The Abdus Salam
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United Nations
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IAEA
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1936-2

Advanced School on Synchrotron and Free Electron Laser Sources and their Multidisciplinary Applications

7 - 25 April 2008

SR Storage Ring
SR Beam Optics

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Elettra, Trieste

SR - Storage Ring
SR - Beam Optics

Accelerators

- Are devices of increasing importance in science and industry
- Usually are big constructions and thus have also a significant public impact.

Synchrotrons

Storage rings

Accelerators are used

- In high energy particle physics to study the internal structure of nuclei and the interactions between elementary particles.

To investigate, one needs a probe with even smaller dimensions, particles have a De Broglie wavelength $\lambda = h/p = hc/E$ (for 10^{-15} m one needs practically 1 GeV)

- In applied science for production of **synchrotron radiation**, isotopes and beams for medical uses and for research in the physics of matter

Motion in Electric and Magnetic Fields

- Governed by Lorentz force

$$\frac{d\vec{p}}{dt} = q\left[\vec{E} + \vec{v} \times \vec{B}\right]$$

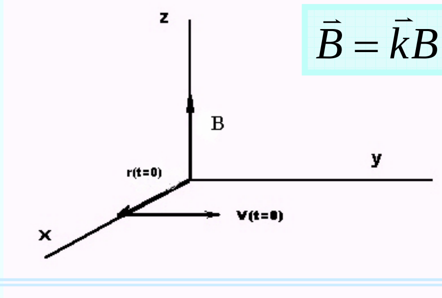
- A magnetic field does not alter a particle's energy

$$E^2 = \vec{p}^{\,2} c^2 + m_0^2 c^4$$

$$\Rightarrow \quad E\frac{dE}{dt} = c^2 \vec{p} \cdot \frac{d\vec{p}}{dt}$$

$$\Rightarrow \quad \frac{dE}{dt} = \frac{qc^2}{E}\vec{p} \cdot \left(\vec{E} + \vec{v} \wedge \vec{B}\right) = \frac{qc^2}{E}\vec{p} \cdot \vec{E}$$

Electric fields accelerate

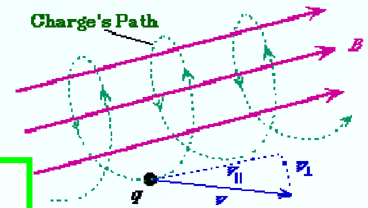


$$\vec{B} = \vec{k}B$$

Magnetic fields confine

$$\frac{d\vec{p}}{dt} = q\vec{v} \times \vec{B} \Rightarrow m\frac{d\vec{v}}{dt} = q\vec{v} \times \vec{B} \Rightarrow \frac{d\vec{v}}{dt} = \omega\vec{v} \times \vec{k}$$

$$\omega = \frac{qB}{m}$$



$$\vec{v}\frac{d\vec{v}}{dt} = \omega\vec{v}(\vec{v} \times \vec{k}) = 0$$

$$\vec{k}\frac{d\vec{v}}{dt} = \omega\vec{k}(\vec{v} \times \vec{k}) = 0$$

SR - Radiation power

When velocity and acceleration are co-linear i.e Electric fields

$$\left(\vec{\beta}\times\dot{\vec{\beta}}\right)=0$$

Liénard equation (1898)

$$P=\frac{2}{3}\frac{e^2}{4\pi\varepsilon_0 c}\gamma^6\left(\dot{\vec{\beta}}^2-\left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2\right)$$

$$\vec{p}=m_0\gamma c\vec{\beta} \qquad \frac{d\gamma}{dt}=\gamma^3\frac{\vec{u}\dot{\vec{u}}}{c^2}$$

$$\dot{\vec{p}}=m_0\gamma^3 c\dot{\vec{\beta}} \qquad \frac{d\vec{p}}{dt}=m_0\gamma^3\dot{\vec{u}}$$

When velocity and acceleration are perpendicular i.e magnetic fields

$$\dot{\beta}=\frac{c\beta^2}{\rho}$$

$$\left(\vec{\beta}\times\dot{\vec{\beta}}\right)=\vec{\beta}\cdot\dot{\vec{\beta}}$$

$$\frac{P}{dE/dt}=\frac{2}{3}\frac{r_e}{mc^2}\left(\frac{dE}{dx}\right)$$

$$P=\frac{2}{3}r_e mc^3\frac{\gamma^4\beta^4}{\rho^2}$$

5.52x10^{-15} m/MeV

$$\delta E\equiv U_0=\frac{2\pi\rho}{\beta c}P=\frac{2}{3}r_e 2\pi mc^2\frac{\gamma^4\beta^3}{\rho}$$

And shows that radiation loss is unimportant in a Linac unless the gain in energy is mc^2=0.511 MeV in a r_e distance (2.82 10^{-15} m) or 2x10^{14} MeV/m (typical gains are 10 MeV/m)

$$\left(\frac{e^2}{4\pi\varepsilon_0}\right)_{mks}=\left(e^2\right)_{gaussian}$$

$$U_0(keV)=88.5\frac{E^4(GeV)}{\rho(m)}$$

Motion under perpendicular $\perp$ constant B-field, E=0

$$\frac{d\vec{v}}{dt} = \omega\vec{v} \times \vec{k}$$

$$\frac{d^2\vec{v}}{dt^2} = \omega\frac{d\vec{v}}{dt} \times \vec{k} = \omega^2(\vec{v} \times \vec{k}) \times \vec{k} = -\omega^2\left[\vec{v} \cdot (\vec{k} \cdot \vec{k}) - \vec{k} \cdot (\vec{k} \cdot \vec{v})\right] = -\omega^2\vec{v}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \left[A\sin(\omega t) + B\cos(\omega t)\right] \quad \Rightarrow \vec{r} = \frac{v_0}{\omega}\left[-i\cos(\omega t) + j\sin(\omega t)\right] + C$$

$$r = \frac{v_0}{\omega} = \frac{v_0 m}{qB}$$

$$\omega = \frac{qB}{m}$$

Magnetic rigidity

$$Br = \frac{v_0 m}{q} = \frac{p}{q}$$

$$qBv = \frac{mv^2}{r}$$

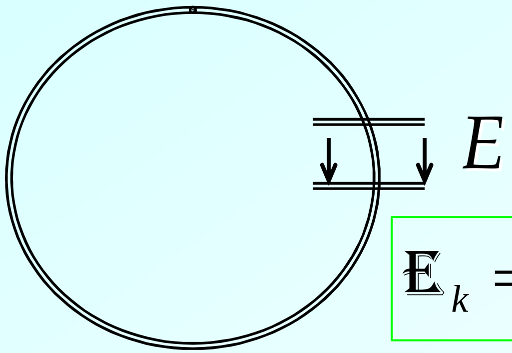
Synchrotrons

- Variation in time of *B*-field to match increase in energy and keep revolution radius constant.

$$\rho = \left| \frac{E}{qcB} \right|$$

$$E\,[GeV] \approx 0.3\, B\,[T]\ \rho\,[m]$$

- Particles can stay for a long time in orbit. Beams of particles need to be focused.
- Principle of frequency modulation and phase stability



$$E_k = qU = \oint E ds$$

But if E static $\oint E ds = \int_S \nabla \times E ds = 0$ *Stokes theorem*

Time varying E fields must be applied

Types of synchrotrons

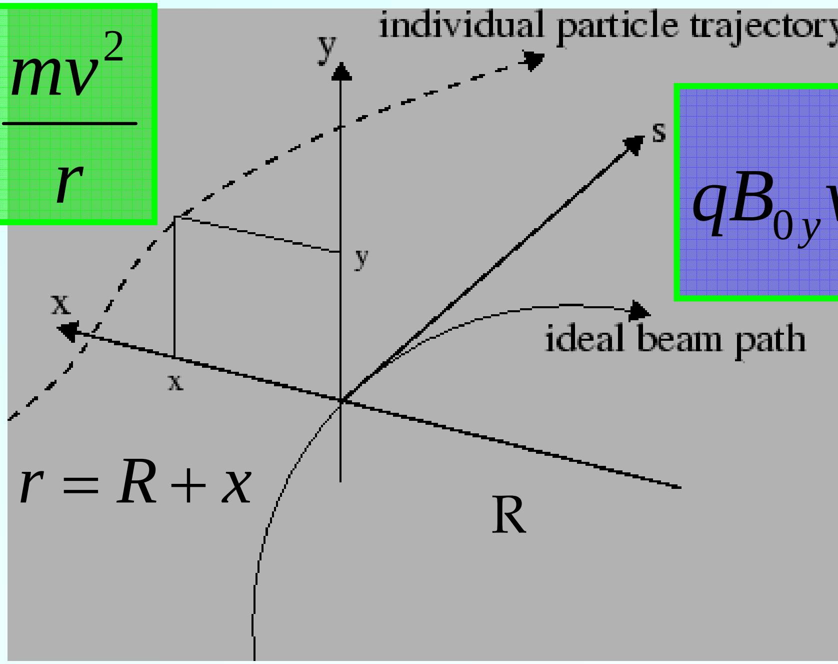
- Boosters to accelerate particles
- **Storage rings**: accumulate particles and keep circulating for long periods; used for high intensity beams to inject into more powerful machines or as synchrotron radiation sources. Storage rings can also accelerate particles (like Elettra)
- Colliders: two beams circulating in opposite directions, made to intersect; maximises energy in centre of mass frame.

Focusing in synchrotrons

- For successful acceleration focusing is needed. Misalignment errors and difficulties in perfect injection cause particles to drift vertically and radially and to hit walls.
- Particles that deviate from the ideal orbit R (also ρ) require a restoring force F_x to keep them close to the orbit

Horizontal plane

$$F_x + qB_y v = \frac{mv^2}{r}$$



$$qB_{0y} v = \frac{mv^2}{R}$$

$$B_y = B_{0y} + \frac{\partial B_y}{\partial x} x = B_{0y}\left(1 + \frac{R}{B_{0y}} \frac{\partial B_y}{\partial x} \frac{x}{R}\right)$$

$$n = -\frac{R}{B_{0y}} \frac{\partial B_y}{\partial x}$$

Field index

$$\frac{1}{x+R} \approx \frac{1}{R}\left(1 - \frac{x}{R}\right)$$

$$F_x \approx \frac{mv^2}{R}\left(1 - \frac{x}{R}\right) - qvB_{y0}\left(1 - n\frac{x}{R}\right)$$

Focusing in synchrotrons

- So for the horizontal plane we have :

$$F_x \approx -\frac{mv^2}{R}\frac{x}{R}(1-n) \Rightarrow \ddot{x} + \omega_x^2 x = 0$$

$$\frac{\omega_x}{\omega_0} \equiv tune$$

Betatron frequency

$$\omega_x = \frac{v}{R}\sqrt{(1-n)} = \omega_0\sqrt{(1-n)}$$

Stable oscillations for n<1

Difficult when increasing the field for higher energies $n = -\frac{R}{B_{0y}}\frac{\partial B_y}{\partial x}$

- Now lets us see what happens in the vertical plane

To achieve stability need a restoring force:

$$F_y = -Cy$$

Therefore a horizontal field component needed

$$B_x = -C'y$$

Lorentz

$$F_y = qvB_x$$

$$B_x = B_{0x} + \frac{\partial B_x}{\partial y} y = \frac{\partial B_y}{\partial x} y = -n \frac{B_y}{R} y$$

$$\boxed{\nabla \times B = 0 \Rightarrow \frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} = 0}$$ Maxwell (Ampere)

$$\boxed{m\ddot{y} + qvn \frac{B_{0y}}{R} y = 0 \Rightarrow m\ddot{y} = -\frac{qRB_{0y}}{v} \frac{v^2}{R^2} ny \Rightarrow m\ddot{y} = -m \frac{v^2}{R^2} ny}$$

$$\boxed{\frac{BqR}{v} = m}$$

$$\boxed{\ddot{y} + \omega_x^2 y = 0 \, with \, \omega_x = \frac{v}{R} \sqrt{n}}$$

Stable oscillations for n>0

$$n = -\frac{R}{B_{0y}} \frac{\partial B_y}{\partial x} \Rightarrow \frac{\partial B_y}{\partial x} \langle 0$$

- Combining one has: $0 < n < 1$ and the condition is obtained by shaping the magnetic poles so that the fields fulfill the condition (called the Weak Focusing Condition) involving very big magnets and big apertures.
- However increasing n strengthens the vertical focusing forces at the expense of the radial
- Later people realized that it is not necessary to simultaneously focus in both planes and this resulted in the strong focusing condition. As a result the modern synchrotron were built and magnets and apertures became much smaller.

