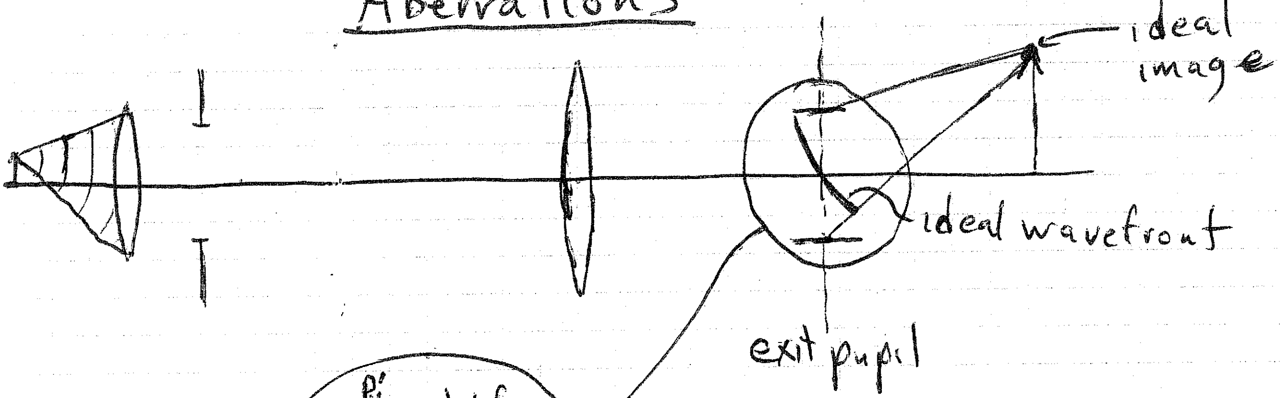


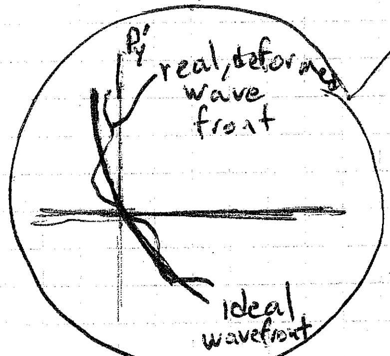
$$\bar{X}_i = \frac{0.609\,\lambda}{NA_i} = \frac{2\pi}{k NA_i}\,0.609$$

the separation between the source points is then

$$\delta_R = \bar{X}_o = \frac{\bar{X}_i}{\bar{M}_T} = \frac{\lambda}{NA_i}\,0.609\,\frac{1}{\bar{M}_T} = \frac{0.609\,\lambda}{NA_o}$$

Aberrations





Let $\underline{\rho} = (\rho_x, \rho_y)$ be normalized coordinates at the pupil.
Let $W(\underline{\rho}, \underline{x}_0)$ be the diference at $\underline{\rho}$ between the real and the ideal wave front for an object point $\underline{x}_0$.

$W(\underline{\rho}, \underline{x}_0)$ is called the <u>wave aberration function</u>.

Expand it in a Taylor series:

$$W(\underline{\rho}, \underline{x}_0) = \underbrace{\cancel{W(\underline{0},\underline{0})}}_{\substack{\text{because}\\ W(\underline{0},\underline{x}_0)=0}} + \underbrace{a \quad \underline{\rho}'\cdot\underline{\rho}'}_{\substack{\text{this term}=0 \text{ if}\\ \text{the axial point is}\\ \text{well focused}}} + \underbrace{b \quad \underline{\rho}'\cdot\underline{x}_0}_{\substack{\text{this term}=0 \text{ if}\\ M_T \text{ is calculated}\\ \text{properly}}} + \underbrace{\cancel{c \quad \underline{x}_0\cdot\underline{x}_0}}_{\substack{\text{because}\\ W(\underline{0},\underline{x}_0)=0}}$$

"3rd" order aberrations:

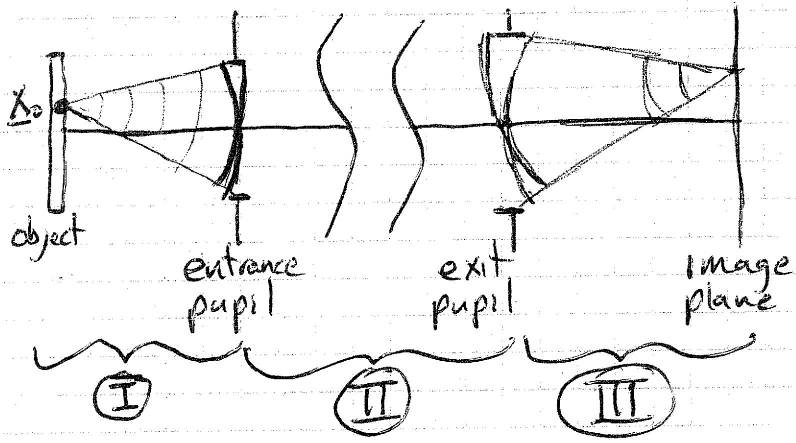
$$+ \underbrace{\left(-\frac{B}{4}\right) |\underline{\rho}|^4}_{\text{spherical}} + \underbrace{F \quad |\underline{\rho}|^2 \underline{\rho}'\cdot\underline{x}_0}_{\text{coma}} + \underbrace{\left(-\frac{D}{2}\right) |\underline{\rho}|^2 |\underline{x}_0|^2}_{\substack{\text{Petzval field}\\ \text{curvature}}} + \underbrace{(-C) \; (\underline{\rho}'\cdot\underline{x}_0)^2}_{\text{Astigmatism}} + \underbrace{E \; \underline{\rho}'\cdot\underline{x}_0 |\underline{x}_0|^2}_{\text{Distortion}} + \underbrace{\cancel{\left(-\frac{A}{4}\right) |\underline{x}_0|^4}}_{\substack{\text{because}\\ W(\underline{0},\underline{x}_0)=0}}$$

+ Higher orders

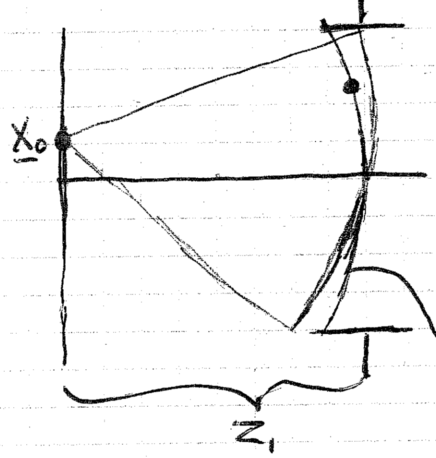
Aberrations can degrade the image significantly. Notice that, for an axial object point ($\underline{x}_0 = \underline{0}$), the only aberration is spherical.

