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Introductory School on Gauge Theory/Gravity Correspondence

19 - 30 May 2008

Holographic QCD

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Holographic QCD

- Introductory school on gauge theory/gravity correspondence, Trieste 2008

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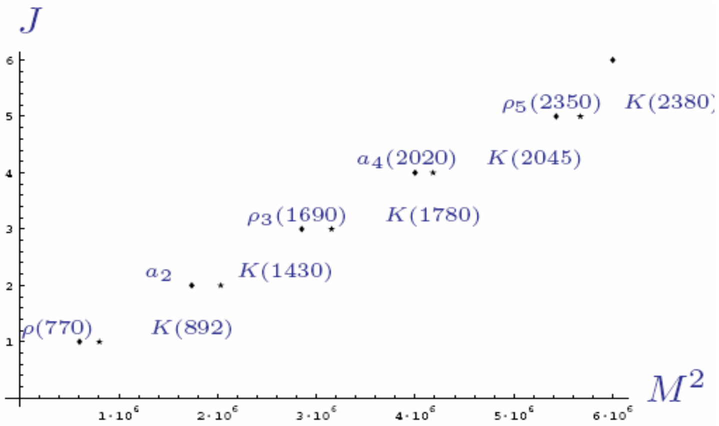
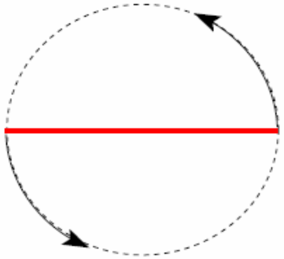
Tel Aviv university

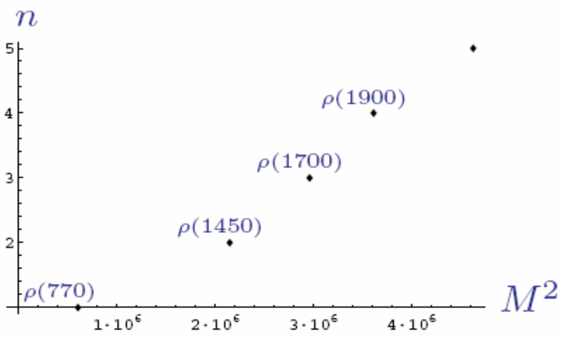
``Old” stringy phenomena in hadron physics

- Quark/anti-quark potentials look stringy too.
  We see Regge trajectories
  both in $J \sim E^2$





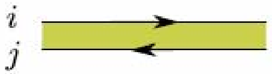
as well as $n \sim E^2$.



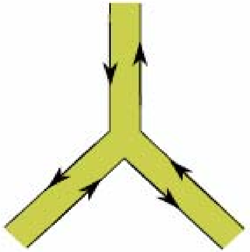
- Should we start at strong coupling ?

‘’t Hooft large N expansion and string worldsheet

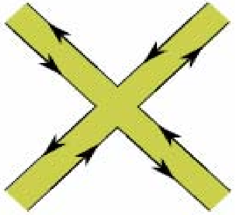
- Organisation of the power series



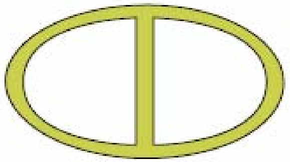
$\sim g^2_{\rm YM}$



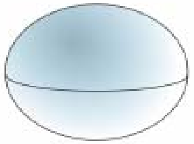
$\sim \dfrac{1}{g^2_{\rm YM}}$

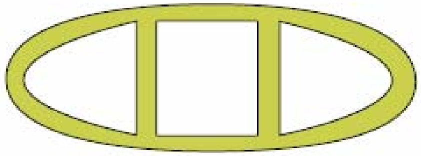


$\sim \dfrac{1}{g^2_{\rm YM}}$

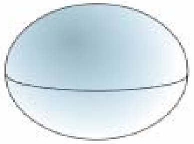


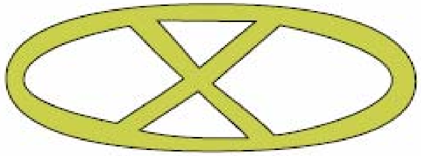
$\sim (g^2_{\rm YM})^{3-2}\, N_c^3 = \lambda\, N_c^2$



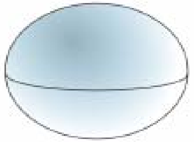


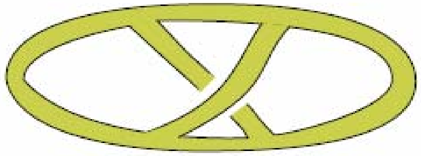
$\sim (g^2_{\rm YM})^{6-4}\, N_c^4 = \lambda^2\, N_c^2$



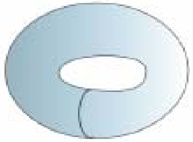


$\sim (g^2_{\rm YM})^{8-5}\, N_c^5 = \lambda^3\, N_c^2$





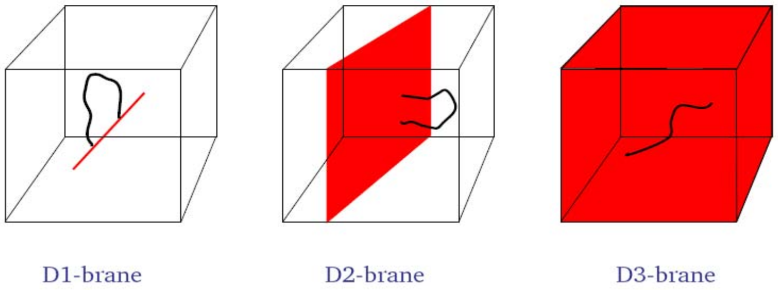
$\sim (g^2_{\rm YM})^{6-4}\, N_c^2 = \lambda^2$



N^{-2} counts *genus*

- The modern view of the string theory of QCD of string hadrons is based on two novel concepts

- *D branes*



- *Maldacena correspondence* between string theory on $AdS_5 x S_5$ and N=4 SYM in 4d.

- In reality color gauge dynamics is *confining* at the strong coupling limit and not conformal
- There is zero supersymmetries

- Our goal is thus to pave the road from the duality of the conformal and maximally supersymmetric theory to *string(gravity)/gauge duality* for

a theory as close as possible to the pure YM theory and QCD and in particular to hadron physics

Holographic QCD

- Outline-

- Lecture I- Confinement from gravity
- Lecture II- Holographic quarks
- Lecture III-Stringy Hadrons
- Lecture IV- Phase diagram of HQCD

Lecture 1-

Confinement from Gravity

- Confining Wilson loop
- Screening ‘t Hooft loop
- Glueball sepctra
- Confining models

Confinement in gauge dynamics

- It is well known that quarks and gluons are confined and hence not asymptotic states.
- There are several manifestations of confinement already in the pure glue theory:
- The *Wilson loop* has an *area law* behavior
- The ‘t Hooft loop admits a screening nature
- The physical states are colorless glueballs with a discrete spectrum and a *mass gap*
- Hence the free energy does not scale with N_c

- When *temperature* is introduced one finds that at low temperature the theory is in a confining phase whereas at high temperature the theory is in a *deconfining* phase

- The latter is characterized by:
- Wilson line which scales with the *perimeter*
- The spectrum is *continuous*
- The free energy behaves like N_c^2
- There is a non trivial expectation to the Polyakov loop