Weak focusing synchrotrons

Simultaneously horizontal and vertical magnetic focusing. Possible in both planes if field lines bend outward due to pole shaping.

But bulky magnets scaled with energy!

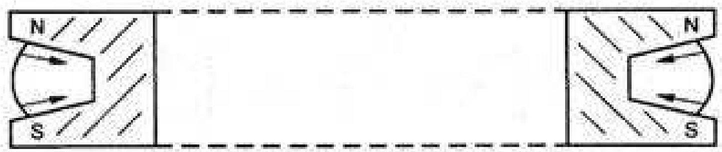


Figure 3.3. Cross section of weak focusing circular accelerator.

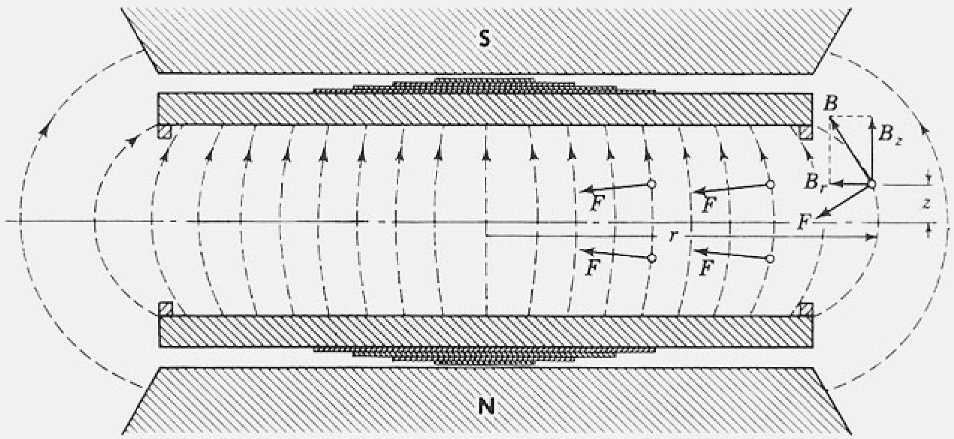


Fig. 6-7. Radially decreasing magnetic field between poles of a cyclotron magnet, showing shims for field correction.

1947 70 MeV electron synchrotron General Electric Co. first observation of SR – resulting to Nobel price.

From 1947 to 1964 many such synchrotrons were built including the 1959 Frascati electron synchrotron of 1.2 GeV

The Dubna synchrotron, the largest of them all with a radius of 28 meters and with a weight of the magnet iron of 36,000 tons

Famous Weak Focusing Synchrotrons

- There were accelerators before the Cosmotron, but this machine was the first accelerator in the world to send particles to energies in the GeV region. The Cosmotron reached its full design energy of 3.3 GeV in January 1953 with some 10^{12} protons per pulse and decommissioned 13 years later.
- It was a 23-m diameter machine, weighing 2,000 tons and composed of 288 C-shaped magnets that guided the protons in a circular path. At that energy, the protons were allowed to strike a target. The fragments of the nuclear collisions were observed in photographs of the telltale trails they left in cloud chambers, or with other detectors.

T. D. Lee, of Columbia University, and C. N. Yang, then of Brookhaven, interpreted results of particle decay experiments at Brookhaven's Cosmotron particle accelerator and discovered that the fundamental and supposedly absolute law of parity conservation had been violated.

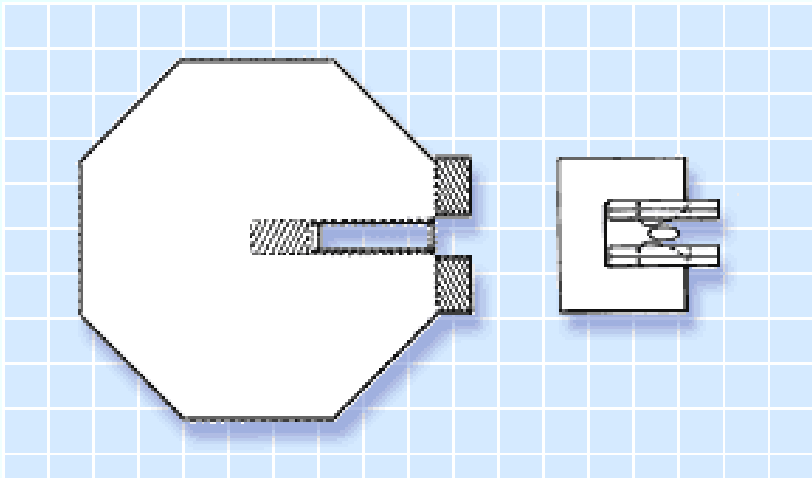
- The Bevatron (6.2GeV) was a weak-focusing synchrotron — at LBNL which began operating in 1954. The antiproton was discovered there in 1955, resulting in the 1959 Nobel Price in physics for Emilio Segrè and Owen Chamberlain. At the time it was built, there was no known way to confine a particle beam to a narrow aperture, so the beam space was about 2 feet by 3 feet in cross section. In order to create anti-protons (mass 938 MeV) in collisions with nucleons in a stationery target while conserving both energy and momentum, a beam proton energy of slightly over 5 GeV is required. The combination of beam aperture and energy required a huge, 10,000 ton iron magnet. It was finally decommissioned in 1994.

Going towards the strong focusing concept

- In ring-shaped accelerators such as the Cosmotron, particles travel through a magnetic field, which keeps them on their circular course by bending their trajectories. As a beam of particles achieves higher energies, the beam remains well focused in the vertical direction, but its trajectory becomes unstable in the horizontal direction, leading to beam loss. This could only be overcome by using more powerful (and far heavier) magnets and drastically increasing the size of the machine.

- In the Cosmotron, all the magnets were C-shaped, with the open side and the magnetic field, facing outward. The breakthrough occurred by alternating the orientation of these magnets, so some of their field gradients faced outward and some inward. Brookhaven physicists found that the net effect of alternating the field gradient was that both the vertical and horizontal focusing of protons could be made strong at the same time, allowing tight control of proton paths in the machine (right). This increased beam intensity while reducing the overall construction cost of a more powerful accelerator.

$$n >> 1$$



The first alternating-gradient synchrotron accelerated electrons to 1.5 GeV. It was built at Cornell University, Ithaca, N.Y. and was completed in 1954.

Strong focusing synchrotrons

- Magnetic field produced by several bending magnets (*dipoles*), increases linearly with momentum.

$$B\rho = \frac{p}{e} \approx \frac{E}{ce} \; so \; E\,[GeV] \approx 0.3\, B\,[T]\;\rho\,[m]$$

- Alternating horizontal focusing (vertical defocusing) and vertical focusing (horizontal defocusing) is provided by special magnets (quadrupoles).

- Practical limitations for magnetic fields => high energies only at large radius
- e.g. **LHC** ***E* = 8 TeV, *B* = 10 T, *r* = 2.7 km**
- But **Elettra** ***E*= 2 *GeV*, *B*= 1.2 *T*, *r*=5.5 *m***

- In a SF- synchrotron, the confining magnetic field comes from a system of several magnetic dipoles forming a closed arc. Dipoles are mounted apart, separated by straight sections/vacuum chambers including equipment for focusing, acceleration, injection, extraction, collimation, experimental areas, vacuum pumps.

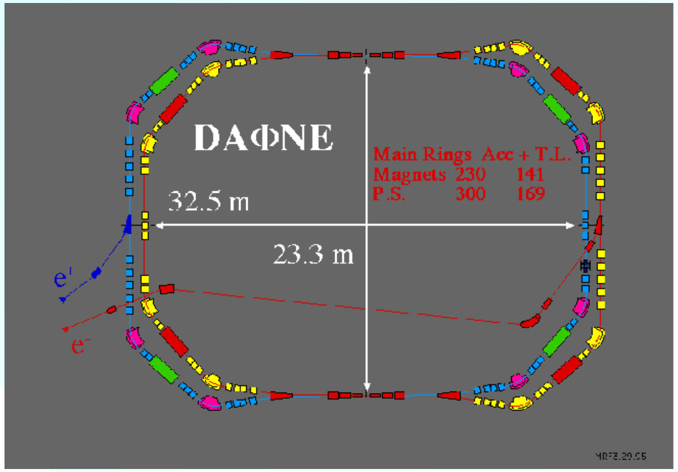
Called Rings but in general are polygons

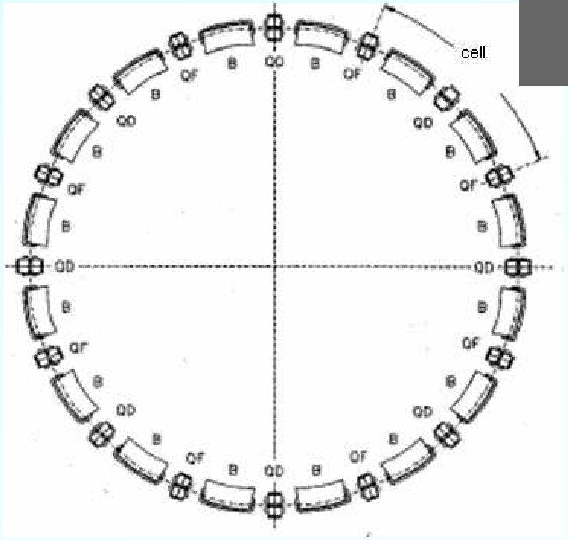
- Mean radius of ring $R > \rho$
- e.g. CERN SPS R = 1100 m, ρ = 225 m

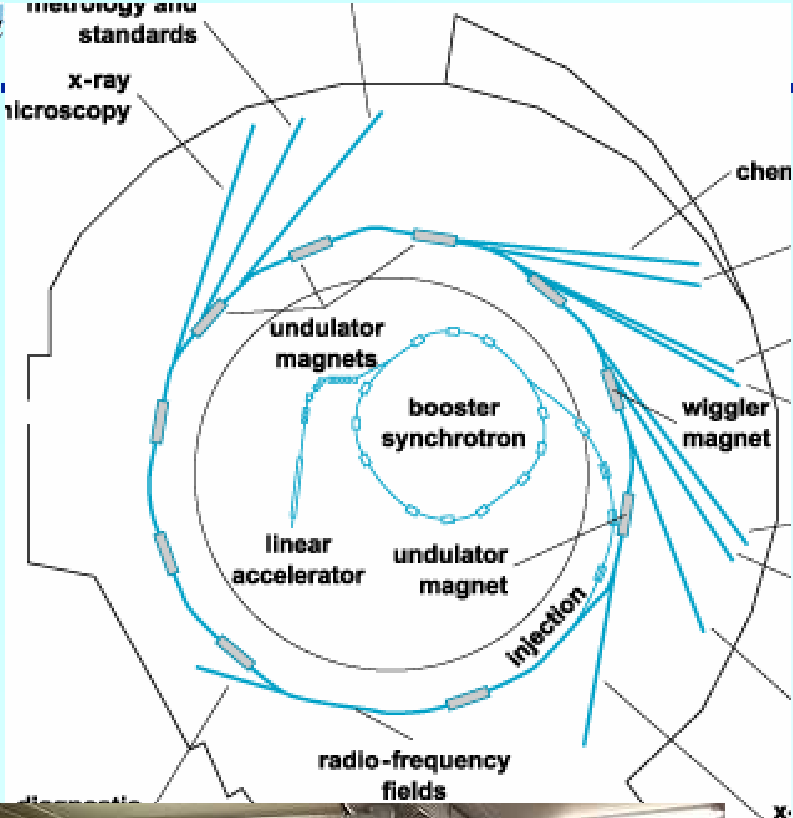
  Elettra R=42.5m , ρ = 5.5 m
- Can also have large machines with a large number of dipoles each of small bending angle.
- e.g. CERN SPS