Simplistic model of imaging systems

Coherent illumination



(I) propagation from object plane to reference wavefront at entrance pupil:



ρ = normalized pupil coordinate

reference wavefront centered at axial object point

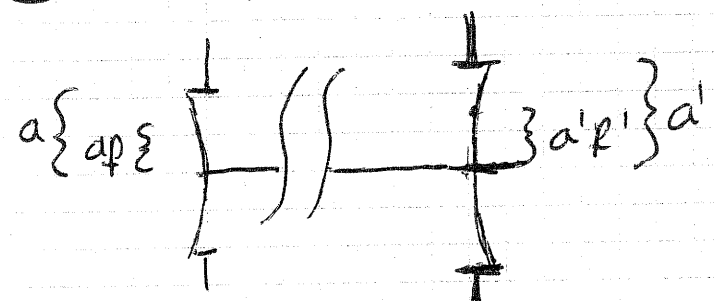
The phase of a spherical wave coming from $\underline{x}_0$ over the reference wavefront is roughly proportional to: $e^{-ik\underline{x}_0\cdot\frac{(a\underline{\rho})}{z_1}} = e^{-ikNA_0\underline{x}_0\cdot\underline{\rho}}$

so, the field at the reference wavefront at the entrance pupil is

$$U_{ent.p.}(\underline{\rho}) \propto \iint_{object} U_{object}(\underline{x}_0)\, e^{-ik\underline{x}_0\cdot\underline{\rho}\, NA_0}\, dx_0 dy_0$$

$$\propto \hat{\mathcal{F}}_{x_0 y_0}\left\{U_{object}\right\}\Big|_{\underline{k}=kNA_0\underline{\rho}}$$

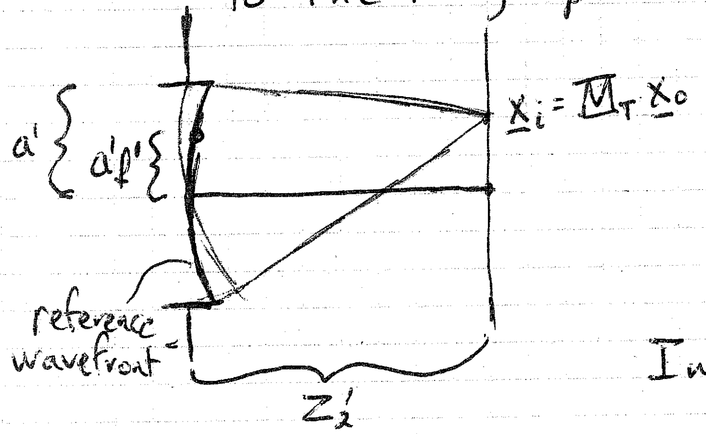
II Propagation from the entrance pupil to the exit pupil



We assume that the system forms a (reversed) image of the entrance pupil at the exit pupil, with a phase containing the aberrations:

$$U_{exit\,p.}(\underline{\rho}') \propto U_{ent.\,p.}(-\underline{\rho})\, \underbrace{e^{ikw(\underline{\rho}', \frac{\underline{X}_i}{M_T})}}_{\text{aberrations}}$$

III Propagation from the reference wavefront at the exit pupil to the image plane.



We assume that all the diffraction effects occur at the (real or virtual) exit pupil.

In analogy to I

$$U_{image}(\underline{X}_i) \propto \iint_{pupil} U_{exit\,p.}(\underline{\rho}')\, e^{-ik \underbrace{NA_i}_{\frac{NA_0}{M_T}} \underline{\rho}' \cdot \underline{X}_i}\, d\rho_x' d\rho_y'$$

Diffraction is accounted for by limiting the integral to $|\underline{\rho}'| \leq 1$. Let $P(\underline{\rho}) = \begin{cases} 1, & |\underline{\rho}| \leq 1 \\ 0, & |\underline{\rho}| > 1 \end{cases}$.

Let $\underline{K} = k NA_0 \underline{\rho}$. Then $\underline{\rho}' = -\underline{\rho} = -\underline{K}/kNA_0$ and

$$\boxed{U_{image}(\underline{X}_i) \propto \hat{f}^{-1}_{\underline{K} \to \frac{\underline{X}_i}{M_T}} \left\{ P\left(\frac{\underline{K}}{kNA_0}\right) e^{ikw\left(\frac{-\underline{K}}{kNA_0}, \frac{\underline{X}_i}{M_T}\right)} \hat{f}_{\underline{X}_0 \to \underline{K}} \left\{ U_{object}(\underline{X}_0) \right\} \right\}}$$

This can also be written

$$U_{image}(\underline{X}_i) \propto \Big[U_{object}(\underline{X}_o) * \underbrace{\hat{\mathcal{F}}^{-1}_{\underline{K}\to\underline{X}_o}\Big\{ P\Big(\frac{\underline{K}}{kNA_o}\Big) e^{ikw\left(\frac{-\underline{K}}{kNA_o}, \frac{\underline{X}_i}{\underline{M}_T}\right)} \Big\}}_{} \Big] \Big|_{\underline{X}_o = \underline{X}_i/\underline{M}_T}$$

for a circular pupil and for $w=0$, this reduces to $\dfrac{J_1(kNA_o|\underline{X}_o|)}{kNA_o|\underline{X}_o|}$

Incoherent Illumination

We have instead

$$I_{image}(\underline{X}_i) \propto \Big[I_{object}(\underline{X}_o) * \Big(\Big| \hat{\mathcal{F}}^{-1}_{\underline{K}\to\underline{X}_o}\Big\{ P\Big(\frac{\underline{K}}{kNA_o}\Big) e^{ikw\left(\frac{-\underline{K}}{kNA_o}, \frac{\underline{X}_i}{\underline{M}_T}\right)} \Big\} \Big|^2 \Big) \Big] \Big|_{\underline{X}_o = \underline{X}_i/\underline{M}_T}$$

This can be written in terms of $\hat{\mathcal{F}}$ as:

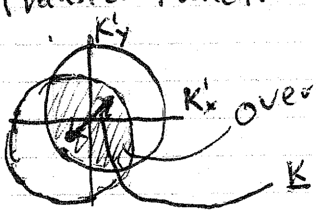
$$I_{image}(\underline{X}_i) \propto \hat{\mathcal{F}}^{-1}_{\underline{K}\to\frac{\underline{X}_i}{\underline{M}_T}} \Big\{ OTF(\underline{K})\, \hat{\mathcal{F}}_{\underline{X}_o\to\underline{K}} \big\{ I_{object}(\underline{X}_o) \big\} \Big\},$$

where the Optical Transfer function is defined as

$$OTF(\underline{K}) = \underset{\text{normalization}}{\frac{1}{N}} \hat{\mathcal{F}}_{\underline{X}_o\to\underline{K}} \Big\{ \Big| \hat{\mathcal{F}}^{-1}_{\underline{K}\to\underline{X}_o} \Big\{ P\Big(\frac{\underline{K}}{kNA_o}\Big) e^{ikw} \Big\} \Big|^2 \Big\}$$

$$= \frac{\iint P^*\Big(\frac{\underline{K}'-\underline{K}}{kNA_o}\Big) P\Big(\frac{\underline{K}'}{kNA_o}\Big) e^{ik\left[w\left(\frac{-\underline{K}'}{kNA_o}, \frac{\underline{X}_i}{\underline{M}_T}\right) - w\left(\frac{\underline{K}-\underline{K}'}{kNA_o}, \frac{\underline{X}_i}{\underline{M}_T}\right)\right]} dK_x' dK_y'}{\iint \Big| P\Big(\frac{\underline{K}'}{kNA_o}\Big) \Big|^2 dK_x' dK_y'}$$

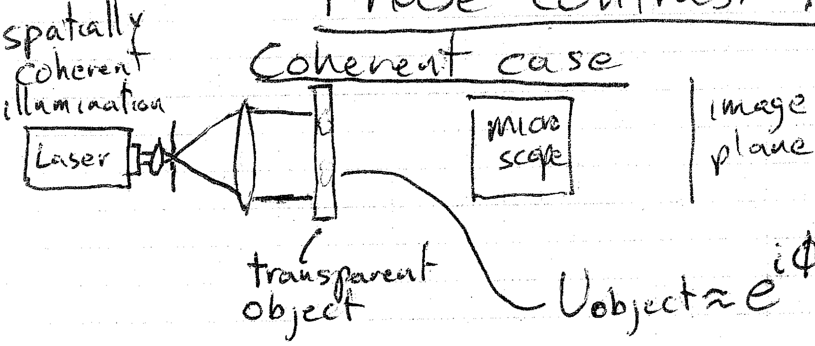
Notice: $OTF(\underline{0}) = 1$.   Modulation Transfer function: $MTF = |OTF|$

For $w=0$, $P = \begin{cases} 0 \\ 1 \end{cases}$, $OTF = MTF \propto$ 

Exercise: Find the expression.

Phase contrast Microscopy

Coherent case



$U_{object} \approx e^{i\phi(\underline{x}_o)}$, for $|\phi| < \pi/2$

The object intensity is approximately uniform, and so is the image, except for the diffraction effects of the pupil.

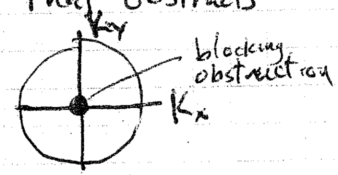
For $|\phi| \ll \pi/2$, $U_{object} = e^{i\phi} \approx 1 + i\phi + \cancel{\ldots}$

$$\hat{\mathcal{F}}_{\underline{x}_o \to \underline{k}} \left\{ U_{object}(\underline{x}_o) \right\} \approx \hat{\mathcal{F}}_{\underline{x}_o \to \underline{k}} \left\{ 1 + i\phi \right\} = \delta(\underline{k}) + i\, \hat{\mathcal{F}}_{\underline{x}_o \to \underline{k}} \left\{ \phi \right\}$$

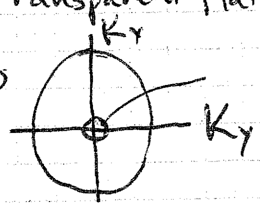
In practice, the object size is finite, so instead of a δ, we have a very narrow distribution, like an Airy pattern (if the field of view of the object is circular).

To increase the visibility, we can do several things:

a) place a block that obstructs the $\delta(\underline{k})$:



b) place instead a thin transparent flat obstruction that introduces a phase of $\pm i$ and a dimming t.



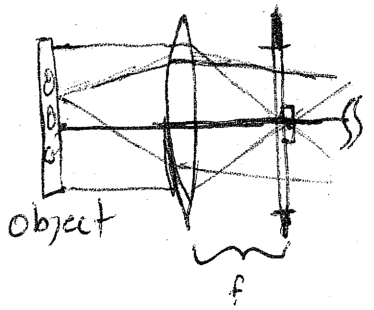
} Phase contrast.

The image will then be, approximately (assuming no aberrations):

$$U_{image}(\underline{x}_i) \propto \hat{\mathcal{F}}^{-1}_{\underline{k} \to \underline{x}_i / M_T} \left\{ P\left(\frac{\underline{k}}{k NA_o}\right) \left[\pm i t\, \delta(\underline{k}) + i\, \hat{\mathcal{F}}_{\underline{x}_o \to \underline{k}} \phi \right] \right\}$$

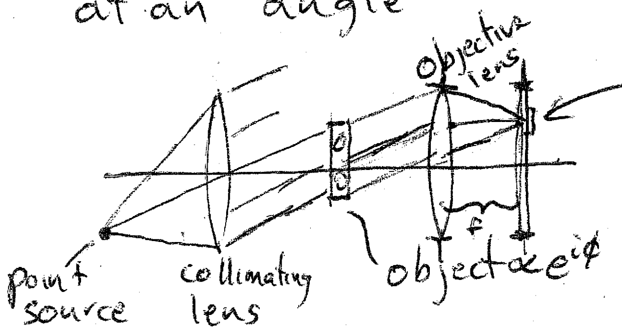
$$\propto \left[\left(\phi(\underline{x}_o) \pm t \right) * \hat{\mathcal{F}}^{-1}_{\underline{k} \to \underline{x}_o} P\left(\frac{\underline{k}}{k NA_o}\right) \right] \Big|_{\underline{x}_o = \underline{x}_i / M_T}$$