Confining Wilson Loop

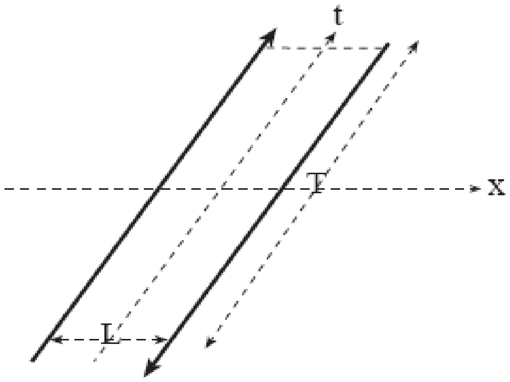
- In SU(N) gauge theories one defines the following gauge invariant operator

$$\boxed{W(C) = \frac{1}{N} Tr P e^{\oint_C A_\mu \dot{x}^\mu(\tau) d\tau}}$$

where C is a some contour

- The quark – antiquark potential can be extracted from a strip Wilson line

$$\boxed{\langle W(C) \rangle = A(L) e^{-TE(L)}.}$$



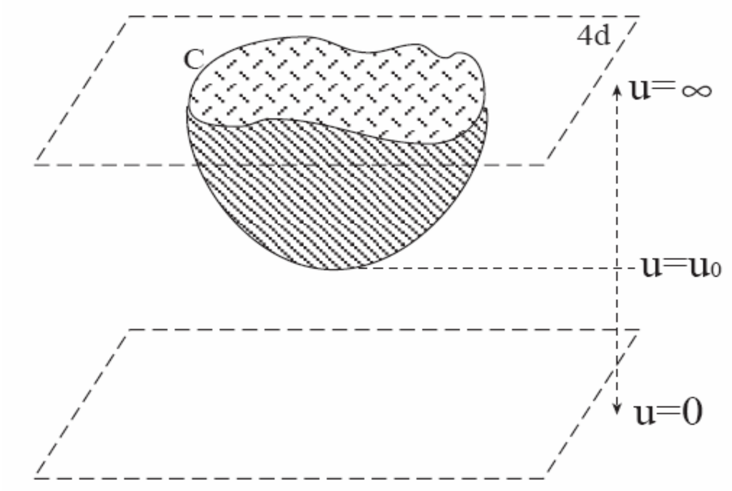
- The signal for confinement is E ~ T_{st} L

Stringy Wilson loop

- The natural stringy gravity dual of the Wilson line (which obeys the loop equation) is

$$\boxed{\langle W(C)\rangle \sim e^{-S_{NG}^{ren}}}$$

where S_{NG}^{ren} is the renormalized Nambu Goto action, namely the renormalized world sheet area



- The basic set up is a d dimensional space-time with the metric

$$ds^2 = -G_{00}(s)dt^2 + G_{x_{||}x_{||}}(s)dx_{||}^2 + G_{ss}(s)ds^2 + G_{x_Tx_T}(s)dx_T^2$$

where $x_{||}$ are p space coordinates on a Dp brane and s and x_T are radial coordinate and transverse directions.

- The corresponding Nambu Goto action is

$$S_{NG} = \int d\sigma d\tau \sqrt{det[\partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu}]}$$

- Upon using the gauge $\tau = t \;\; \sigma = x$ NG action is

$$S_{NG} = T \cdot \int dx \sqrt{f^2(s(x)) + g^2(s(x))(\partial_x s)^2}$$

where

$$f^2(s(x)) \equiv G_{00}(s(x))G_{x_{||}x_{||}}(s(x)) \qquad g^2(s(x)) \equiv G_{00}(s(x))G_{ss}(s(x))$$

- The Hamiltonian equation of motion is

$$\frac{ds}{dx} = \pm \frac{f(s)}{g(s)} \cdot \frac{\sqrt{f^2(s) - f^2(s_0)}}{f(s_0)}$$

- The separation distance between the string endpoints (the quark antiquark)

$$L = \int dx = 2 \int_{s_0}^{s_1} \frac{g(s)}{f(s)} \frac{f(s_0)}{\sqrt{f^2(s) - f^2(s_0)}} ds$$

- The NG action diverges. It is renormalized by

(i) regularizing the integral $\int^{\infty} \longrightarrow \int^{s_{max}}$

(ii) subtracting the quark masses

$$m_q = \int_0^{s_1} g(s) ds$$

•So that the remormalized quark antiquark potential is

$$E = f(s_0) \cdot L + 2 \int_{s_0}^{s_1} \frac{g(s)}{f(s)} (\sqrt{f^2(s) - f^2(s_0)} - f(s)) ds$$

$$- 2 \int_0^{s_0} g(s) ds$$

Theorem 1 *Let S_{NG} be the NG action defined above, with functions $f(s), g(s)$ such that:*

1. *$f(s)$ is analytic for $0 < s < \infty$. At $s = 0$, (we take here that the minimum of f is at $s = 0$) its expansion is:*

   $$f(s) = f(0) + a_k s^k + O(s^{k+1})$$

   with $k > 0$, $a_k > 0$.

2. *$g(s)$ is smooth for $0 < s < \infty$. At $s = 0$, its expansion is:*

   $$g(s) = b_j s^j + O(s^{j+1})$$

   with $j > -1$, $b_j > 0$.

3. *$f(s), g(s) \geq 0$ for $0 \leq s < \infty$.*

4. *$f'(s) > 0$ for $0 < s < \infty$.*

5. *$\int^{\infty} g(s)/f^2(s) ds < \infty$.*

Then for (large enough) L there will be an even geodesic line asymptoting from both sides to $s = \infty$, and $x = \pm L/2$. The associated potential is

1. *if $f(0) > 0$, then*
   - (a) *if $k = 2(j+1)$,*

     $$\boxed{E = f(0) \cdot L - 2\kappa + O((\log L)^{\beta} e^{-\alpha L})}$$

   - (b) *if $k > 2(j+1)$,*

     $$\boxed{E = f(0) \cdot L - 2\kappa - d \cdot L^{-\frac{k+2(j+1)}{k-2(j+1)}} + O(L^{\gamma}).}$$

   where $\gamma = -\frac{k+2(j+1)}{k-2(j+1)} - \frac{1}{k/2-j}$ and β and κ, α d and $C_{n,m}$ are positive constants determined by the string configuration.

   In particular, there is

   $$\boxed{\textit{linear confinement}}$$

2. *if* $f(0) = 0$, *then if* $k > j + 1$,

$$\boxed{E = -d' \cdot L^{-\frac{j+1}{k-j-1}} \quad + O(L^{\gamma'})}$$

where $\gamma' = -\frac{j+1}{k-j-1} - \frac{2k-j-1}{(2k-j)(k-j-1)}$ *and* d' *is a coefficient determined by the classical configuration.*

In particular,

$$\boxed{\textit{there is no confinement}}$$

Sufficient conditions for confinement

- We thus conclude that the a sufficient condition for confinement is if either

$$\text{(i) } f \textbf{ has a minimum at } s_{min} \textbf{ and } f(s_{min}) \neq 0$$

$$\text{(ii) } g \textbf{ diverges at } s_{div} \textbf{ and } f(s_{div}) \neq 0$$

Model	Nambu-Goto Lagrangian	Energy
$AdS_5 \times S^5$[8]	$\sqrt{U^4/R^4 + (U')^2}$	$-\frac{2\sqrt{2}\pi^{3/2}R^2}{\Gamma(\frac{1}{4})^4} \cdot L^{-1}$
non-conformal D_p brane[43, 16] (16 susy)	$\sqrt{(U/R)^{7-p} + (U')^2}$	$-d' \cdot L^{-2/(5-p)}$ $+O(l^{-2/(5-p)-2(6-p)/(5-p)(7-p)})$
Pure YM in 4d at finite temperature [16, 15]	$\sqrt{(U/R)^4(1-(U_T/U)^4) + (U')^2}$	$\sim L^{-1}(1-c(LT)^4)$ for $L << L_c$ full screening $L > L_c$

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Dual model of Pure YM in 3d [12, 16, 32, 31]	$\sqrt{(U/R)^4 + (U')^2(1-(U_T/U)^4)^{-1}}$	$\frac{U_T^2}{2\pi R^2} \cdot L - 2\kappa + O((\log L)^\beta \, e^{-\alpha L})$
Dual model of Pure YM in 4d	$\sqrt{(U/R)^3 + (U')^2(1-(U_T/U)^3)^{-1}}$	$\frac{U_T^{3/2}}{2\pi R^{3/2}} \cdot L - 2\kappa + O((\log L)^\beta \, e^{-\alpha L})$

Rotating D_3 [44]	$\sqrt{C}\sqrt{\frac{U^6}{U_0^4}\Delta+(U')^2\frac{U^2\Delta}{1-a^4/U^4-U_0^6/U^6}}$	$4/3\frac{U_T^2}{R^2}CL+...$
D_3+D_{-1} [45]	$\sqrt{(U^4/R^4+q)+(U')^2(1+qR^4/U^4)}$	qL+...
$MQCD$ [18]	$2\sqrt{2\zeta}\sqrt{\cosh(s/R_{11})}\sqrt{1+s'^2}$	$E=2\sqrt{2\zeta}\cdot L-2\kappa$ $+O((\log L)^{\beta}\ e^{-1/\sqrt{2}R_{11}L})$
't Hooft loop [13, 17]	$\frac{1}{g_{YM}^2}\sqrt{(U/R)^3(1-(U_T/U)^3)+(U')^2}$	full screening of monopole pair

Quantum fluctuations

- Introduce quantum fluctuations around the classical configuration

$$x^\mu(\sigma,\tau) = x^\mu_{cl}(\sigma,\tau) + \xi^\mu(\sigma,\tau)$$

- Consequently the quantum corrected *Wilson loop* reads

$$\langle W\rangle = \mathrm{e}^{-TE_{cl}(L)} \int \prod_a d\xi_a \exp\left(- \int d^2\sigma \sum_a \xi^a \mathcal{O}^a \xi^a \right)$$

where ξ are the fluctuations left after gauge fixing.