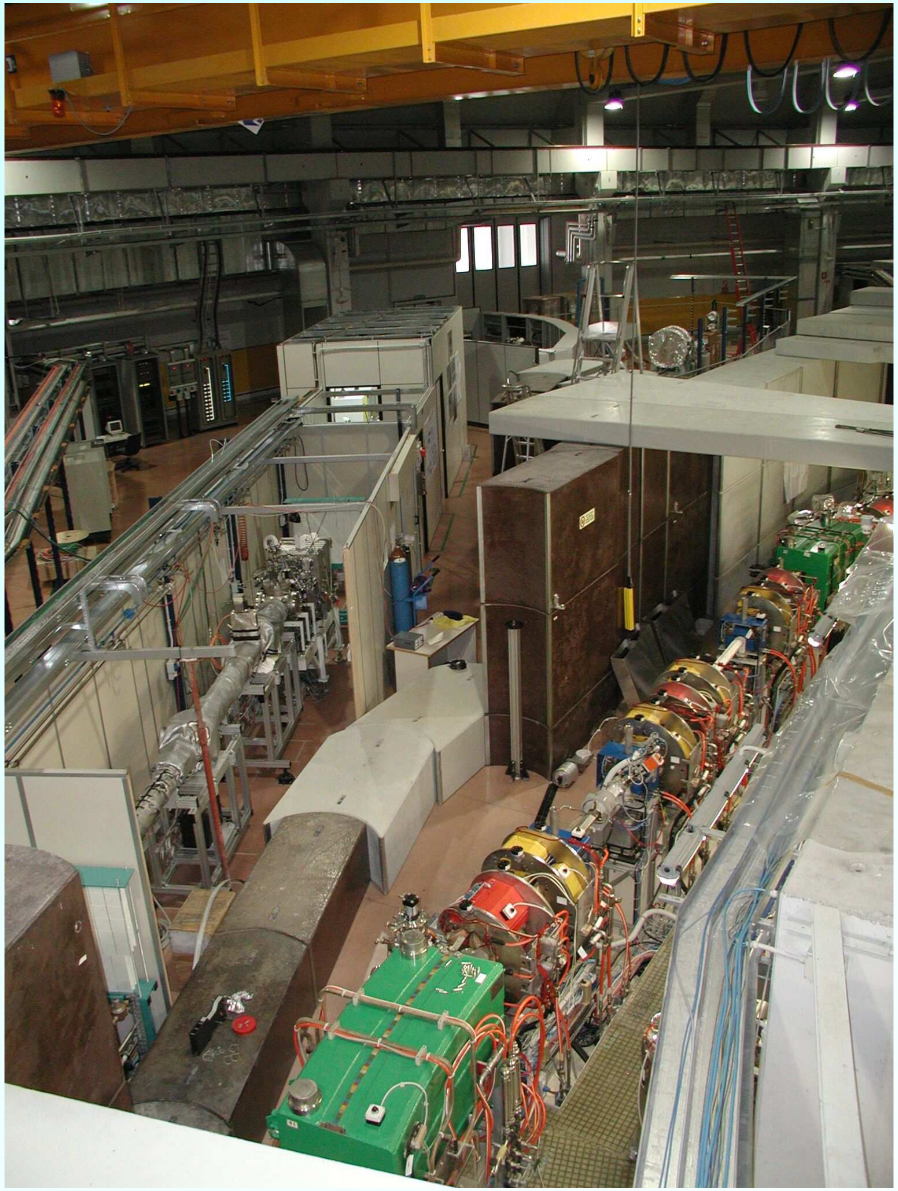
  744 dipole magnets, 6.26 m long, angle $\theta = 0.48^o$
- Elettra

  24 dipoles 1.5 m long with angle $\theta = 15^o$






standards
x-ray
undulator magnets
booster synchrotron
wiggler magnet
linear accelerator
undulator magnet
injection
radio-frequency fields



Dipole Confinement

$$B_y(x) = B_{y0} + \frac{dB_y}{dx}x + \frac{1}{2!}\frac{d^2B_y}{dx^2}x^2 + \frac{1}{3!}\frac{d^3B_y}{dx^3}x^3 + \ldots$$

$$F_x \approx \frac{mv^2}{R}(1-\frac{x}{R}) - qvB_{y0}(1-n\frac{x}{R}) \qquad qB_{0y}v = \frac{mv^2}{R}$$

$$\ddot{x} + \omega_x^2 x = 0 \Rightarrow x''v^2 + \omega_x^2 x = 0 \Rightarrow x'' + \frac{\omega_x^2}{v^2}x = 0$$

$$\boxed{\frac{\omega_x^2}{v^2} = \frac{1}{\rho^2} + \frac{1}{B\rho}\frac{\partial B_y}{\partial x} = \frac{1}{\rho^2} - k} \qquad \boxed{k = \frac{n}{\rho^2}}$$

gradient

$$\boxed{k = \frac{g}{B\rho}}$$

$$x'' + (\frac{1}{\rho^2} - k_x)x = 0$$

No vertical focusing if $k_y = 0$

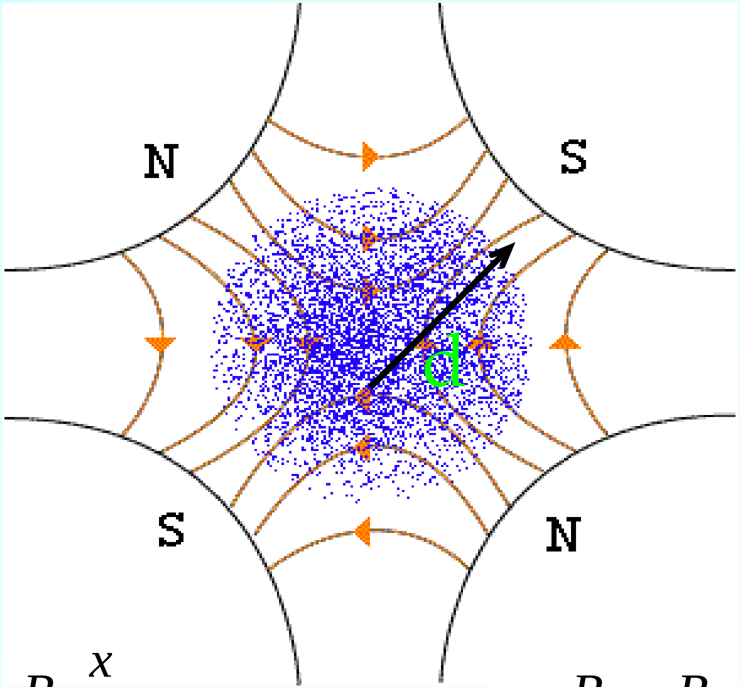
For a pure dipole K_x=0, but still there is focusing

$$\boxed{x'' = -\frac{1}{\rho^2}x}$$

$$y'' + k_y x = 0 \Rightarrow y'' = 0$$

Quadrupoles focus

- Quadrupoles focus horizontally, defocus vertically or vice versa. Forces are linearly proportional to displacement from axis.
- A succession of opposed elements enable particles to follow stable trajectories, making small (betatron) oscillations about the design orbit. Net effect is focusing!



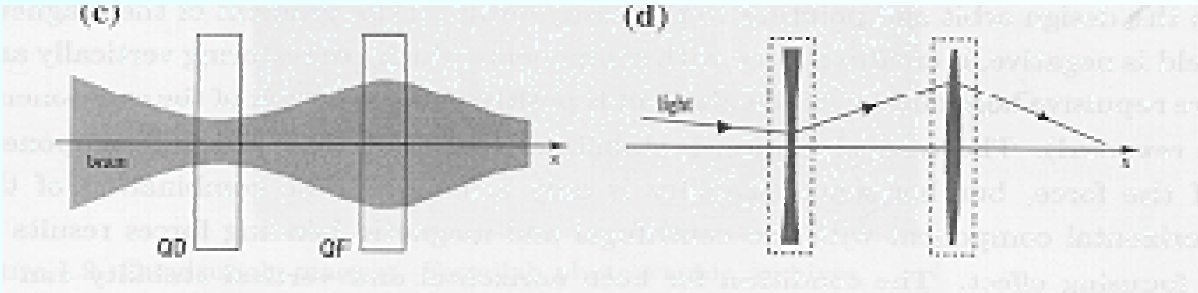
$$\ddot{x} = -\frac{qv}{m} B_y \Rightarrow \ddot{x} + \frac{qv}{m} B_0 \frac{x}{d} = 0$$

$$B_y = B_0 \frac{x}{d} \qquad B_x = B_0 \frac{y}{d}$$

$$\ddot{y} = \frac{qv}{m} B_x \Rightarrow \ddot{y} - \frac{qv}{m} B_0 \frac{y}{d} = 0$$

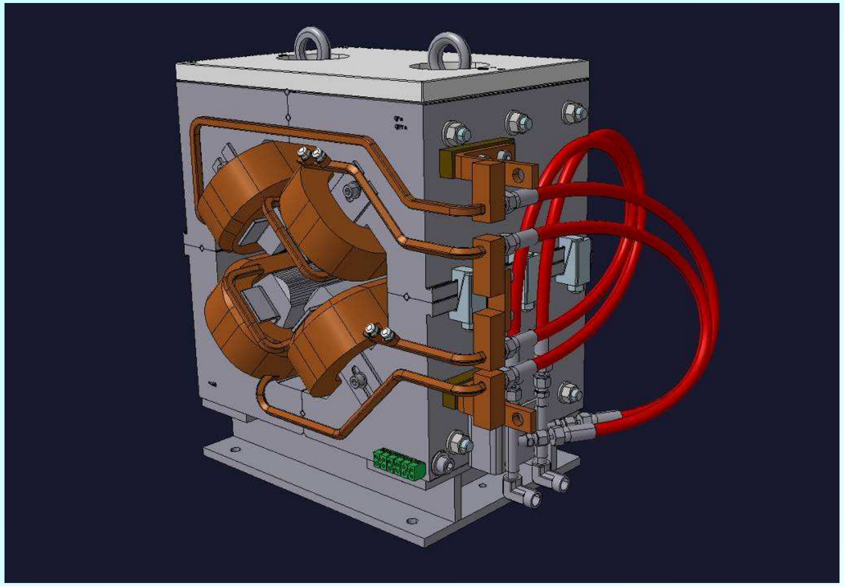
$$\boxed{x'' = -\frac{q}{p} B_0 \frac{x}{d}} \qquad \boxed{y'' = \frac{q}{p} B_0 \frac{y}{d}}$$

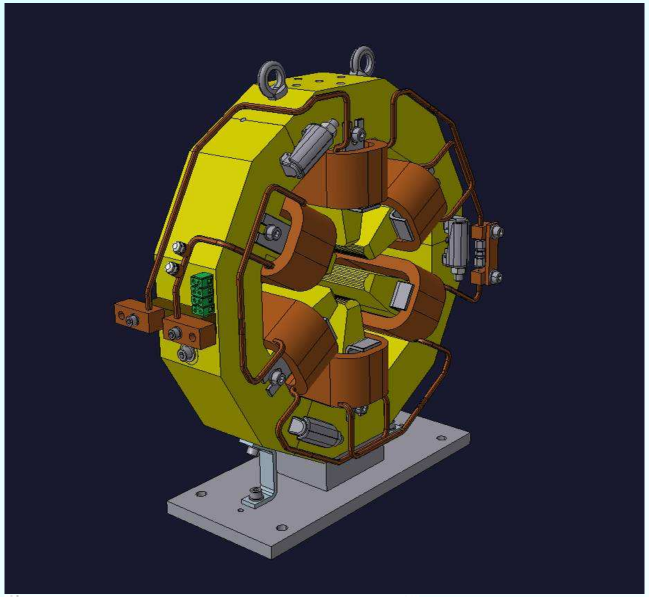
focusing defocusing

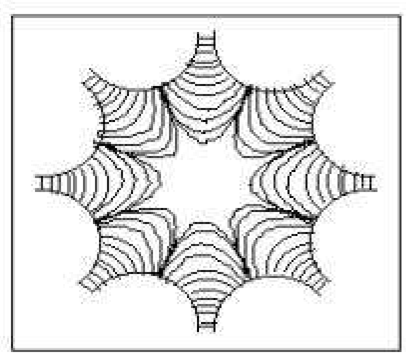


Focusing Elements

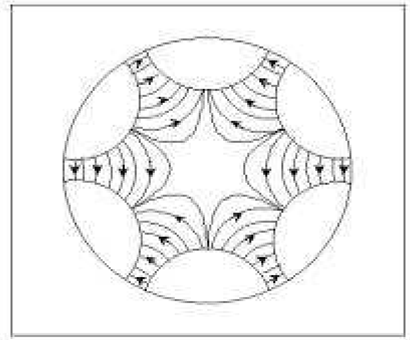
$$B_y(x) = B_{y0} + \frac{dB_y}{dx}x + \frac{1}{2!}\frac{d^2B_y}{dx^2}x^2 + \frac{1}{3!}\frac{d^3B_y}{dx^3}x^3 + \ldots$$