This is implemented by placing the phase retarder mask on the back focal plane of the objective lens:



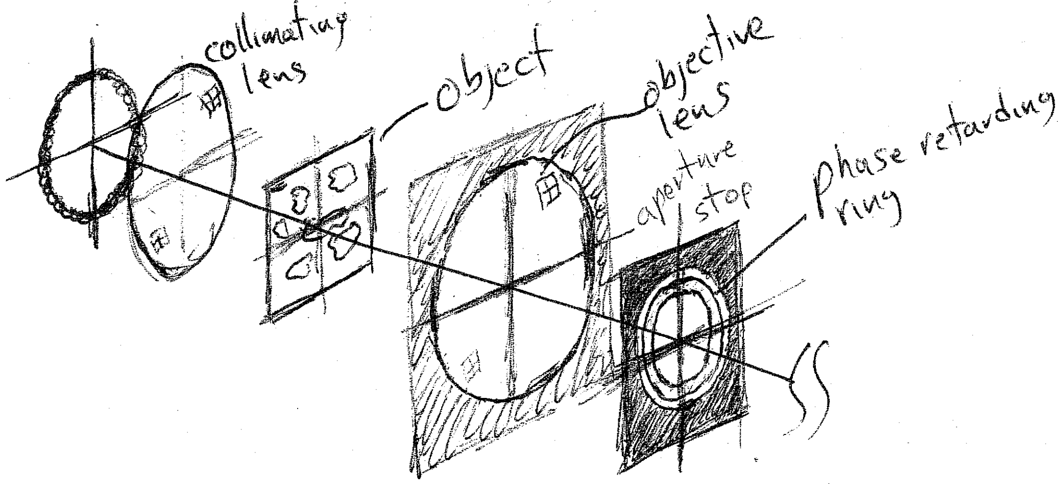
phase contrast with incoherent illumination

Suppose that, instead of illuminating with a collimated beam in the $\hat{z}$ direction, we use one at an angle



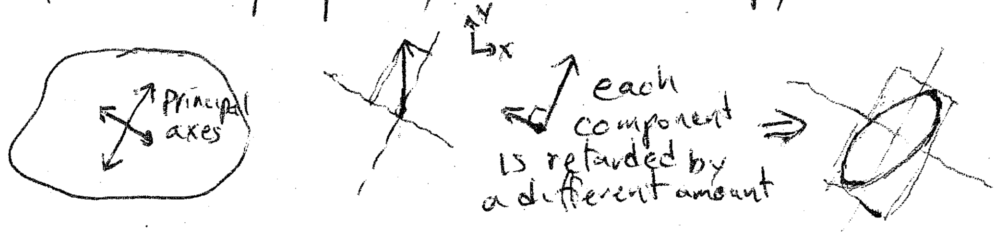
The phase retarding spot would have to be moved off-axis.

Now consider a superposition of mutually incoherent point sources distributed over a ring:



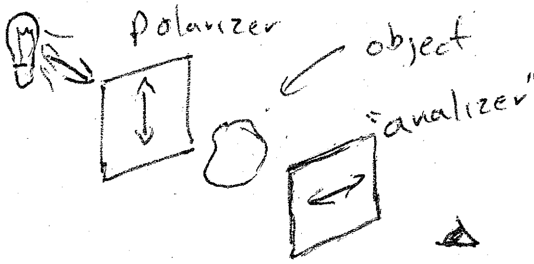
Anisotropy

Many crystals (quartz, calcite) and biological objects present the property of anisotropy:



the anisotropic medium turns the linear polarization into elliptic, i.e. it introduces an x component

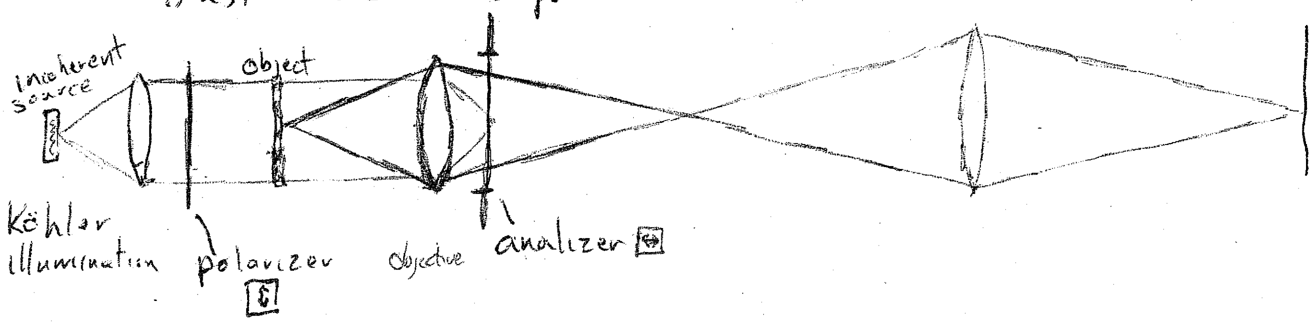
Polariscope



If the is not anisotropic, then no light is transmitted.

Polarization microscopy

Just insert a polarizer and analizer



Differential Interference Contrast (DIC) Microscopy

Consider again a transparent object with uniform amplitude:

$$\text{object} \propto e^{i\phi(x_0,y_0)}$$

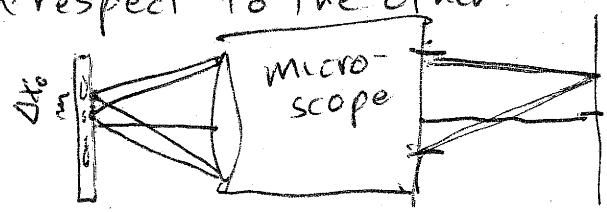
Another way to induce an amplitude variation in the image is to somehow take a directional derivative (say, with respect to x_0)

$$\frac{\partial\,\text{object}}{\partial x_0} \propto i\frac{\partial\phi}{\partial x_0}\, e^{i\phi}$$