- The corresponding correction to the free energy is

$$F_B = -\log \mathcal{Z}_{(2)} = -\sum_a \frac{1}{2} \log\det \mathcal{O}_a$$

Wilson loop in flat space-time

- Consider the bosonic action of a string in flat space-time with boundary conditions fixed at

$$x(\sigma = 0) = 0 \qquad x(\sigma = \pi) = L$$

- The static NG action reads

$$S_{NG} = T_{st} \int dx \sqrt{1 + (\partial_x u)^2}$$

- The classical quark antiquark potential is

$$V(L) = T_{st} L$$

- Thus this Wilson loop associates with *linear confinement*

Bosonic quantum fluctuations

- The action of the bosonic fluctuations is

$$S_{(2)} = \frac{1}{2}\int d\sigma d\tau \sum_{i=1}^{D-2}\left[(\partial_\sigma \xi_i)^2 + (\partial_\tau \xi_i)^2\right]$$

- The Eigenvalues of the $\mathcal{O}^a$ are

$$\lambda_{n,m} = (\frac{n\pi}{L})^2 + (\frac{m\pi}{T})^2$$

- The free energy is given by

$$-\frac{2}{D-2}F_B = \log\prod_{nm}\lambda_{n,m} = T\frac{\pi}{2L}\sum_n n + O(L)$$

- *Zeta function regularization* yields the following potential

$$\boxed{\Delta V(L) = -\frac{1}{T}F_B = -(D-2)\frac{\pi}{24}\cdot\frac{1}{L}}$$

- This is the famous Luscher term

Fermionic fluctuations

- For the *Green Schwarz superstring* there are also fermionic fluctuations.
- The fermionic part of the k gauged fixed GS action is

$$S_F^{flat} = 2i \int d\sigma d\tau \bar{\psi} \Gamma^i \partial_i \psi$$

Weyl Majorana spinor SO(9,1) gamma matices i,j=1,2

- Thus the fermionic operator is

$$\hat{\mathcal{O}}_F = D_F = \Gamma^i \partial_i$$

and squaring it gives

$$(\hat{\mathcal{O}}_F)^2 = \Delta = \partial_x^2 - \partial_t^2$$

- The total B and F contribution to the free energy is

$$F = 8 \times \left(-\frac{1}{2} \log \det \Delta + \log \det D_F \right) = 0$$

- No QM correction and vanishing *Luscher term*

Can the WL be evaluated exactly?

- The space-time energy of a bosonic string is

$$E^2 = P^2 + 4(L_0 - a) = (LT_{st})^2 + 4(L_0 - a)$$

- Hence for the lowest tachyonic state $L_0=0$ we get

$$E^2 = P^2 + m_{tach}{}^2 = (LT_{st})^2 - T_{st}\frac{\pi(D-2)}{12}$$

- The energy is therefore

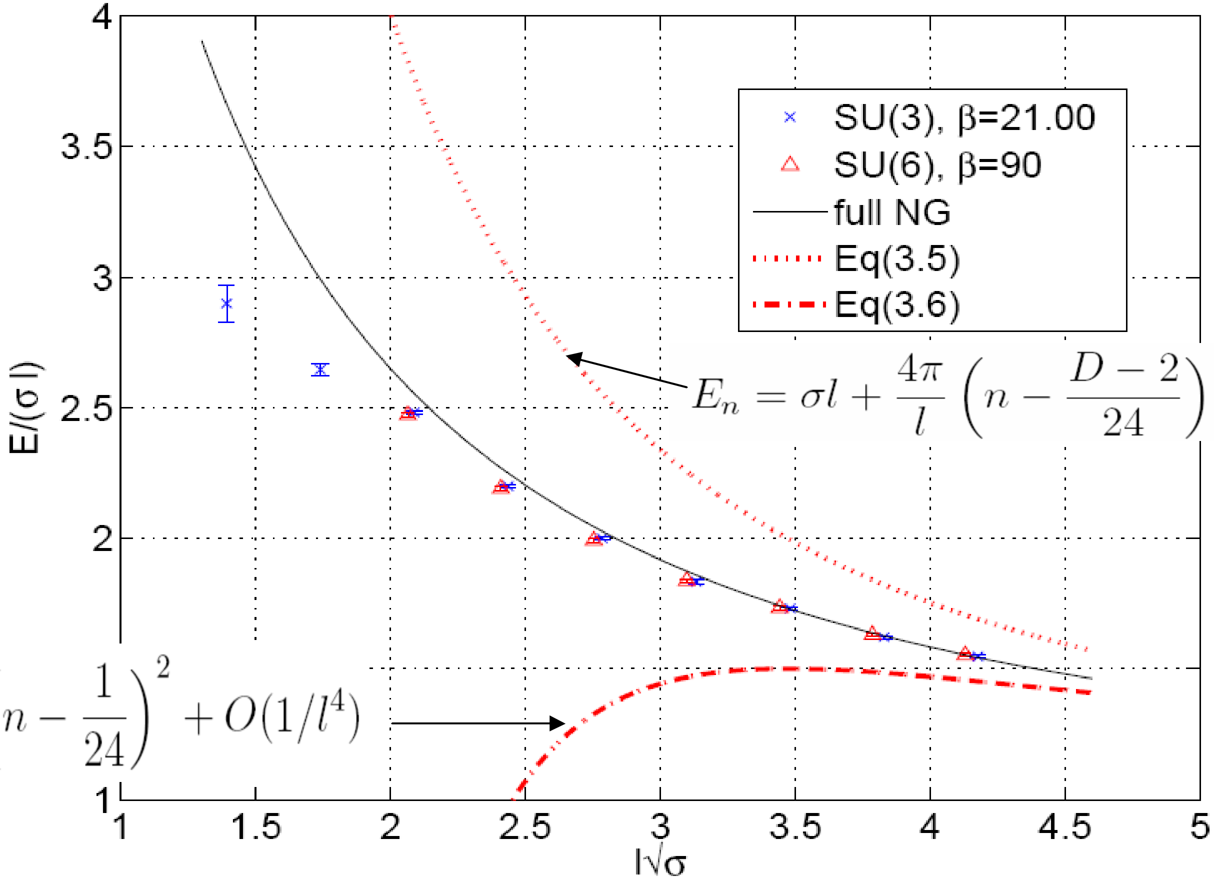
$$\boxed{V(L) = T_{st}L\sqrt{1 - \frac{\pi(D-2)}{12}\frac{1}{T_{st}L^2}}}$$