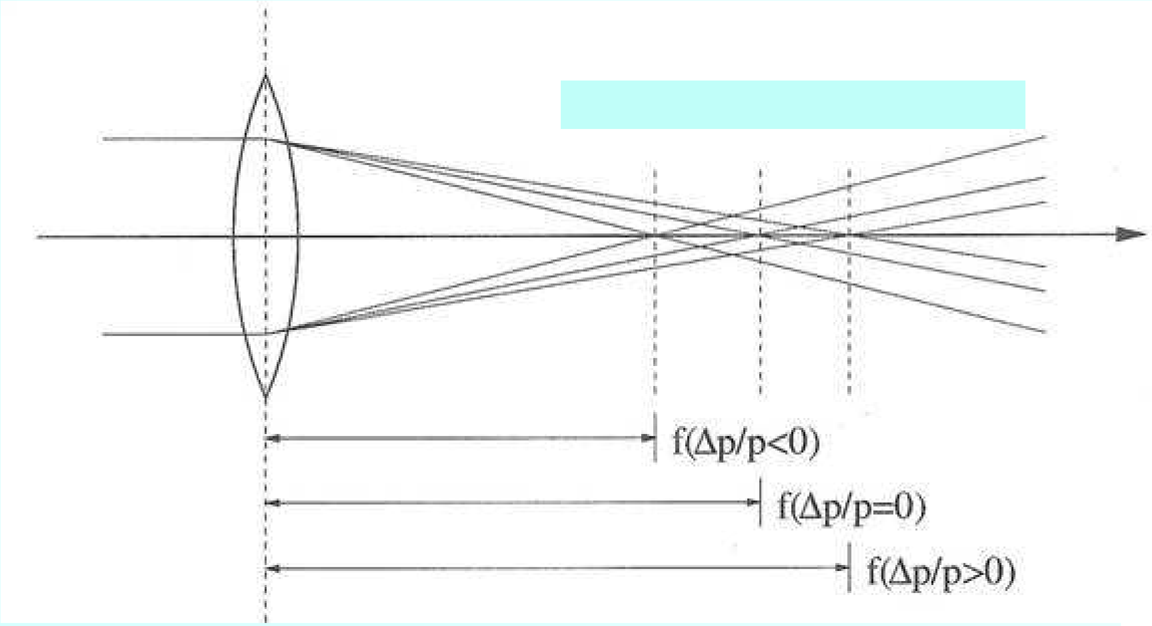


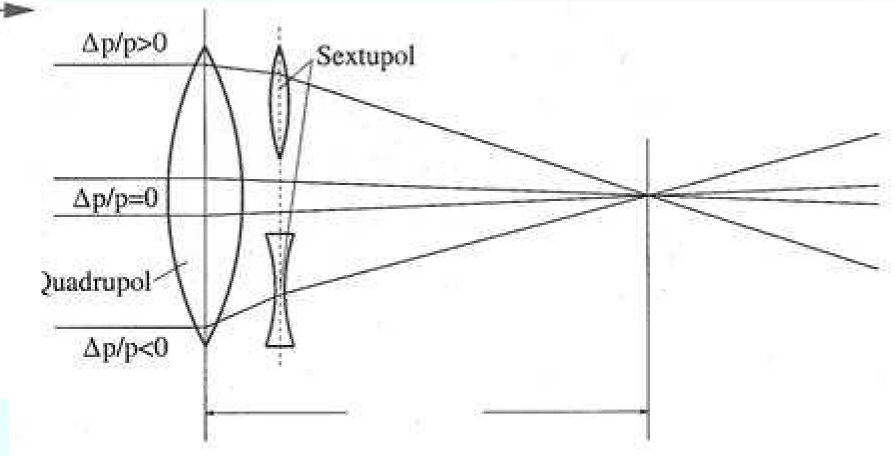
$$B(x) = \frac{b_3 x^3}{3!}$$



$$B(x) = \frac{b_2 x^2}{2!}$$

Chromatic Aberration (Focal length of the lens is dependent upon energy)

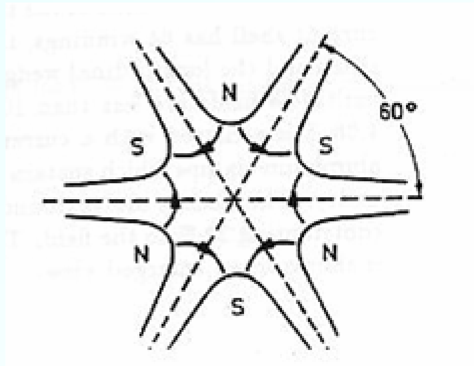


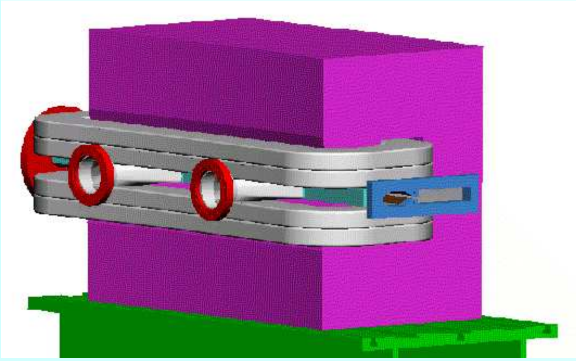


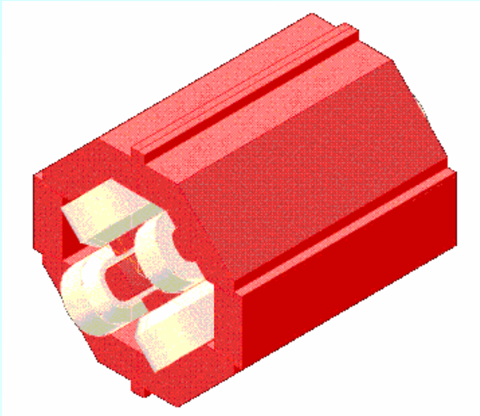
The sextupole gives a position dependent Quadrupole

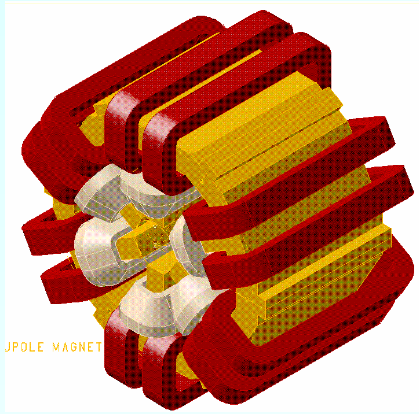
$$B_x = 2kxy$$
$$B_y = k(x^2 - y^2)$$

Sextupoles are used to correct longitudinal momentum errors.

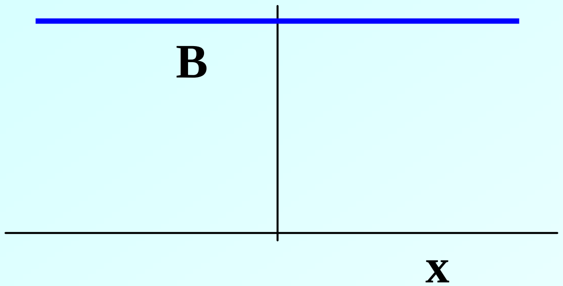




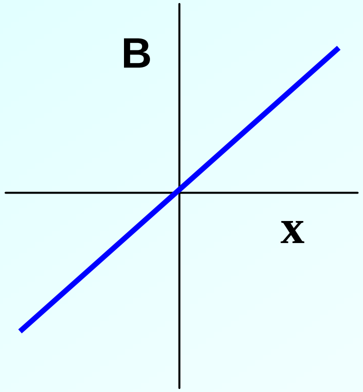




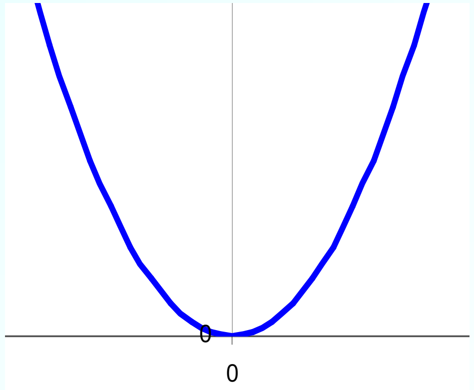
JPOLE MAGNET



B
x



B
x

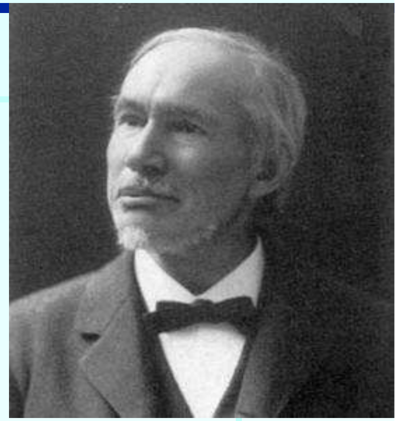


0
0

Hill's Equation

- Equation of transverse motion
  - Drift: $$x'' = 0, \quad y'' = 0$$
  - Solenoid: $$x'' + 2k\,y' + k'\,y = 0, \quad y'' - 2k\,x' - k'\,x = 0$$
  - Dipole: $$x'' + \frac{1}{\rho^2}x = 0, \quad y'' = 0$$
  - Quadrupole: $$x'' + k\,x = 0, \quad y'' - k\,y = 0$$
  - Sextupole: $$x'' + k\left(x^2 - y^2\right) = 0, \quad y'' - 2kxy = 0$$
- Hill's Equation:

$$x'' + k_x(s)\,x = 0, \quad y'' + k_y(s)\,y = 0$$



George Hill

In the late 19th century, George Hill first derived a set of equations that approximately govern the motion of a small mass in a hierarchical 3-body system.