The idea of DIC is to optically approximate a derivative. Recall

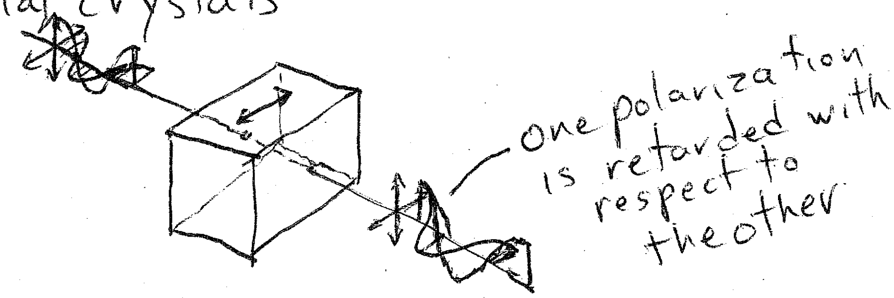
$$\frac{\partial U}{\partial x_0} = \lim_{\Delta x_0 \to 0} \frac{U(x_0+\frac{\Delta x_0}{2}, y_0) - U(x_0-\frac{\Delta x_0}{2}, y_0)}{\Delta x_0}$$

Then, the microscope must somehow image two slightly displaced object points to the same image point, where one of the images is out of phase with respect to the other:

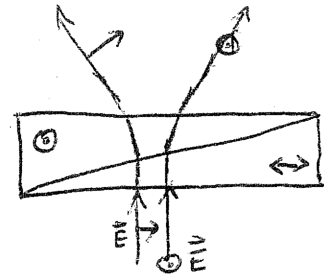


DIC combines the two images by using polarization.

Uniaxial crystals

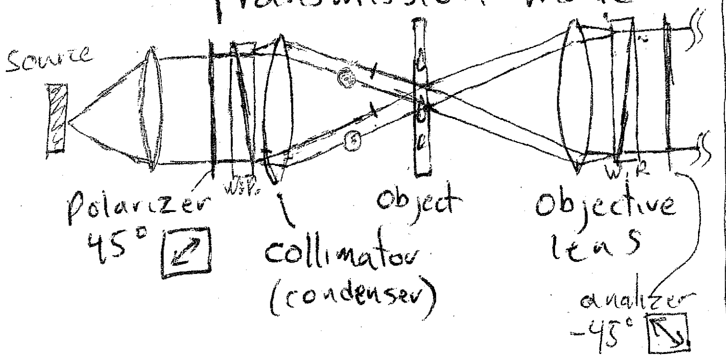


Wollaston prism

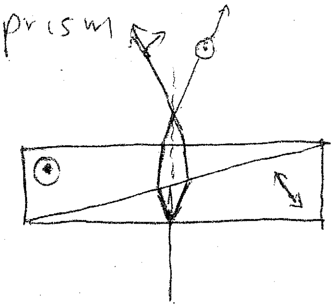


two wedges of identical crystal, with axes at 90°

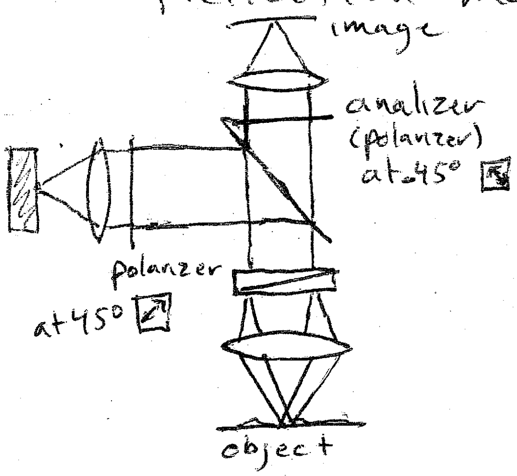
Transmission mode



can replace the Wollaston prism with a Nomarski prism



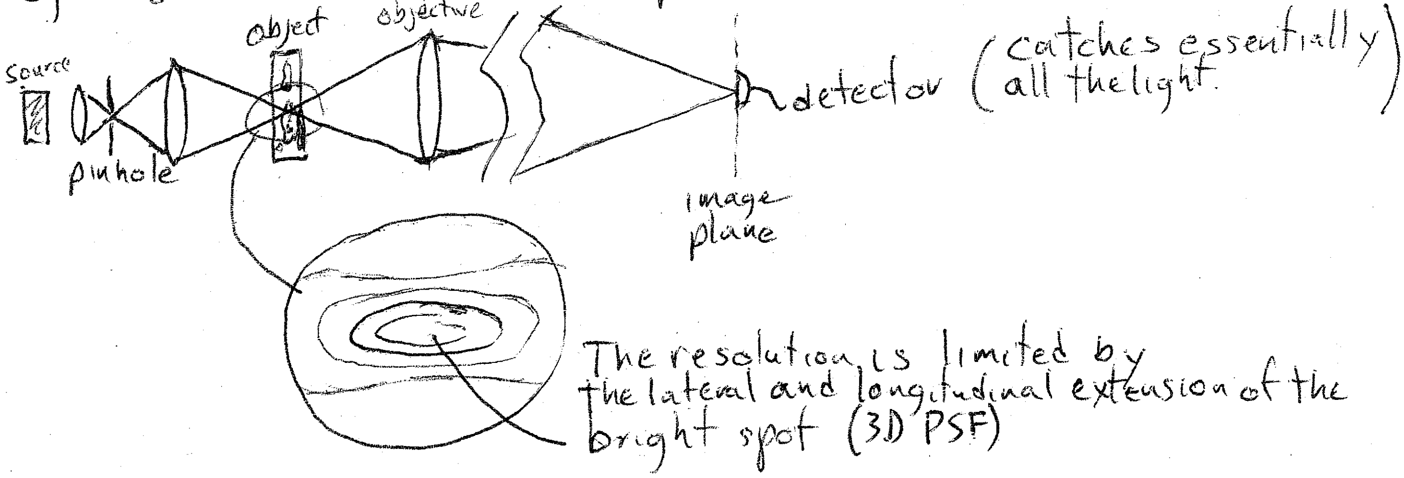
Reflection mode



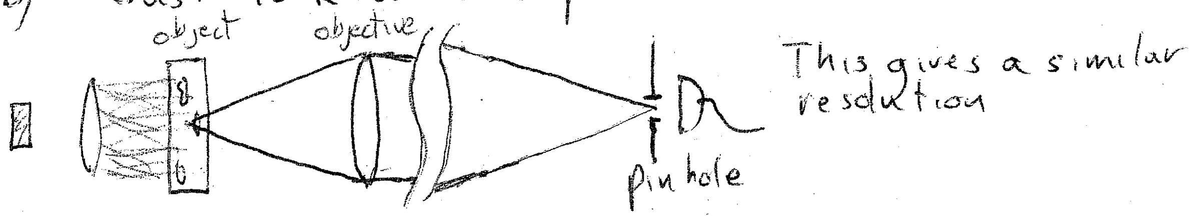
Scanning Microscopes

We only look at one "point" of the object at a time. We can do that in several ways:

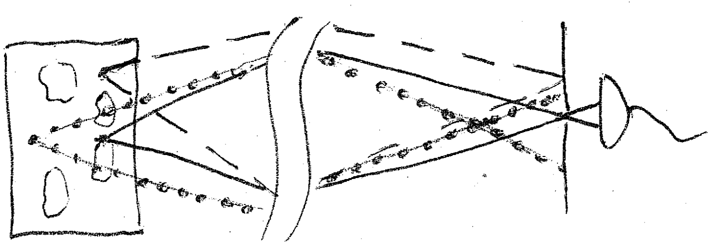
a) Just illuminate one point:



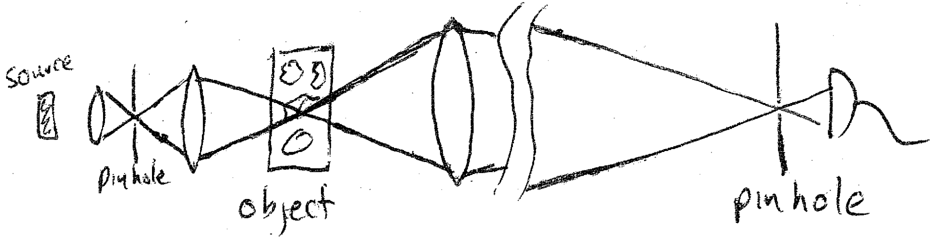
b) Just look at one point:



Both a) & b) can resolve points laterally and longitudinally. For b):



c) Confocal microscopy: does both things.

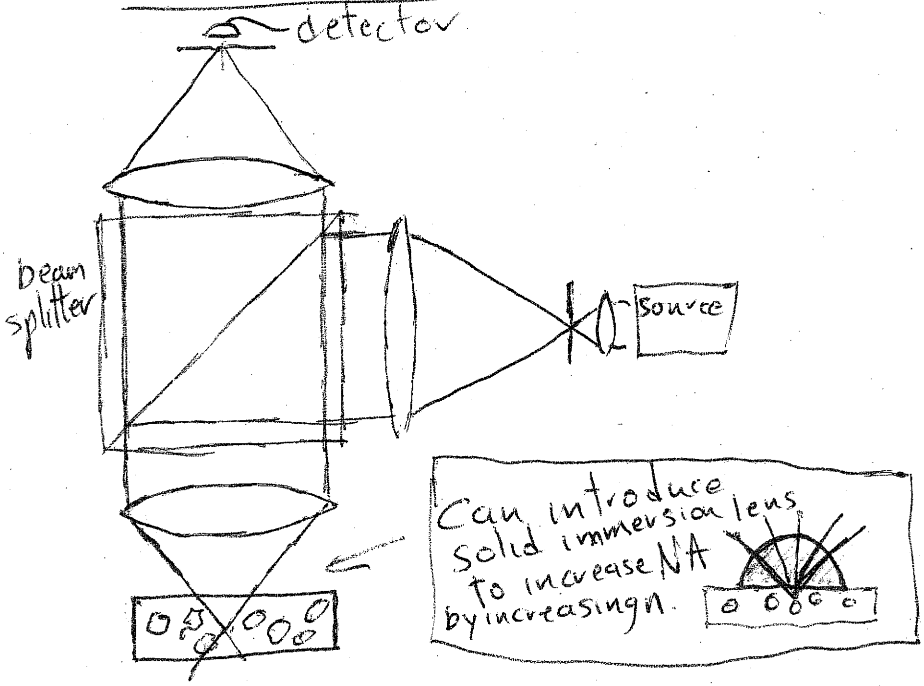


so we mostly illuminate one point, and we filter most of the light not coming from that point.

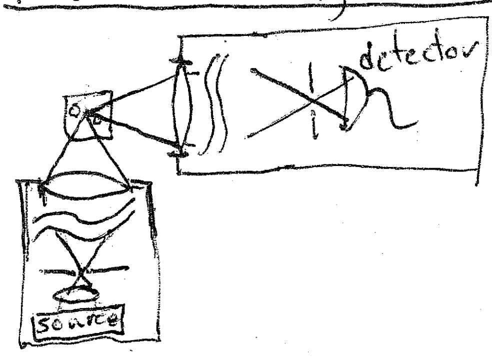
The total PSF is ≈ the excitation PSF x the detection PSF.

Other Configurations

Reflection



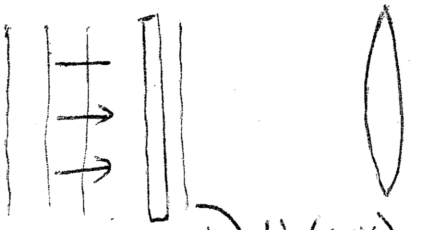
Theta configuration



Question: can you explain what is the advantage of this configuration? Why?

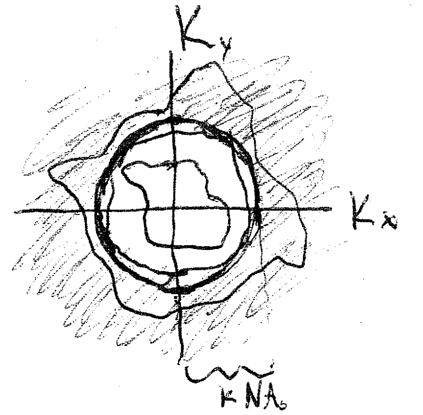
Why does local illumination give more resolution?

a) coherent collimated illumination



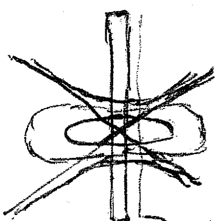
$T(x_0,y_0)$ $U_0(x_0,y_0) \propto T(x_0,y_0)$