- When expanded we recover the linear and Luscher terms

$$\sim T_{st}L - \pi\frac{(D-2)}{24}\frac{1}{L} + \ldots$$

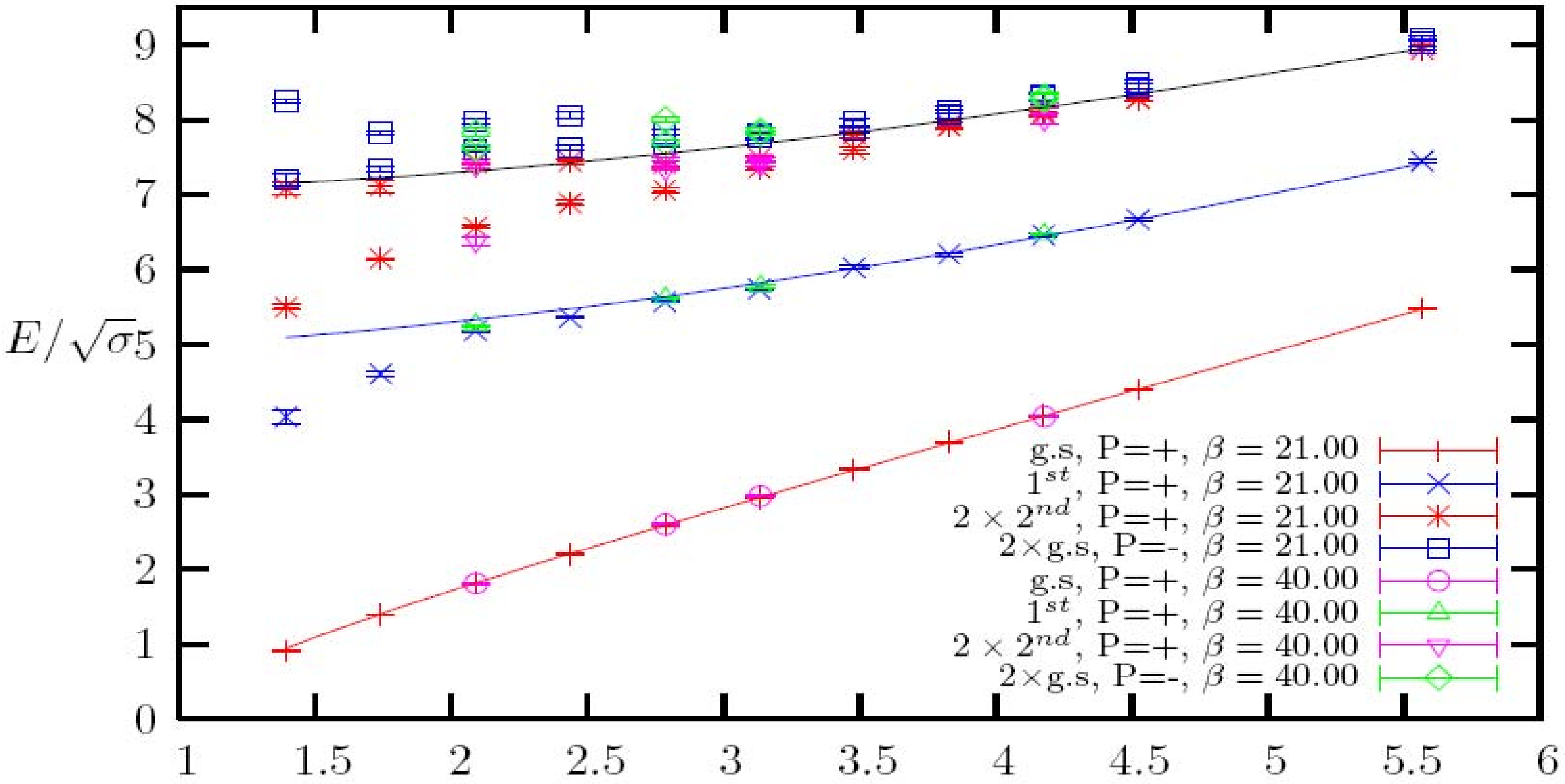
Comparing the exact bosonic string picture to lattice calculations

- The quark antiquark potential for SU(3) and SU(6) gauge theories in 2+1 dimensions were extracted in lattice calculations.



$$E_n = \sigma l + \frac{4\pi}{l}\left(n - \frac{D-2}{24}\right) + O(1/l^2),$$

$$E_n = \sigma l + \frac{4\pi}{l}\left(n - \frac{1}{24}\right) - \frac{8\pi^2}{\sigma\, l^3}\left(n - \frac{1}{24}\right)^2 + O(1/l^4)$$



The energies of the lowest 7 states as a function of L

In comparison to the NG predictions.

Lessons from the comparison

(i) The NG string fits nicely (ii) The string “resides” in “flat space time”

Back to the fluctuations for a confining background

- As a prototype we take the *near extremal Ads5xs5* which in the limit of small radius is the dual to 3d *pure YM* theory.
- The bosonic operators are

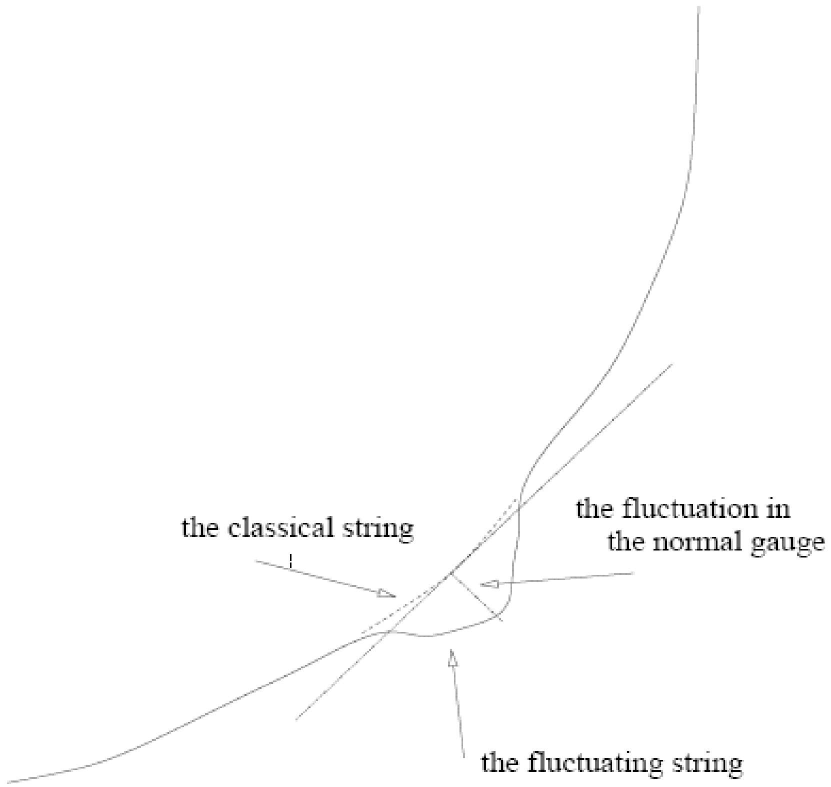
.

$$\hat{\mathcal{O}}_y \longrightarrow \frac{u_T^2}{2}\left[\partial_x^2+\partial_t^2\right]$$

$$\hat{\mathcal{O}}_\theta \longrightarrow \frac{R^2}{2}\left[\partial_x^2+\partial_t^2\right]$$

$$\hat{\mathcal{O}}_z \longrightarrow 2u_T^2 e^{-2u_T L}\left[\partial_x^2+\partial_t^2\right]$$

$$\hat{\mathcal{O}}_n \longrightarrow \left[\frac{4u_T^2}{2R^4}+\frac{1}{2}\partial_x^2+\frac{1}{2}\partial_t^2\right]$$



The operators of the transverse fluctuations correspond to massless modes but the *longitudinal* normal mode is a *massive* mode. So altogether there are 7 bosonic massless modes

- Had the fermionic modes been those of flat space time then the total coefficient in front of the Luscher term whould have been +8-7=+1.
- This means a *repulsive* “Culomb” like potential.This contradicts gauge dynamics.
- However, in the near extremal $AdS_5 \times S^5$ case

due to the coupling to the RR flux the square of the fermionic operator is

$$\hat{\mathcal{O}}^2_\psi = \frac{u_T^2}{2}\left[\partial_x^2 + \partial_t^2 + (\frac{U_T}{R^2})^2\right]$$