Solutions

- The general solution of the homogeneous differential equation $u''+Ku=0$ where u is x or y is:

$for\ K>0$

$$C(s)=\cos(\sqrt{K})s \ and \ S(s)=\frac{1}{\sqrt{|K|}}\sin(\sqrt{K})s$$

$$\boxed{K=\frac{1}{\rho^2}-k}$$

Horizontal

$$\boxed{k=\frac{1}{B\rho}\frac{\partial B_y}{\partial x}}$$

$$\boxed{K=k}$$

Vertical

$for\ K<0$

$$C(s)=\cosh(\sqrt{K})s \ and \ S(s)=\frac{1}{\sqrt{|K|}}\sinh(\sqrt{K})s$$

$$\boxed{K(s)=K(s+C)}$$

Any arbitrary solution is linear combination of these principal solutions

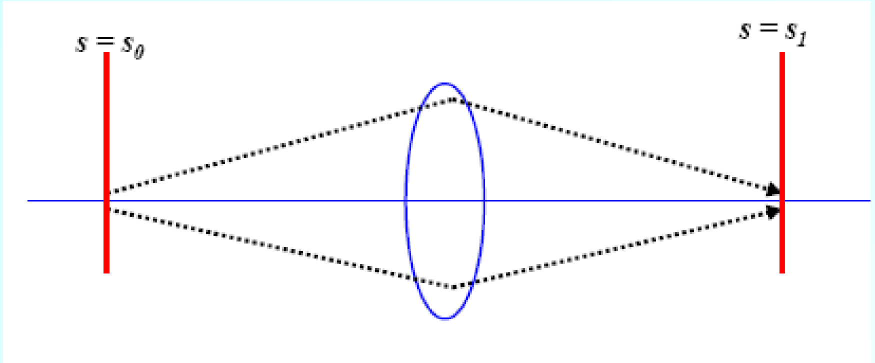
$$u(s)=C(s)u_0+S(s)u_0^{'}$$
$$u'(s)=C'(s)u_0+S'(s)u_0^{'}$$

M

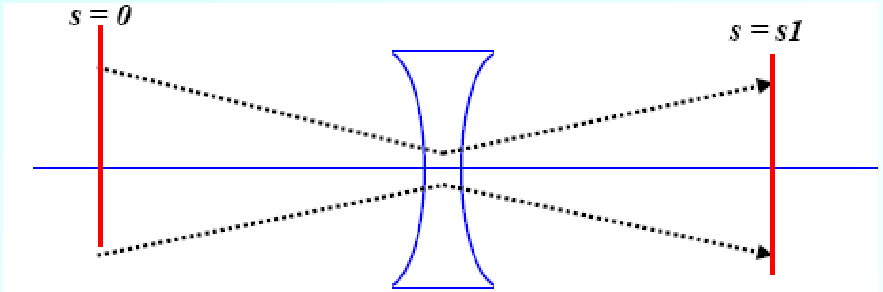
$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \begin{pmatrix} u_0 \\ u_0^{'} \end{pmatrix}$$

and for stable solutions the determinant should be 1

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|k|}l) & \frac{1}{\sqrt{|k|}}\sin(\sqrt{|k|}l) \\ -\sqrt{|k|}\sin(\sqrt{|k|}l) & \cos(\sqrt{|k|}l) \end{pmatrix}$$

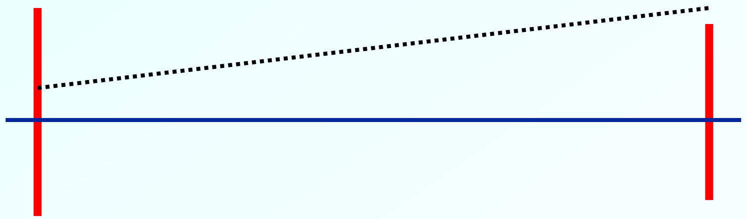


$$M_{def} = \begin{pmatrix} \cosh(\sqrt{|k|}l) & \frac{1}{\sqrt{|k|}}\sinh(\sqrt{|k|}l) \\ \sqrt{|k|}\sinh(\sqrt{|k|}l) & \cosh(\sqrt{|k|}l) \end{pmatrix}$$



$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

In a drift space of length ℓ, x' is unaltered but $x \rightarrow x+\ell x'$



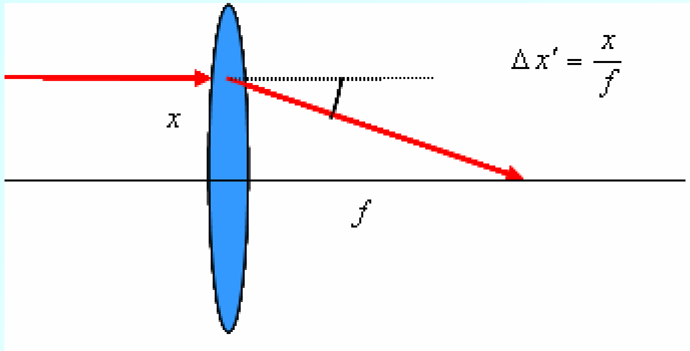
Need to focus

Thin Lens Analogy

$$M_{dip} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} \end{pmatrix}$$

If $\boxed{f \equiv \frac{1}{kl} >> l}$ *and for l -> 0 keeping kl =const.*

$$M_y = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \qquad M_x = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$



$$\begin{pmatrix} x \\ x' \end{pmatrix}_{out} = M_x \begin{pmatrix} x \\ x' \end{pmatrix}_{in}$$

In an F-drift-D system combined effect is

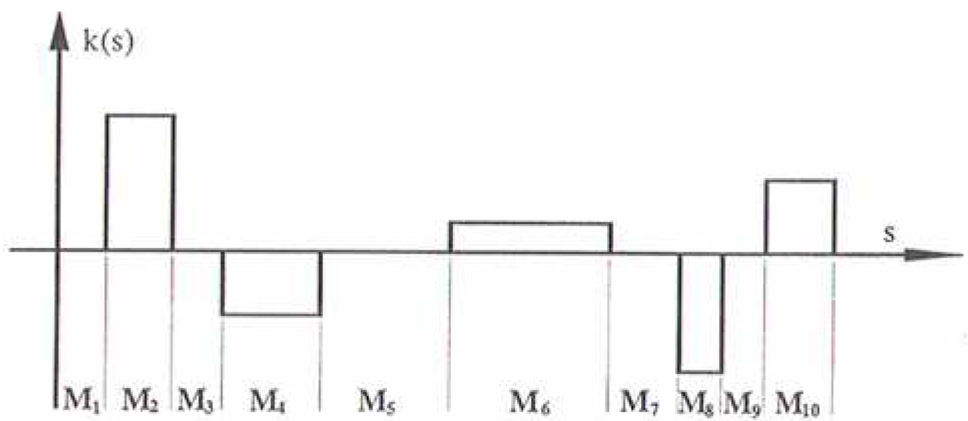
$$\begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} = \begin{pmatrix} 1 - l/f & l \\ -l/f^2 & 1 + l/f \end{pmatrix}$$

Thin lens of focal length f^2/ℓ, focusing overall, if $\ell < f$.

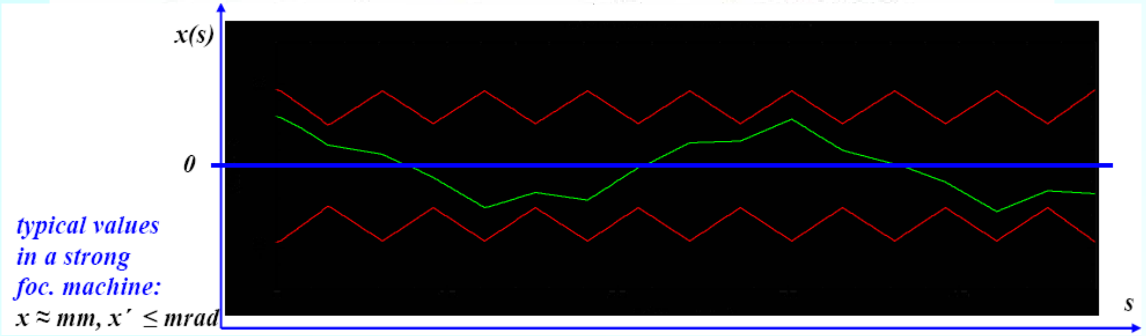
Same for D-drift-F ($f \to -f$), so system of AG lenses can focus in both planes simultaneously

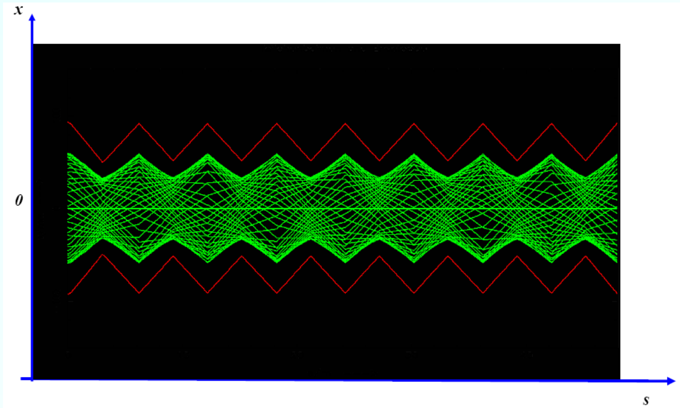
Since the lattice is a combination of elements and drifts the total outcome can be taken by matrix multiplication.

$$\mathcal{M} = \mathcal{M}_{10} \ldots \mathcal{M}_5 \, \mathcal{M}_4 \, \mathcal{M}_3 \, \mathcal{M}_2 \, \mathcal{M}_1$$



$$\begin{pmatrix} x \\ x' \end{pmatrix}_{out} = M_{10} \ldots M_3 M_2 M_1 \begin{pmatrix} x \\ x' \end{pmatrix}_{in}$$





One or many turns

Beta functions

- The Hill's equations describe in a piece wise way the beam position along the accelerator since K=0 for drifts, K=constant in quadrupoles etc.
- To overcome the problem we try the following solution:

$$u(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos(\varphi(s) + \varphi_0)$$

which is similar to the solution of an harmonic oscillator. Differentiating twice and inserting to Hill's equation we obtain the following relations:

$$\beta\varphi' = 1 \, so \, \varphi(s) = \int_0^s \frac{ds}{\beta(s)} + \varphi_0$$

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + \beta^2 k = 1$$

or

$$\beta'' + 2k\beta - 2\gamma = 0$$

$$a = -\frac{1}{2}\beta' \, and \, \gamma = (1 + a^2)/\beta$$

Tune $Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$

$$x_\beta(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos(\varphi(s)+\varphi_0)$$

$$x'_\beta(s) = -\sqrt{\varepsilon}\,\frac{\alpha}{\sqrt{\beta(s)}}\cos(\varphi(s)+\varphi_0) - \frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\sin(\varphi(s)+\varphi_0)$$

where $\beta(s)$ is the beta function,

$\alpha(s)$ is the alpha function,

$\varphi(s)$ is the betatron phase, and

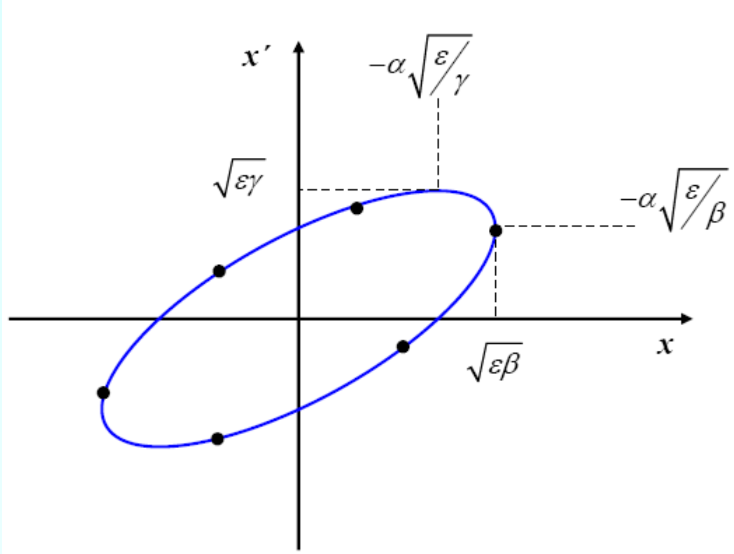
ε is an action variable

Define the Betatron or Twiss or lattice functions (Courant-Snyder parameters)

$$\boxed{\begin{aligned} \beta(s) &\equiv w^2(s) \\ \alpha(s) &\equiv -\frac{1}{2}\frac{d\beta(s)}{ds} \\ \gamma(s) &\equiv \frac{1+\alpha^2(s)}{\beta(s)} \end{aligned}}$$

Eliminating the angles $\gamma u^2 + 2\alpha uu' + \beta u'^2 = \varepsilon$ *Courant Snyder invariant*

The equation of an ellipse in phase space with area **πε**
The twiss functions have a geometric meaning