$\hat{f}_{x_0} U_0$



The system only accepts spatial frequencies $|\underline{K}| < kNA_0$

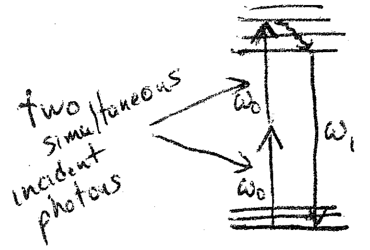
b) Localized (focused) illumination



$T(\underline{x}_0)$ $U_0(x_0,y_0) \propto \dfrac{J_1(kNA_{illum}|\underline{x}_0|)}{kNA_{illum}|\underline{x}_0|} T(\underline{x}_0)$

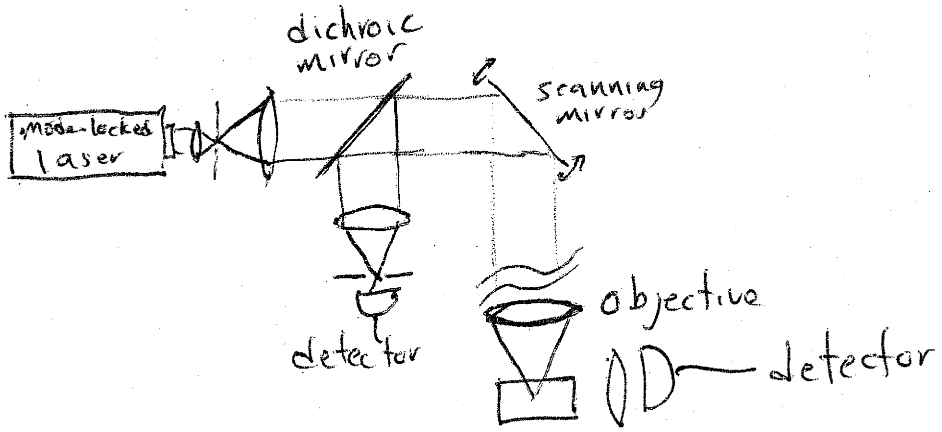
Questions: What is now the form of $\hat{f}_{x_0} U_0$? Do higher frequencies enter now the optical system? How large?

Two-photon microscopy



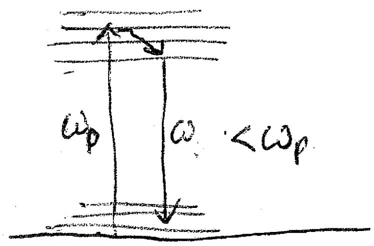
ω_1 is a bit less than $2\omega_0$, so λ_1 is a bit more than $\frac{\lambda_0}{2}$.
The two incident photons must be nearly simultaneous, so the probability of emission (and therefore the intensity I_1) is proportional to the square of the incident intensity: $I_1 \propto I_0^2$

Must use high-power pulsed lasers.



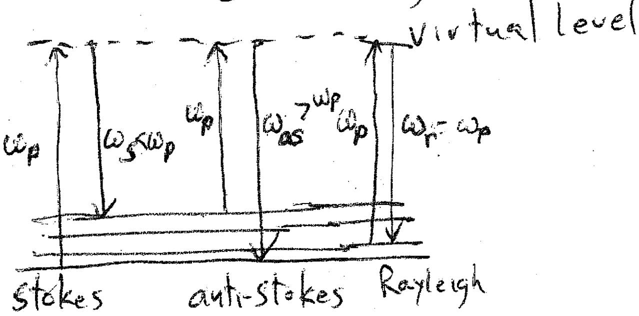
because $I_1 \propto I_0^2$, the PSF is <u>enhanced</u>.

Fluorescence



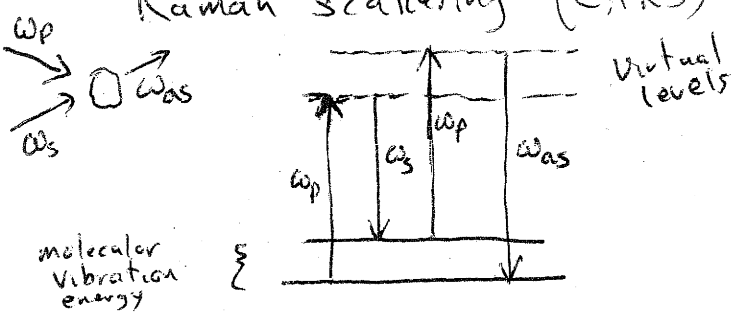
needs the use
of dyes

Raman scattering



Very unlikely events,
small signals

Coherent anti-stokes Raman scattering (CARS)



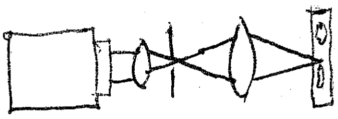
More likely than Raman,
but, like 2-photon, requires pulsed
lasers. Resolution similar
to 2-photon.

Can be implemented in confocal microscope,
using dichroic mirror as beam-splitter.

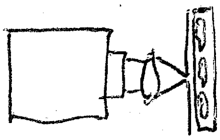
Near-field scanning optical microscopy

a) Near field illumination

Instead of using the focalized illumination of confocal microscopy



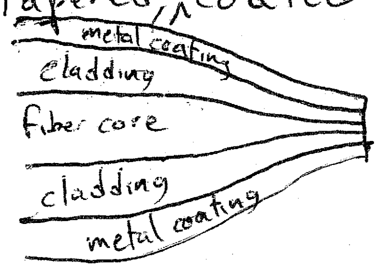
Consider placing the pinhole right at the object.



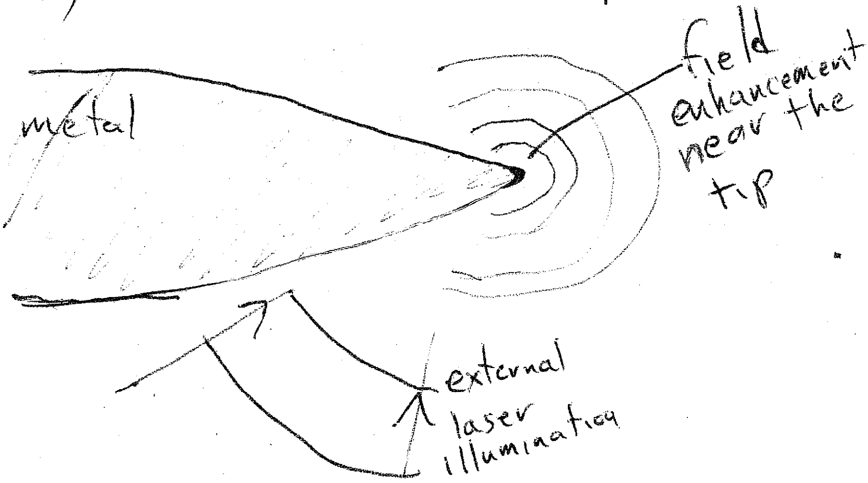
If the aperture is very small (smaller than $\frac{\lambda}{NA_{illum}}$) then more spatial frequency components of the object can be coupled with the observation system, even frequency components for which $|\underline{k}| > k$!