- Thus we get an *attarctive Luscher* term

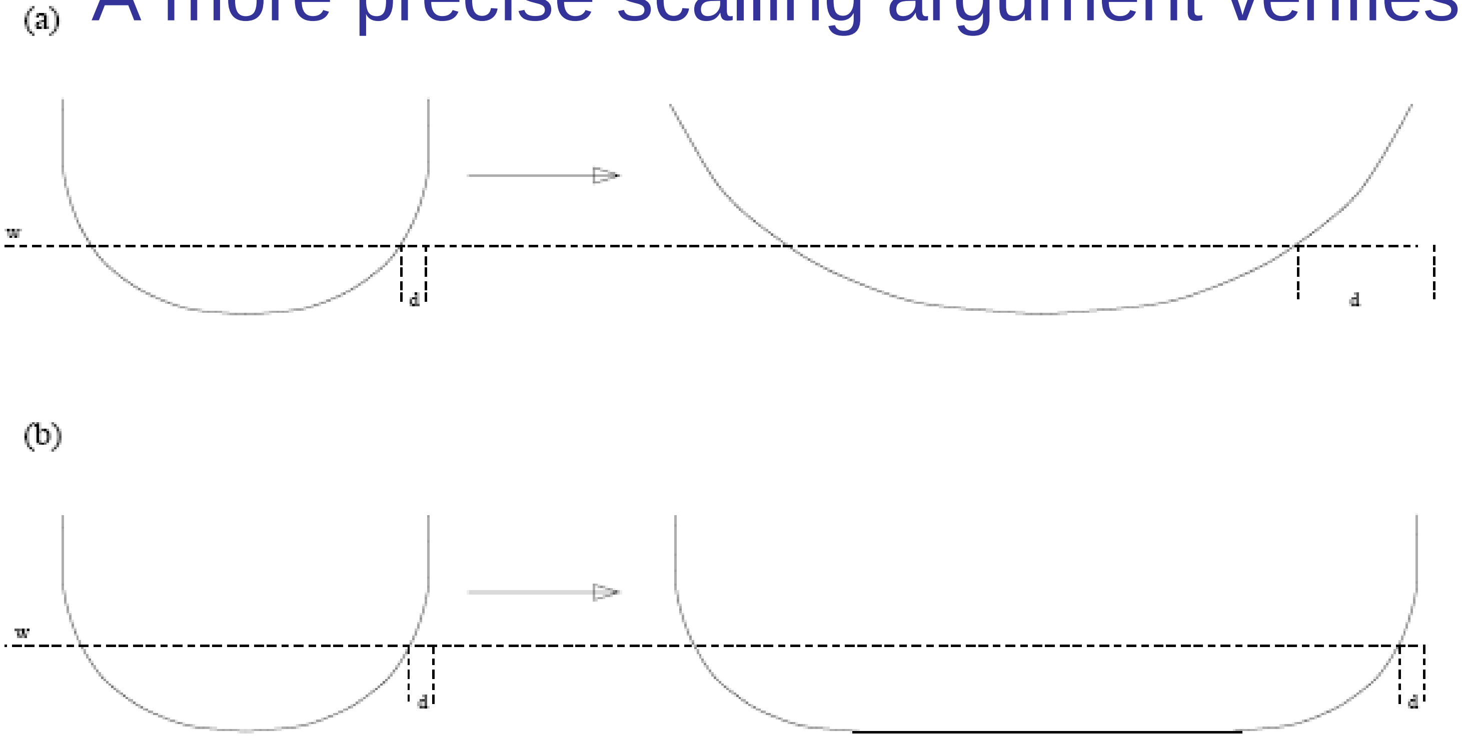
$$-7\frac{\pi}{24}\frac{1}{L}$$

Does the confining WL behave like a QCD WL?

Yes!

Effectively it behaves like a string in flat space time. This is due to the |_| shape and the cancelation of the vertical segments.

A more precise scalling argument verifies this



$AdS_5 \times S^5$

confining

The ‘t Hooft loop –monopole anti-monopole potential

- It is well known that in a theory were the quarks are confined, the *monopoles are screened* (and wise versa).

- Just as a Wilson loop with a fundamental string describes the quark anti-quark pair, the hodge dual of a string namely a *D1 brane* is the stringy description of a ‘t Hooft loop

- In Witten’s model (or its non-critical analog) we are in type IIA so the role of the D1 brane is played by a D2 brane that wraps the compactified cycle.

- The action of a D2 brane is the DBI action. The worldvolume of it is along (t,x,τ).

$$S=\frac{1}{(2\pi\alpha')^{3/2}}\int d\tau d\sigma_1 d\sigma_2 e^{-\phi_0}\sqrt{\det h_{\rm ind}}=\beta\int_{-L/2}^{L/2}dx\frac{u}{R_{AdS}}\sqrt{u'^2+\left(\frac{u}{R_{AdS}}\right)^4 f(u)},$$

- Similar to the Wilson line we solve the e.o.m

$$\left(\frac{u}{R_{AdS}}\right)^5 f(u)\left(u'^2+\left(\frac{u}{R_{AdS}}\right)^4 f(u)\right)^{-1/2}=\text{const}$$

- The separation distance is

$$L=\int dx=2\int_{u_\Lambda}^{\infty}\frac{du}{u'}=2\frac{R^2_{Ads}}{u_{\min}}\epsilon^{1/2}\int_1^{\infty}dy\frac{y}{\sqrt{(y^5-1+\epsilon)(y^5-\epsilon y-1+\epsilon)}},$$

- Again we have to *substruct the masses* of the monopole pair, namely of D2 brane that strech from the boundary to the horizon

- The renormalized energy is

$$\Delta E \sim \beta \frac{u_{\min}^2}{u_\Lambda}\left[\int_1^\infty dy\, y^2 \left(\sqrt{\frac{y^5-1+\epsilon}{y^5-\epsilon y-1+\epsilon}}-1\right)-\frac{1}{3}\left(1-\left(\frac{u_\Lambda}{u_{\min}}\right)^3\right)\right]$$

- Since this energy is positive it means that the configuration of two parallel D2 branes is favorable and hence the system is *screened*

Glueball spectra

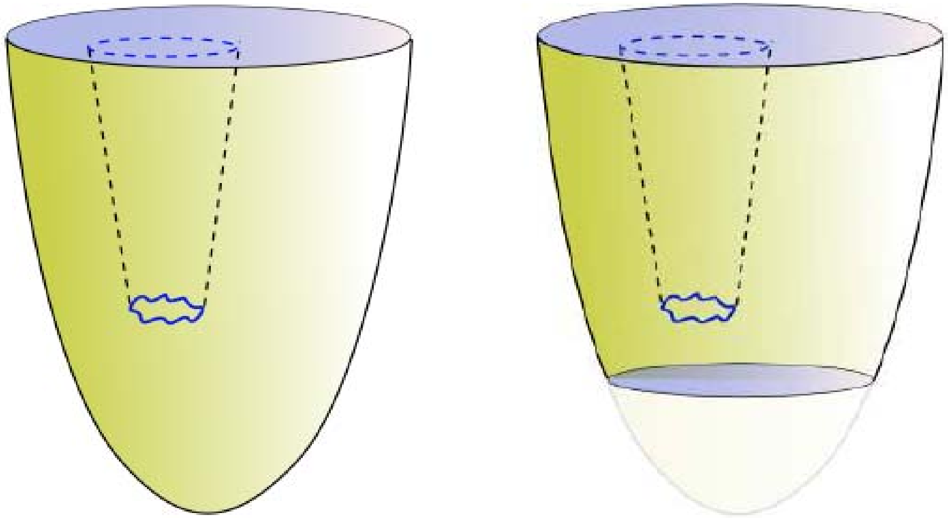
- Confining gauge dynamics implies a *discrete* spectrum with a *mass gap*.
- In the gauge/gravity duality one measures the

the spectrum of glueballs using the spectrum of

the fluctuations of the bulk fields of the gravity background: graviton, dilaton, NS and RR forms

- For instance the Tr[F^2] glueball corresponds to the *dilaton* (or other scalar bulk operators)

Does the fluctuation of a bulk operator in front of a wall yield a discrete spectrum of glueballs with a mass gap



conformal
anti de-Sitter

confining
extra scale ("wall")

- Thus to determine the 0^{++} spectrum we solve the linearized e.o.m of the dilaton

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\,\phi\right) = 0$$

- This of course depend on the background metric.

We can take the YM3 , YM4, non-critical, KS, MN, models.

Here we use a formulation that fits the first three models

$$\begin{aligned} ds^2 &= \frac{r^2}{L^2}\left(f(r)d\tau^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right) + \frac{L^2}{r^2}f^{-1}(r)dr^2 \\ &\text{with } f(r) = \left(1 - \frac{R^{p+1}}{r^{p+1}}\right) \end{aligned}$$

- We take the following ansatz

$$\phi = b(r)e^{ik\cdot x} \qquad\qquad k^2 = -M^2$$

- Substituting this into the e.o.m we get

$$\frac{\partial^2 b(r)}{\partial r^2} + \frac{(p+2)r^{p+1} - R^{p+1}}{r\left(r^{p+1} - R^{p+1}\right)}\frac{\partial b(r)}{\partial r} + \frac{M^2L^2r^{p-2}}{r\left(r^{p+1} - R^{p+1}\right)}b(r) = 0$$

- One way to extract the spectrum of M is to use the WKB approximation
- We change the coordinates

$$b(r) = \beta(r)\chi(r) \qquad \beta(r) = \sqrt{\frac{r-R}{r(r^{p+1}-R^{p+1})}} \quad r = R(1+e^y)$$