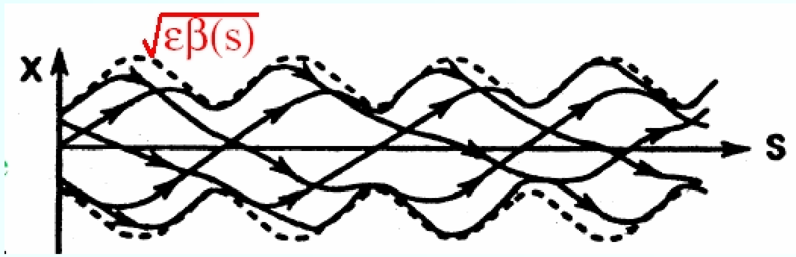


The beam size envelope is

$$E(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$

The beam divergence is

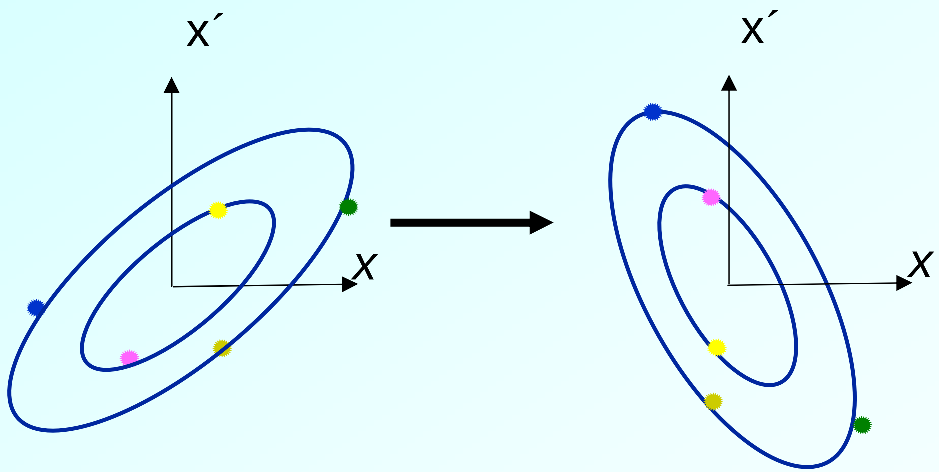
$$A(s) = \sqrt{\varepsilon}\sqrt{\gamma(s)}$$

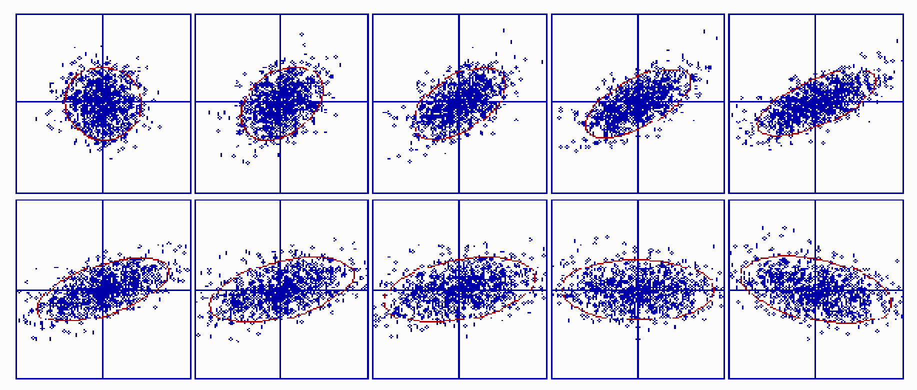


Emanuel Karantzoulis

Transverse Phase Space

- Under linear forces, any particle moves on an ellipse in phase space (x,x´).
- Ellipse rotates in magnets and shears between magnets, but its area is preserved: *Emittance*





$$\varepsilon_{rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2}$$

$$x(s)=\sqrt{\varepsilon}\sqrt{\beta_s}\left(\cos\psi_s\cos\phi-\sin\psi_s\sin\phi\right)$$

$$x'(s)=\frac{-\sqrt{\varepsilon}}{\sqrt{\beta_s}}\left[\alpha_s\cos\psi_s\cos\phi-\alpha_s\sin\psi_s\sin\phi+\sin\psi_s\cos\phi+\cos\psi_s\sin\phi\right]$$

starting at point s(0) = s_0, where we put Ψ(0) = 0

$$\left.\begin{aligned}\cos\phi&=\frac{x_0}{\sqrt{\varepsilon\beta_0}}\quad,\\ \sin\phi&=-\frac{1}{\sqrt{\varepsilon}}\left(x_0'\sqrt{\beta_0}+\frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right)\end{aligned}\right\}$$

inserting above …

$$x(s)=\sqrt{\frac{\beta_s}{\beta_0}}\left\{\cos\psi_s+\alpha_0\sin\psi_s\right\}x_0+\left\{\sqrt{\beta_s\beta_0}\sin\psi_s\right\}x_0'$$

$$x'(s)=\frac{1}{\sqrt{\beta_s\beta_0}}\left\{(\alpha_0-\alpha_s)\cos\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s\right\}x_0+\sqrt{\frac{\beta_0}{\beta_s}}\left\{\cos\psi_s-\alpha_s\sin\psi_s\right\}x_0'$$

$$\begin{pmatrix}x\\x'\end{pmatrix}_s=M\begin{pmatrix}x\\x'\end{pmatrix}_0$$

$$M=\begin{pmatrix}\sqrt{\dfrac{\beta_s}{\beta_0}}\left(\cos\psi_s+\alpha_0\sin\psi_s\right) & \sqrt{\beta_s\beta_0}\sin\psi_s\\ \dfrac{(\alpha_0-\alpha_s)\cos\psi_s-(1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\dfrac{\beta_0}{\beta s}}\left(\cos\psi_s-\alpha_s\sin\psi_s\right)\end{pmatrix}$$

Transport of the twiss parameters in terms of the transfer matrix elements

$$\begin{pmatrix} \beta \\ a \\ \gamma \end{pmatrix} = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & (S'C+SC') & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ a_0 \\ \gamma_0 \end{pmatrix}$$

To propagate particle coordinates along the lattice …

$$R_{oneturn} = \begin{pmatrix} \cos\varphi + a\sin\varphi & \beta\sin\varphi \\ -\gamma\sin\varphi & \cos\varphi - a\sin\varphi \end{pmatrix}$$

betatron tune, $Q = \phi/(2*\pi)$

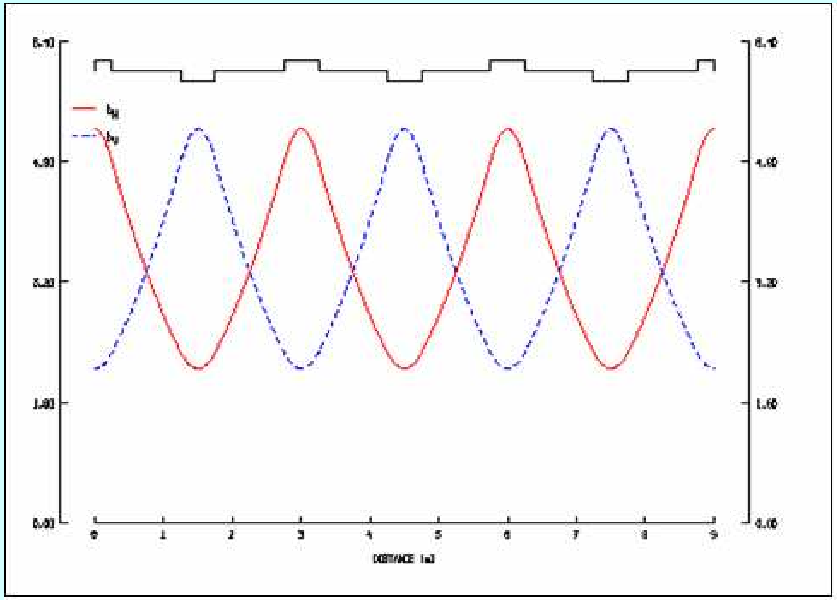
Stable solutions (emittance conservation) imposes

$$\boxed{|R| = 1}$$

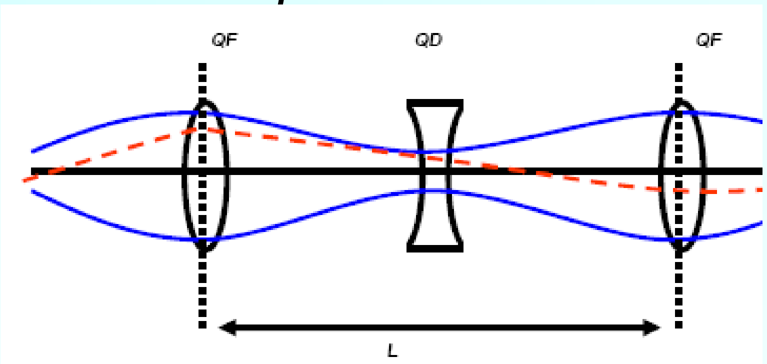
For long term stability
ϕ is real

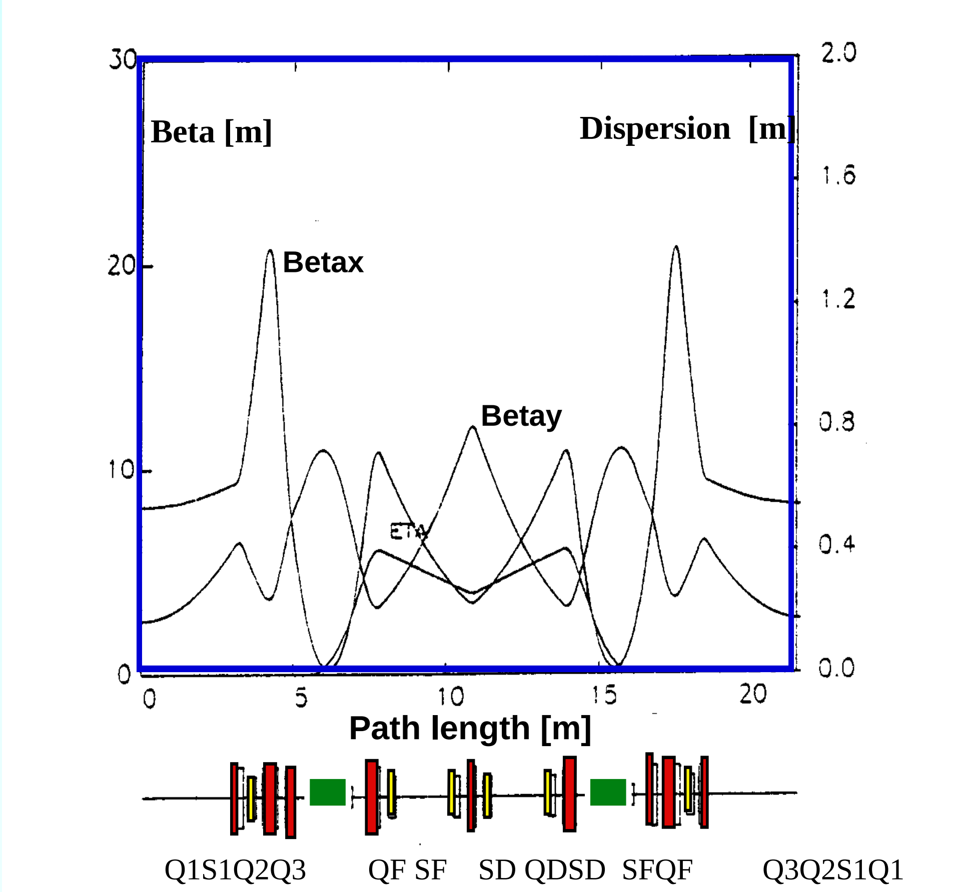
$$\boxed{Tr|R| = 2|\cos\varphi| < 2}$$

Examples of beta functions



Matched beam oscillations in a simple FODO cell





Elettra's double bend achromat

$$M_{fodo} = M_{qf} M_{ld} M_{qd} M_{ld} M_{qf}$$

$$M_{qf} = \begin{pmatrix} \cos(\sqrt{|k|}l) & \frac{1}{\sqrt{|k|}}\sin(\sqrt{|k|}l) \\ -\sqrt{|k|}\sin(\sqrt{|k|}l) & \cos(\sqrt{|k|}l) \end{pmatrix}$$

k=0.541m^{-2},l$_q$=0.5 m, l$_d$=2. 5 m

$$M_{FoDo} = \begin{pmatrix} 0.707 & 8.206 \\ -0.061 & 0.707 \end{pmatrix}$$

$$M_{qd} = \begin{pmatrix} \cosh(\sqrt{|k|}l) & \frac{1}{\sqrt{|k|}}\sinh(\sqrt{|k|}l) \\ \sqrt{|k|}\sinh(\sqrt{|k|}l) & \cosh(\sqrt{|k|}l) \end{pmatrix}$$

$$M_{drift} = \begin{pmatrix} 1 & l_d \\ 0 & 1 \end{pmatrix}$$

Is motion stable? Trace(M)=1.415 < 2

Phase advance per cell

$$R_{oneturn} = \begin{pmatrix} \cos\varphi + a\sin\varphi & \beta\sin\varphi \\ -\gamma\sin\varphi & \cos\varphi - a\sin\varphi \end{pmatrix} \Rightarrow \cos\varphi = \frac{1}{2}Tr(R) = 0.707$$

ϕ=45

Beta function: $\beta = \dfrac{M_{1,2}}{\sin\varphi} = 11.61$

Insertions-drifts

What happens to the beam parameters in a drift if we don't focus?