In practice, instead of an aperture, what are used are:

1) Tapered, metal-coated fibers

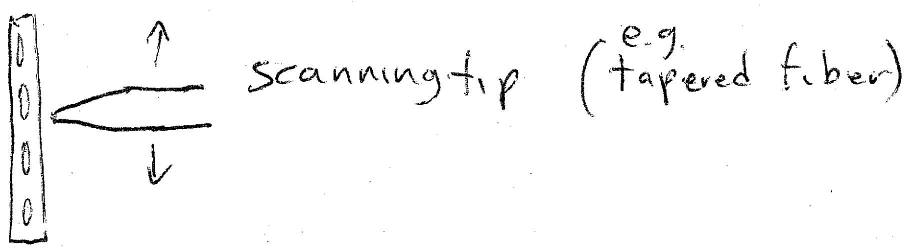


2) Resonant metal tips

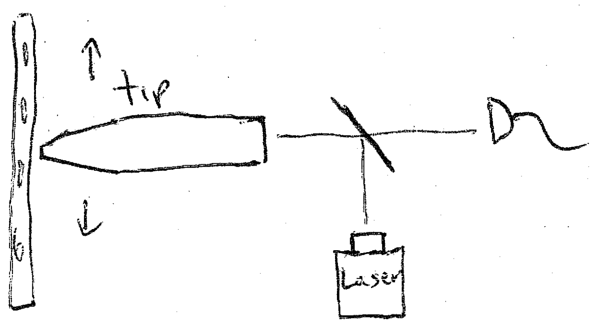


b) near field detection

We can also replace the imaging system with a near-field probe that measures the field very close to the object, detecting even the evanescent waves

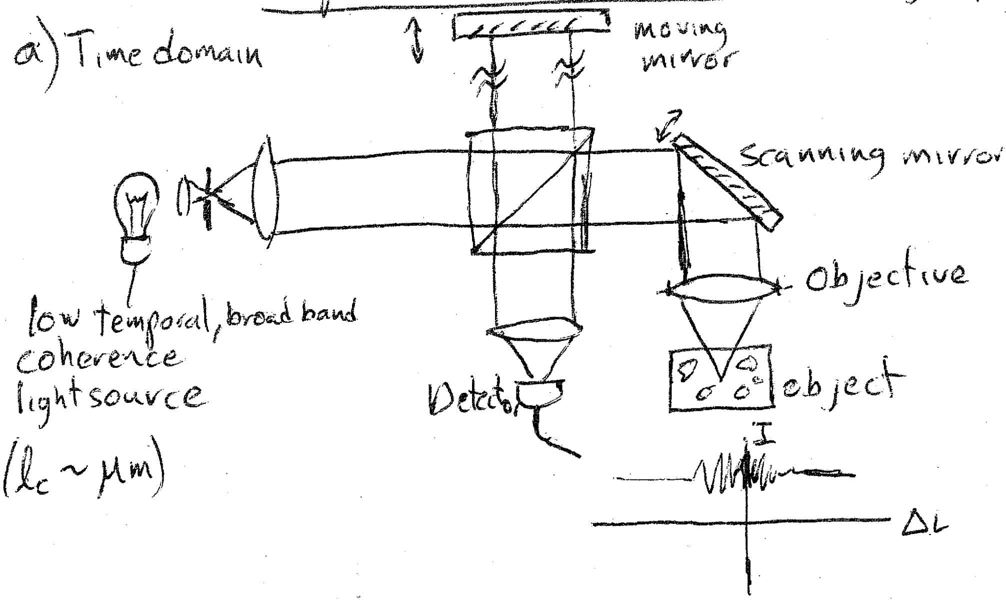


c) near field illumination and detection.



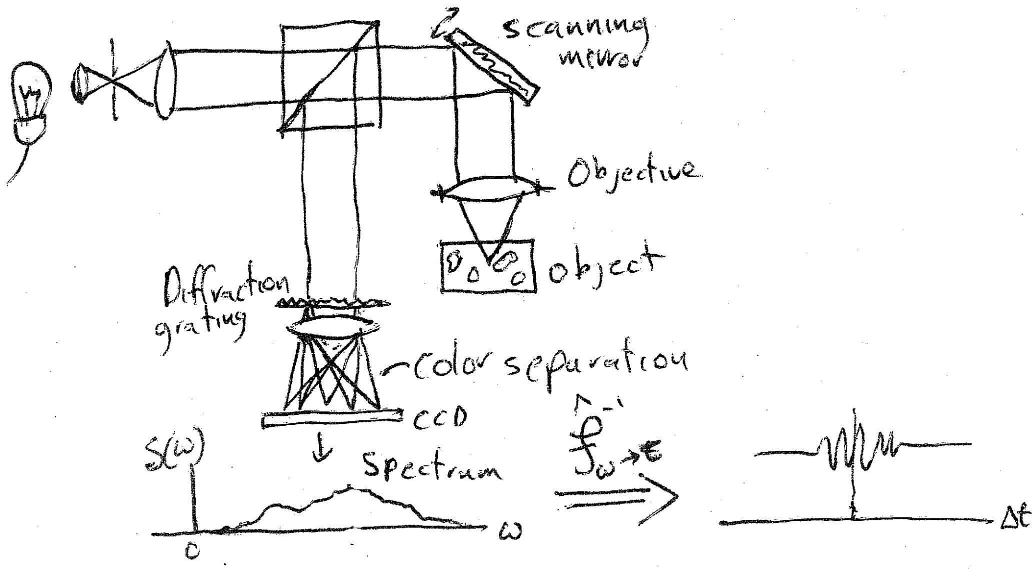
Optical Coherence tomography (OCT)

a) Time domain



$(l_c \sim \mu m)$

b) Frequency domain



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