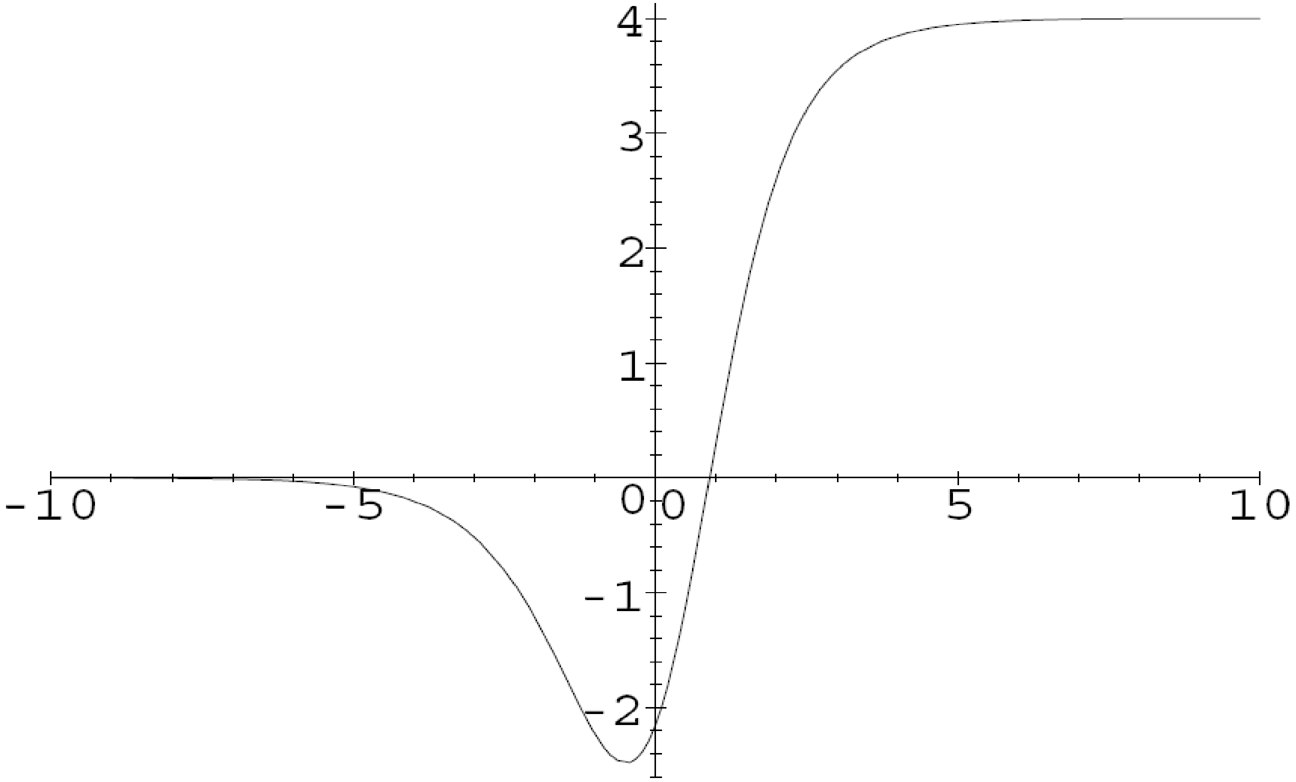
- The schroedinger equation we find reads

$$-\chi''(y) + V(y)\,\chi(y) = 0$$

- with the well potential

$$\begin{aligned} V(y) &= \frac{1}{4} + \frac{e^{2y}\left(p(p+2)\,(1+e^y)^{2(p+1)} - 2p(p+2)\,(1+e^y)^{p+1} - 1\right)}{4\,(1+e^y)^2\left((1+e^y)^{p+1}-1\right)^2} \\ &\quad -\frac{M^2L^4}{R^2}\,\frac{e^{2y}\,(1+e^y)^{p-3}}{(1+e^y)^{p+1}-1} \ . \end{aligned}$$

In a figure the potential looks simpler



- We expand the potential and find its turning points

$$r_+ = R + \frac{2}{p+1} M L^2 \qquad \text{and} \qquad r_- = R \ .$$

- The WKB equation for a bound state is

$$\left(n - \frac{1}{2}\right)\pi = \int_{y_-}^{y_+} \sqrt{V(y)}dy$$

- From which we read the spectrum of the 0^{++} glueballs

$$M^2(p) = n\left(n + \frac{p-1}{2}\right)\frac{16\pi^3}{\beta^2}\left(\frac{\Gamma\left(\frac{p+3}{2(p+1)}\right)}{\Gamma\left(\frac{1}{p+1}\right)}\right)^2 + O(n^0)$$

$$\begin{aligned} M^2(p=3) &\simeq \frac{56.67}{\beta^2} n\,(n+1) + O(n^0) \\ M^2(p=5) &\simeq \frac{29.36}{\beta^2} n\,(n+2) + O(n^0) \end{aligned}$$

The spectrum of the 2^{++} glueballs

- The 2^{++} glueball associate with the fluctuations of the *metric*
- The corresponding e.o.m are the linearized Einstein equations

$$\frac{1}{2}\nabla_a\nabla_b h^c{}_c + \frac{1}{2}\nabla^2 h_{ab} - \nabla^c\nabla_{(a}h_{b)c} - \frac{p+1}{L^2}h_{ab} = 0$$

- We parametrize the fluctuations as

$$H_{ab} = \varepsilon_{ab}\frac{r^2}{L^2}H(r)$$

- We use a transverse gauge

$$H_{a\mu}k^\mu = 0.$$

- The equation of motion for $H(r)$ is identical to that of $b(r)$ and hence

$$\frac{M_{2^{++}}}{M_{0^{++}}} = 1$$

- There are glueballs states associated with the fluctuations of all the bulk fields