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \begin{pmatrix} u_0 \\ u_0' \end{pmatrix} \qquad M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} \beta \\ a \\ \gamma \end{pmatrix} = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & (S'C+SC') & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ a_0 \\ \gamma_0 \end{pmatrix}$$

$$\beta(s) = \beta_0 - 2\alpha_0 l + \gamma_0 l^2$$
$$\alpha(s) = \alpha_0 - \gamma_0 l$$
$$\gamma(s) = \gamma_0$$

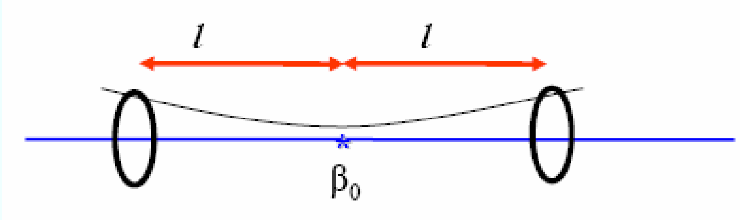
Assuming that we are in as symmetry point in the middle of the drift i.e. a_0=0

$$\boxed{\beta(s) = \beta_0 + \frac{l^2}{\beta_0}}$$

A bad consequence of phase space conservation. One can only optimize:

$$\frac{d\hat{\beta}(s)}{d\beta_0} = 1 - \frac{l^2}{\beta_0^{\,2}} = 0 \Rightarrow \beta_0 = l \text{ and } \hat{\beta} = 2\beta_0$$

Important for Light sources

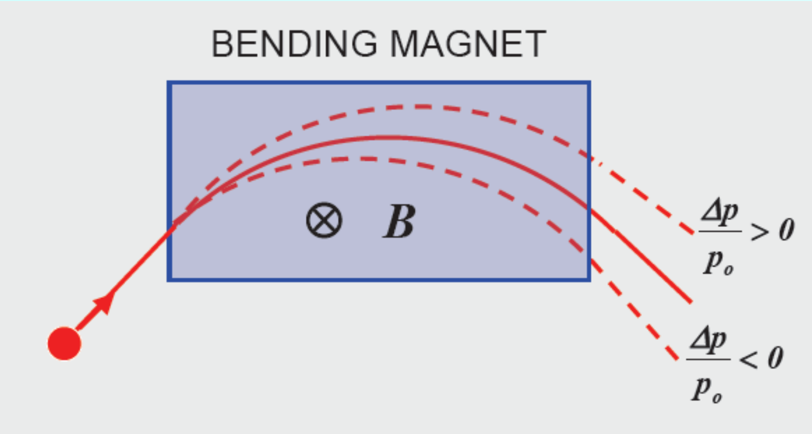


Dispersion

Up to now we have assumed $\Delta p/p=0$ but reality is different since not all particles see the same accelerating voltage. For example at Elettra $\Delta p/p=7\times10^{-4}$ but usually in SR in general is larger.

$$mv^2\frac{d^2x}{ds^2}=\frac{mv^2}{\rho}(1-\frac{x}{\rho})-qvB_{y0}(1-n\frac{x}{\rho})$$

$$x''=\frac{1}{\rho}(1-\frac{x}{\rho})-\frac{qB_{y0}}{mv}(1-n\frac{x}{\rho})$$

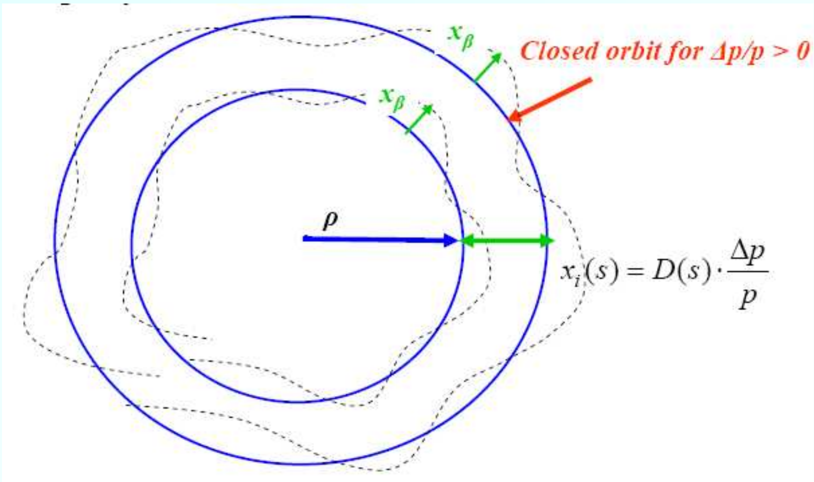


$$mv=p_0+\Delta p$$

$$x''+(\frac{1}{\rho^2}-k)x=\frac{\Delta p}{p}\frac{1}{\rho}$$

$$x(s)=x_\beta(s)+x_d(s)$$

Particles with different energies are moving on different orbits in a bending magnet -> dispersion orbit



$$D(s)=\frac{x_d(s)}{\Delta p/p}$$

Dispersion and Momentum compaction – Transition

$$\begin{pmatrix} x \\ x' \\ \frac{\Delta p}{p} \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \frac{\Delta p}{p} \end{pmatrix}_0$$

$$D(s) = S(s)\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})d\tilde{s} - C(s)\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})d\tilde{s}$$

$$M_{dip} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} \end{pmatrix}$$

In a bending the dispersion is as

$$D(s) = \rho(1 - \cos\frac{s}{\rho})$$

$$D'(s) = \sin\frac{s}{\rho}$$

The momentum compaction factor is measure for the change in equilibrium radius with momentum.

$$\alpha = \frac{\Delta R/R_0}{\Delta p/p_0} \qquad \alpha = \frac{1}{C_0}\oint \frac{D(s)}{\rho} ds$$

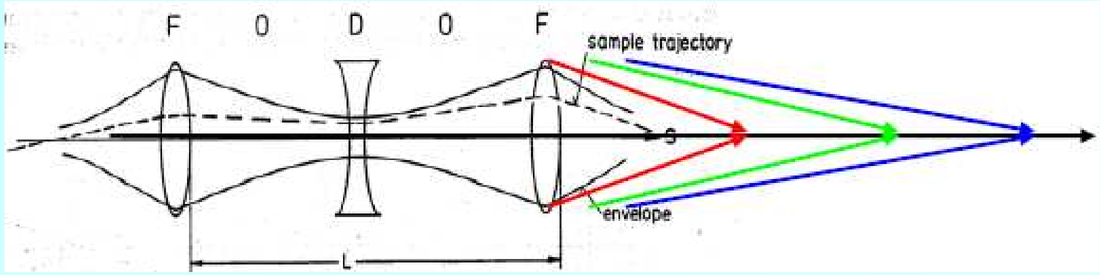
$$\boxed{\frac{\Delta R}{R_0} = \alpha\left(\frac{\Delta p}{p_0} - \frac{\Delta B}{B_0}\right)} \qquad \boxed{\frac{\Delta p}{p_0} = \frac{1}{\alpha}\frac{\Delta R}{R_0} + \frac{\Delta B}{B_0}}$$

$$\frac{\Delta T}{T_0} = \frac{\Delta R}{R_0} - \frac{\Delta v}{v_0} = \left(\alpha - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p_0} = \eta\frac{\Delta p}{p_0}$$

Transition Energy

$$\eta = \alpha - \frac{1}{\gamma^2} = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$$

Chromaticity



Higher energy

Correct

Lower energy

$$k=\frac{eg}{p}=\frac{eg}{p_0+\Delta p}\approx\frac{eg}{p_0}\left(1-\frac{\Delta p}{p_0}\right)=k_0+\Delta k$$

$$\Delta k=-\frac{\Delta p}{p_0}k_0$$

…which acts like quadrupole error and leads to a tune spread

$$\Delta Q=\int_s^{s+l}\frac{\Delta k(s)\beta(s)}{4\pi}ds\approx\frac{\Delta k l_{quad}\bar{\beta}}{4\pi}=-\frac{1}{4\pi}\frac{\Delta p}{p_0}k_0 l_{quad}\bar{\beta}$$

$$\Delta Q=\frac{\Delta p}{p_0}\xi$$

Elettra -42 h

and -14 v

Chromaticity is created by the lattice itself

$$\xi=-\frac{1}{4\pi}*\oint k(s)\beta(s)ds$$

Use sextupoles to correct

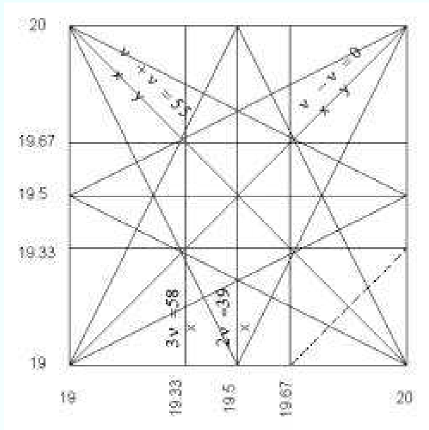
Resonances in Circular Accelerators

- Circular machines are very sensitive to field errors or misalignments since the particles traverse the focusing lattice many times. Resonant-type instabilities occur when the errors or misalignments are encountered at the same phase of the betatron oscillations during each revolution (i.e., whenever there is an integral relationship between betatron frequency and orbital frequency). A realistic error field $\Delta B_y(x,s)$ must be added to the ideal field configuration

$$\frac{q}{p}B_y(x,s) = \frac{1}{R(s)} + \kappa_x(s)x(s) + \frac{q}{p}\Delta B_y$$

$$\theta = s/R$$

$$x''(s) + \kappa_x(s)x(s) = \frac{1}{R}\frac{\Delta B_y(x,s)}{B_{y0}}$$



$$\frac{d^2x}{d\theta^2} + Q_x^2 x = a_0 + a_1 x + a_2 x^2 + b_1 xy + \ldots$$

$$mQ_x + nQ_y = p$$

Here **m,n,p** are integers and

$$|m| + |n| = l$$

Is the order of resonance

Dipole field errors : $Q = p =$ integer

Quadrupole filed errors : $Q = p/2$

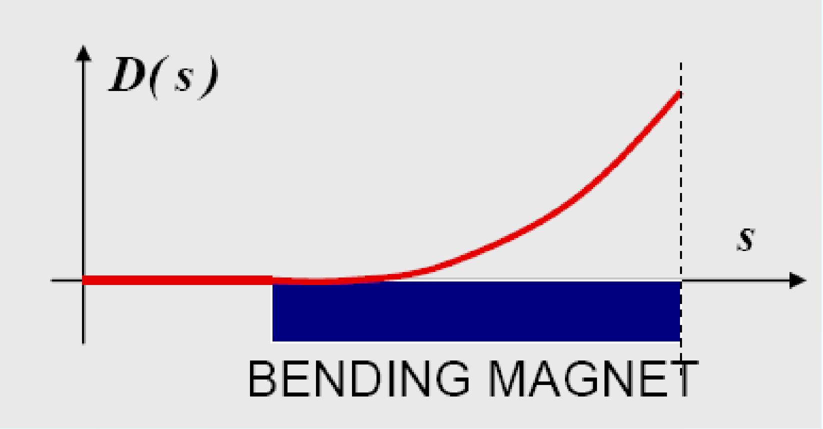
Optimizing the brilliance

$$B = \frac{F}{(4\pi)^2 \kappa\varepsilon^2}$$

$$\varepsilon_x = C_q \frac{\gamma^2}{J_x} \frac{\langle H \rangle_{bend}}{\rho}$$

$$H = D_x^2 + (D_x a_x + D_x' \beta_x)^2$$

One possibility minimize H-> dispersion ***(example bending magnet)***



$$< H >_{bend} = \frac{1}{L}\int_0^L H(s)ds \ -> \min$$

$$D(s) = \rho\left(1 - \cos\frac{s}{\rho}\right)$$

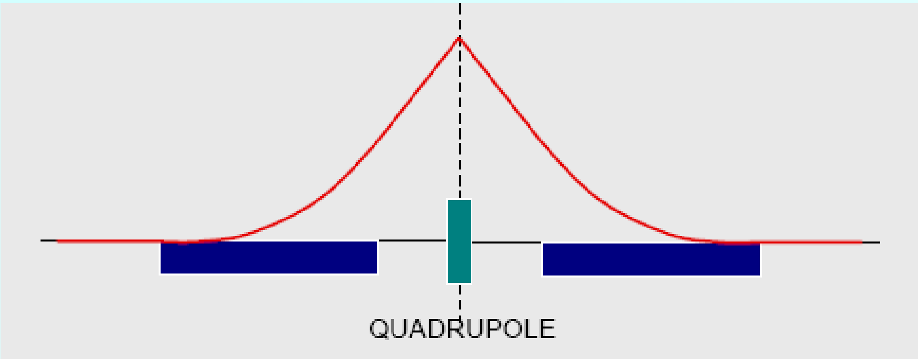
$$D'(s) = \sin\frac{s}{\rho}$$

$$< H(s) >= \frac{L^2}{\rho^2}\left[\frac{1}{3}\beta_0 - \frac{1}{4}\alpha_0 L + \frac{1}{20}\gamma_0 L^2\right]$$

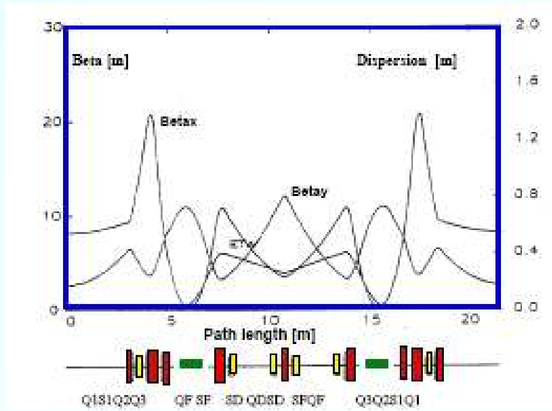
Solving for $\frac{\partial H}{\partial \beta_0} = 0$ *and* $\frac{\partial H}{\partial a_0} = 0$ Leads to $\beta_0 = 2L\sqrt{\frac{3}{5}}$ $a_0 = \sqrt{15}$

$$\varepsilon_x = C_q \frac{\gamma^2}{J_x} K \left(\frac{L}{\rho} \right)^3 = C_q \frac{\gamma^2}{J_x} K \phi^3$$

<u>*K=0.065*</u>



This is an achromat of 2 bending magnets <u>*called DBA*</u>



At least two quads are needed to match a=0 and D′=0 in the symetry point

Ring Concepts-longitudinal dynamics

$$\tau = \frac{2\pi R}{v} \approx \frac{L}{c}$$

$$\frac{\omega}{2\pi} = \frac{1}{\tau} \approx \frac{c}{L}$$

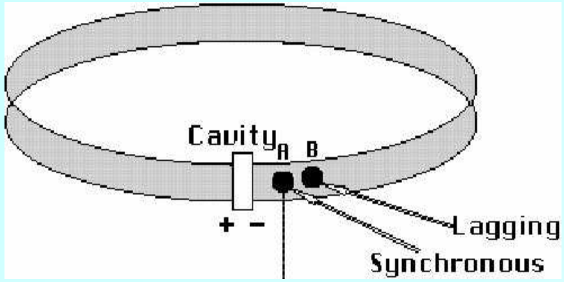
$$\omega_{rf} = h\omega \approx \frac{hc}{L}$$

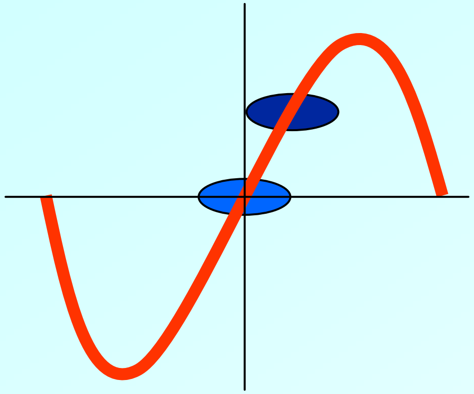
$$\rho B = \frac{p}{q}$$

Magnetic rigidity

- Important concepts in rings:
  - Revolution period τ
  - Revolution frequency ω
- RF cavities with fields oscillating at a multiple of the revolution frequency. $\omega_{rf}=h\omega_0$
  - h is the harmonic number=> how many long. places
- Energy increase ΔE when particles pass RF cavities $\Rightarrow$ can increase energy only so far as can increase B-field in dipoles to keep constant ρ.

Longitudinal motion - effects of the RF Cavity





- Cavity set up so that particle at the centre of bunch, called the synchronous particle, acquires just the right amount of energy.
- Particles see voltage $V_0 \sin 2\pi\omega_{rf} t = V_0 \sin \varphi(t)$
- In case of no acceleration, synchronous particle has $\varphi_s = 0$
  - Particles arriving early see $\varphi < \varphi_s$
  - Particles arriving late see $\varphi > \varphi_s$
  - energy of those in advance is decreased relative to the synchronous particle and vice versa.
- To accelerate, make $0 < \varphi_s < \pi$ so that synchronous particle gains energy

$$\Delta E_s = qV_0 \sin \varphi_s$$

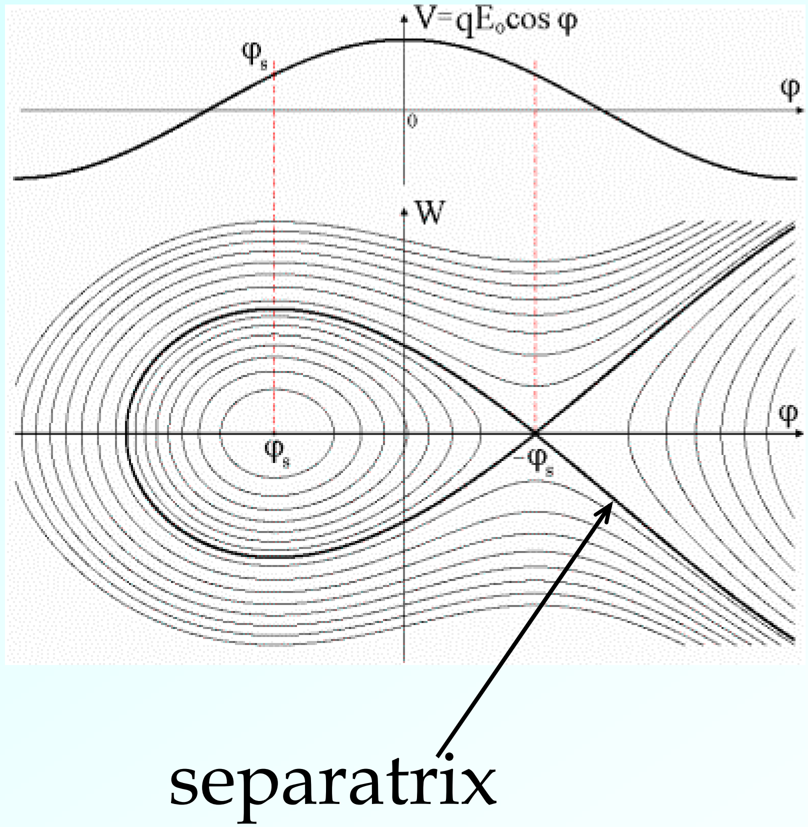
$$\frac{dE}{dt} = qV \sin(\varphi_s + \Delta\varphi) \quad \Delta E = E - E_s$$

$$\frac{d\Delta E}{dt} = qVf_0 (\sin(\varphi_s + \Delta\varphi) - \sin\varphi_s) \approx \Delta\varphi\, qVf_0 \cos\varphi_s$$

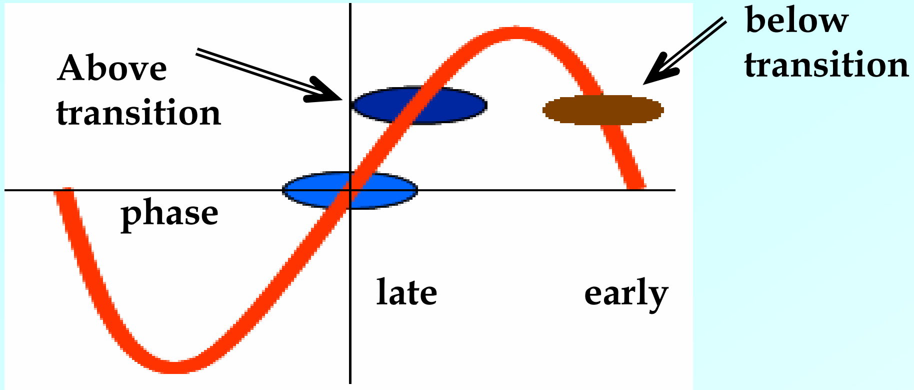
$$\Delta\varphi = -\omega_{rf} \Delta t \Rightarrow \frac{d\Delta\varphi}{dt} = -\omega_{rf} \frac{\Delta t}{T_0} = -h\omega_0 \frac{\Delta C}{C_0} = -h\omega_0 \alpha \frac{\Delta E}{E_s}$$

Limits of Stability

- Phase space is a useful for understanding the behaviour of a particle beam.
- Longitudinally, not all particles are stable. There is a limit to the stable region (the separatrix or "bucket") and, at high intensity, it is important to design the machine so that all particles are confined within this region and are "trapped".
- Similar notion exists for the transverse plane – dynamic acceptance



Phase stability and bunching



Going to difference equations:

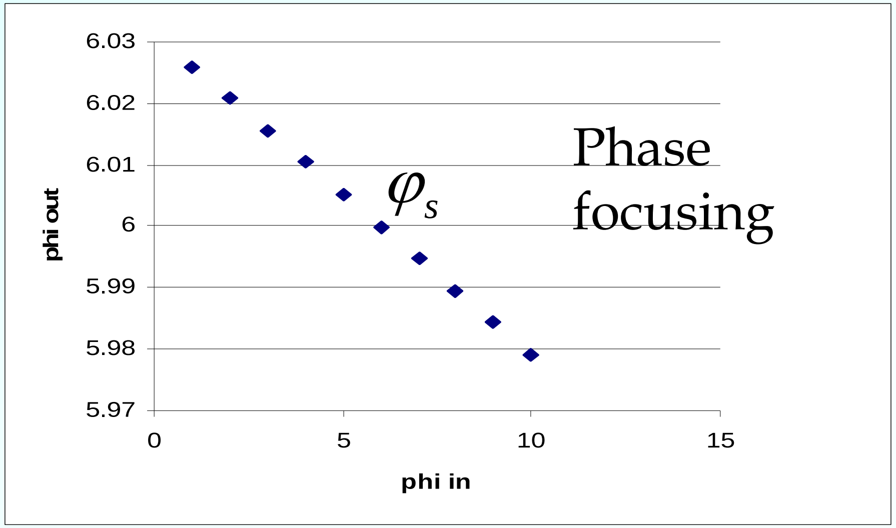
$$\Delta E = qV \approx \Delta\varphi\, qV \cos\varphi_s$$

$$\Delta\varphi \approx -2\pi h\alpha \frac{\Delta E}{E_s}$$

$$\varphi_{out} \approx \varphi_s - 2\pi h\alpha \frac{qV \cos\varphi_s}{E_s}(\varphi_{in} - \varphi_s)$$

Below transition the energy gain increases mainly the velocity of the particle which then arrives earlier

Above transition the energy increases mainly the mass of the particle which enlarges its orbit radius and therefore arrives later



Other issues… too many

- Lifetime (scattering theory)

  Beam particles get lost due to various reasons:

  quantum excitations due to SR

  beam – rest gas scattering

  beam-beam scattering

- Collective effects, beam instabilities (perturbation theory, Vlasov equations, Fokker Plank equations)

  Beam electromagnetically interacts with environment that feed backs on itself or other beam….

And many other issues that can not be covered here…..

A good ref : Introduction to Accelerator Physics (CAS CERN school) Zakopane, Poland, 1-13 October 2006

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