- Alltogether there are 6 different type of glueball states

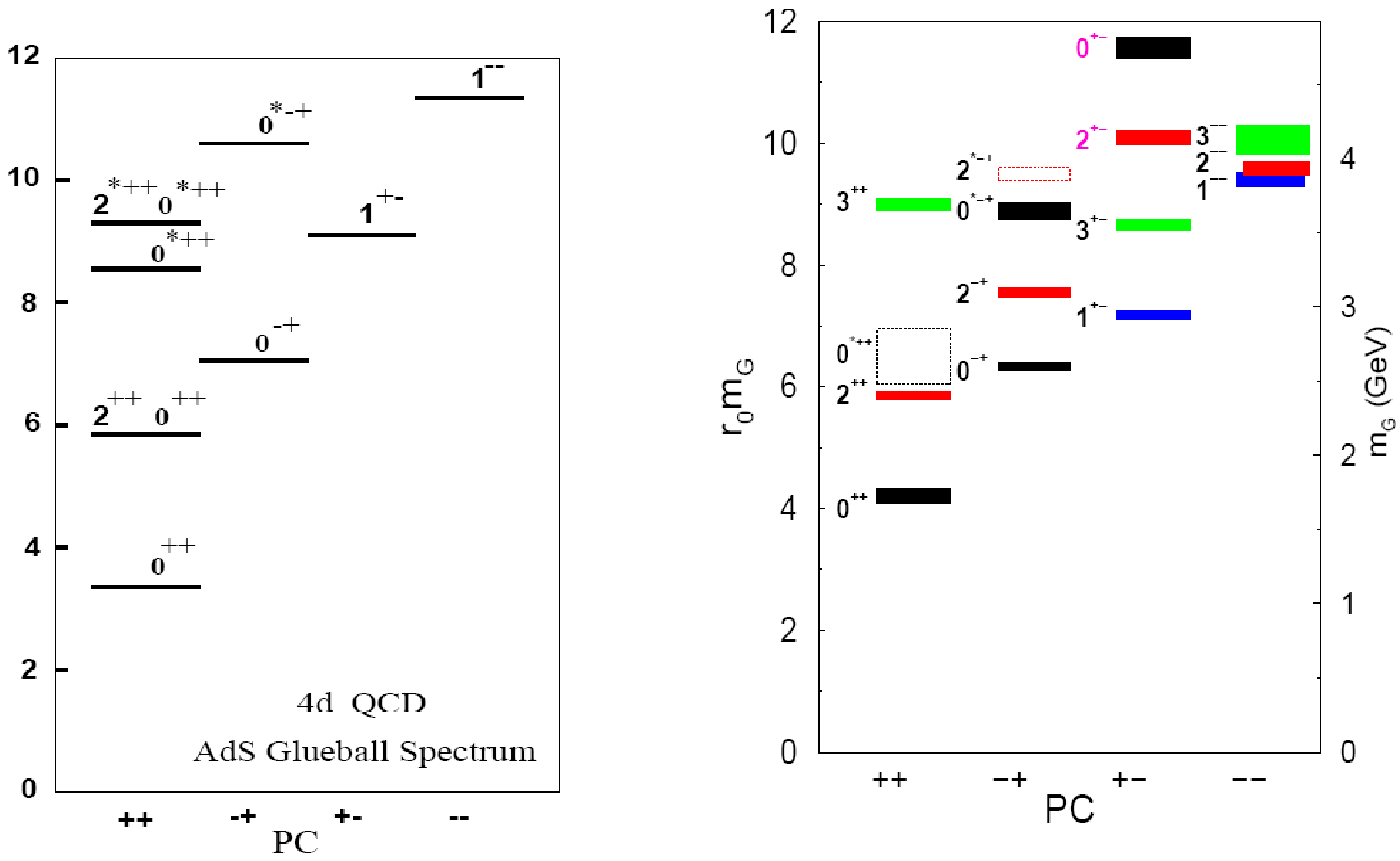
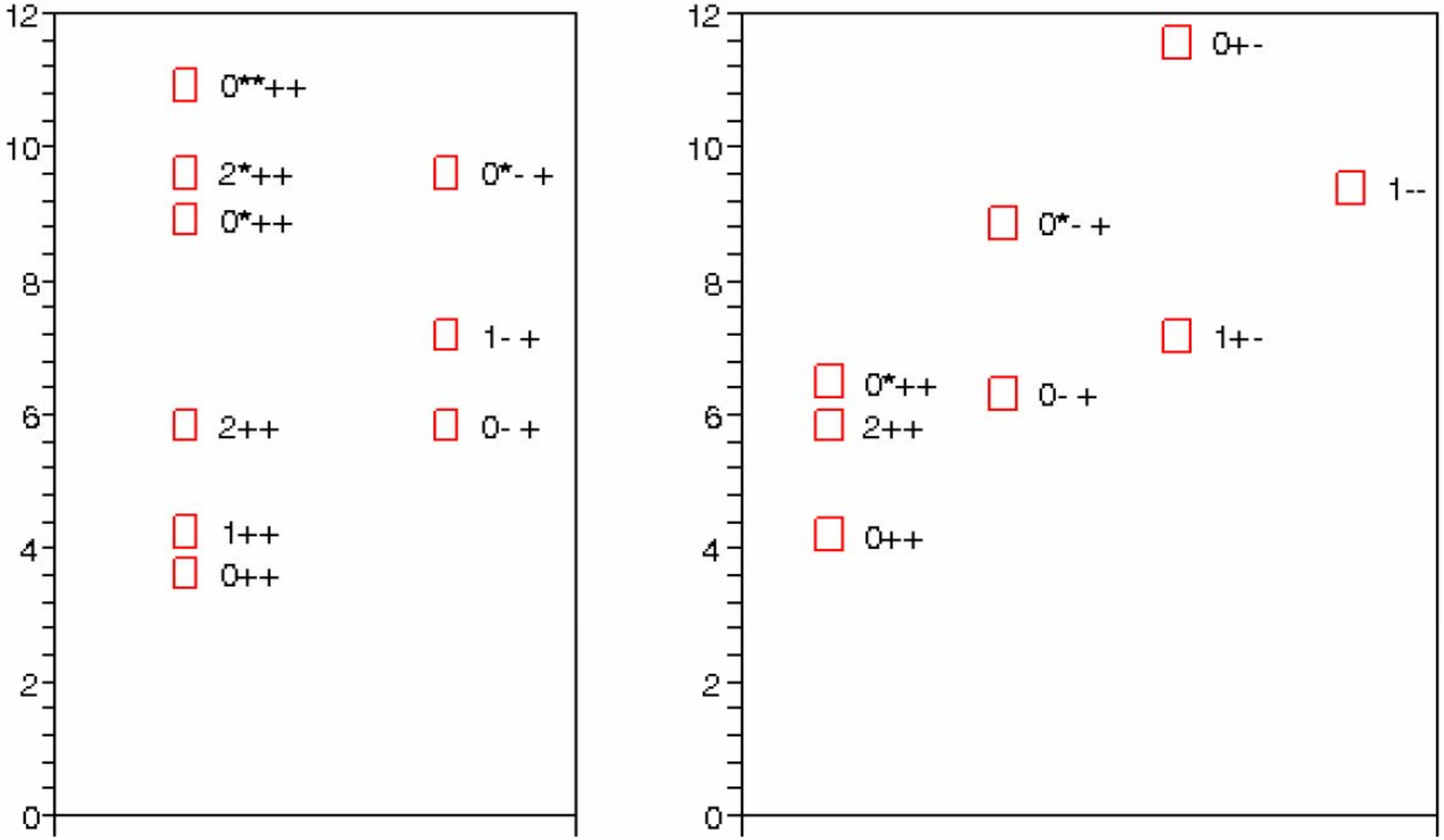


Figure 2: The AdS glueball spectrum for QCD_4 in strong coupling (left) compared with the lattice spectrum [5] for pure SU(3) QCD (right). The AdS cut-off scale is adjusted to set the lowest 2^{++} tensor state to the lattice results in units of the hadronic scale $1/r_0 = 410$ Mev.



igure 3: The AdS glueball spectrum for YM_4 computed in the framework of non-critical superravity (left) and the corresponding lattice results (right). The AdS scale is adjusted to set the west 2^{++} state to the the lattice result in units of the hadronic scale $1/r_0 = 410\text{MeV}$.

- There is a *remarkable correspondence* of the overall mass and spin structure between the gravity models and lattice calculations.

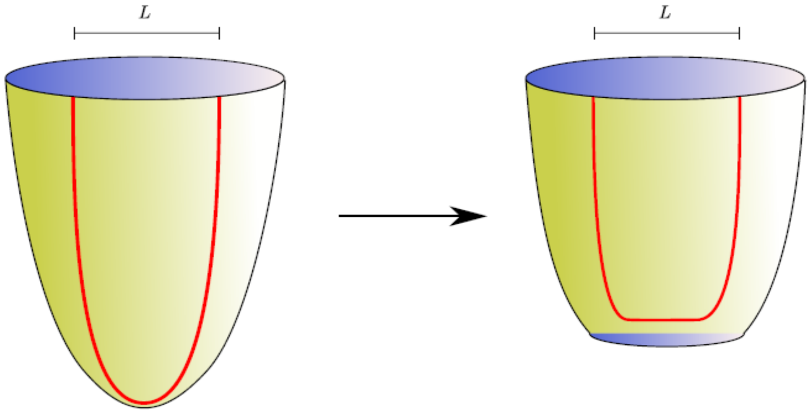
- Nevertheless it does not make sense to claim that there is an agreement to few percent for isolated mass ratios.

- One has to device a mechanism to get rid of many “spurious” states from gravity that do not show up in QCD due KK and other modes.
- One obvious shortcoming of the gravity models is the *absence of glueballs of spin higher than 2.*

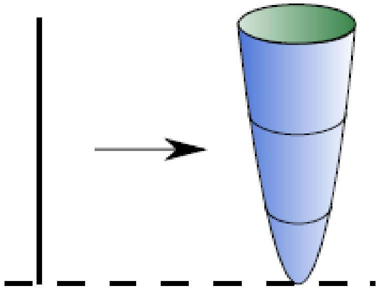
Witten’s model-

a prototype of confining model

- We have seen that a way to get a confining background is to cut the radial direction and introduce a *scale*.
- One approach is indeed to cut by hand an Ads space. This is not a solution of the SUGRA equations of motion. People use it to examine phenomenological properties (AdsQCD)



- The apparoch of Witten was to compactify one coordinate and find a “cigar-like” solution.



- One imposes *anti-periodic* boundary conditions on fermions. This *kills supersymmetry*.
- In the dual gauge theory the *gauginos* and the *scalars* acquire a *mass ~T* and hence in the large T limit they decouple and we are left only with the gauge fields.
- Since in large T β goes to zero so that we loose one space dimension and we end up with a pure gauge theory in p-1 space dimensions.
- The gravity theory associated with D3 branes namely the Ads5xs5 case compactified on a circle is dual to pure YM theory in 3d
- The same mechanism for D4 branes yields

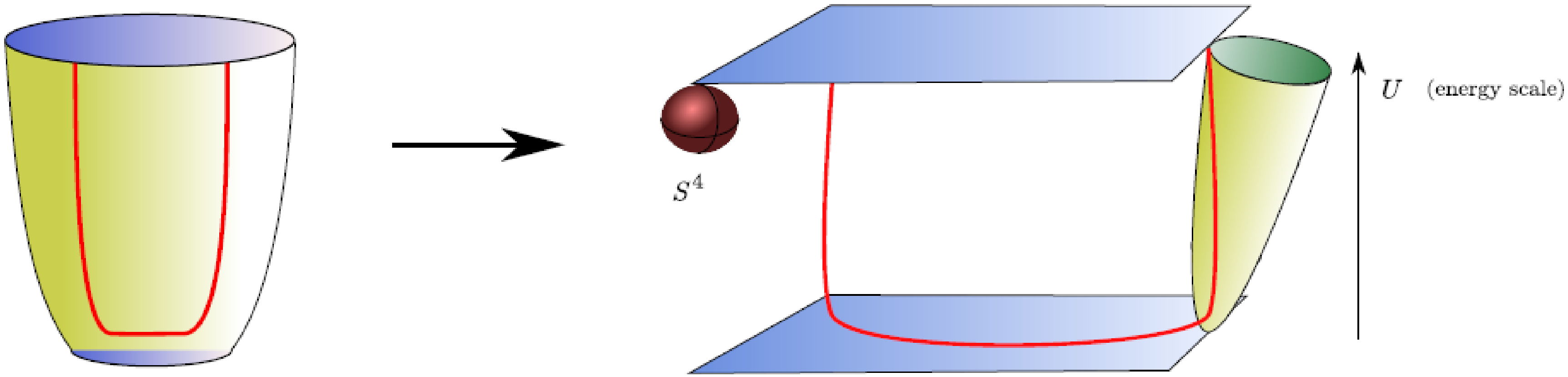
a dual theory of Pure YM in 4d.

$$\mathrm{d}s^2 = \left(\frac{U}{R}\right)^{3/2} \left[\eta_{\mu\nu}\mathrm{d}X^\mu \mathrm{d}X^\nu + f(U)\mathrm{d}\theta^2\right] + \left(\frac{R}{U}\right)^{3/2} \left[\frac{\mathrm{d}U^2}{f(U)} + U^2 \mathrm{d}\Omega_4\right]$$

world-volume
our 3+1 world

$f(U) = 1 - \left(\frac{U_\Lambda}{U}\right)^3$
θ is a compact
Kaluza-Klein circle

U: radial direction
bounded from
below $U \geq U_\Lambda$



- The gauge theory and sugra parametrs are related via

$$g_5^2 = (2\pi)^2 g_s l_s, \qquad g_4^2 = \frac{g_5^2}{2\pi R} = 3\sqrt{\pi}\left(\frac{g_s u_\Lambda}{N_c l_s}\right)^{1/2}, \qquad M_{gb} = \frac{1}{R},$$

$$T_{st} = \frac{1}{2\pi l_s^2}\sqrt{g_{tt}g_{xx}}\Big|_{u=u_\Lambda} = \frac{1}{2\pi l_s^2}\left(\frac{u_\Lambda}{R_{D4}}\right)^{3/2} = \frac{2}{27\pi}\frac{g_4^2 N_c}{R^2} = \frac{\lambda_5}{27\pi^2 R^3},$$

5d coupling

4d coupling

glueball mass

String tension

- The gravity picture is **valid** only provided that $\lambda_5 \gg R$

- At energies $E \ll 1/R$ the theory is **effectively 4d**.

- However it is not really QCD since $M_{gb} \sim M_{KK}$

- In the opposite limit of $\lambda_5 \ll R$ **we approach QCD**

- Other confining backgrounds like the Maldacena Nunez (*MN*) and Klebanov Strassler (*KS*) will be discussed by Bertolini
- I would like now to introduce the notion of non-critical string backgrounds and in particular *nonciritical* confining backgrounds.

Summary of lecture 1.

- *Wilson loop* as a basic guiding line in constructing confining backgrounds
- *Discrete* spectrum with a *mass gap* for glueballs
- *Witten’s model* dual to contaminated pure YM theory
- Finding non-critical string/sugra models

Lecture 2: **Fundamental quarks**

- So far we have discussed the gravity duals of gauge dynamics without fundamental quarks

- ‘t Hooft taught us that the Feynmann diagrams of SU(N_c) YM theory in the large N_c limit reorganize themselves into a genus expansion of closed string theory.

- Adding *fundamental quarks* is described by adding *boundaries* to the Reimann surfaces,namely one adds an *open string* sector.

- Lets go for a moment from the SUGRA background back to the brane configuration.

- If we add to the original stack of Nc D3 (or D4) branes another set of Nf Dp branes there will be strings connecting the original D3 and Dp branes.

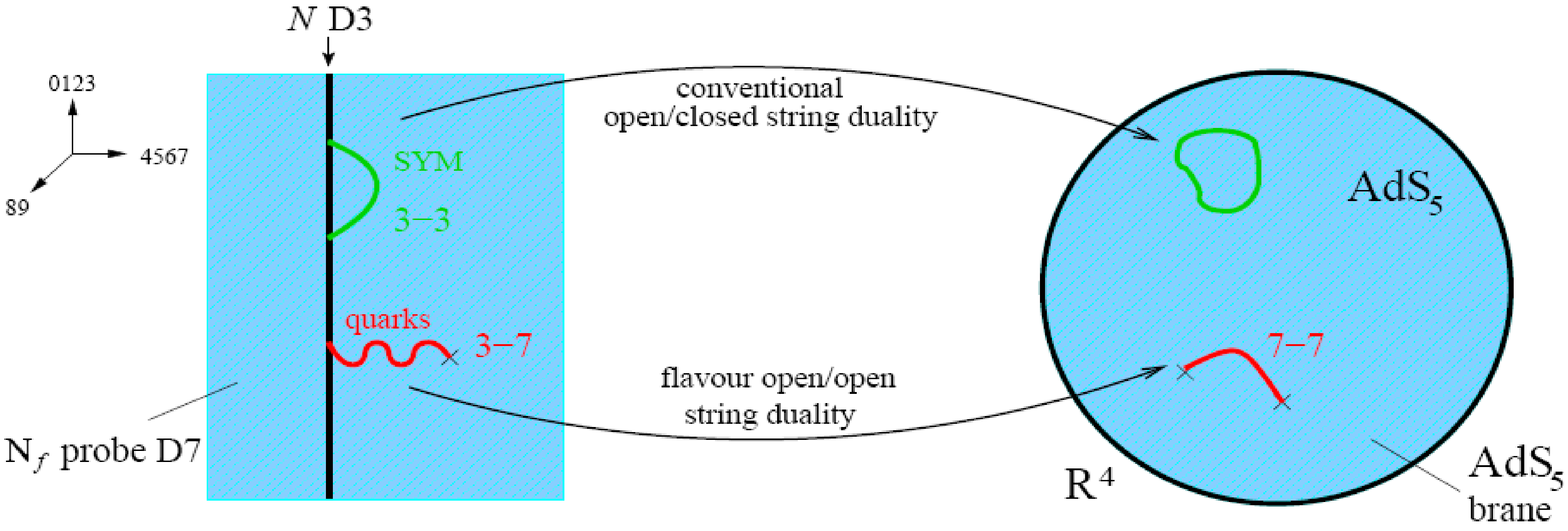
- These strings map in the dual field theory to *bifundamental "quarks*" that transform as the(Nc, Nf) representation of the U(Nc)xU(Nf) gauge symmetry.

- For Nc >> Nf the U(Nf) can be treated as a global symmetry and hence we got fundamental quarks.

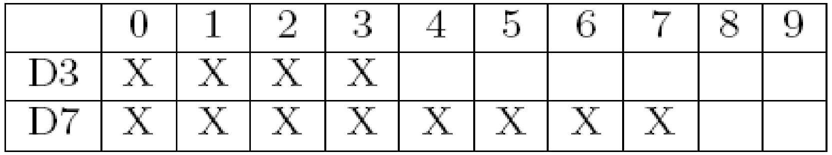
- Coming back to the SUGRA background, in the case of $N_c >> N_f$ we can safely *neglect the backreaction* of the additional branes on the background. Thus we have introduced infact flavor *probe branes* into a background gravity model dual of a YM (SYM) theory. This is the gravity analog of using a *quenched* approximation in lattice gauge theories.

- *Karch and Katz* introduced Nf D7 probe branes to the Ads5xs5 background in such a way that the D7 branes wrap an S3 inside the S5. This system is stable since in spite of the fact that the slipping has a negative m^2 it is higher than the *BF* bound.

In addition to the closed strings that associate with the *glueballs* we have now also open strings associated with *mesons*

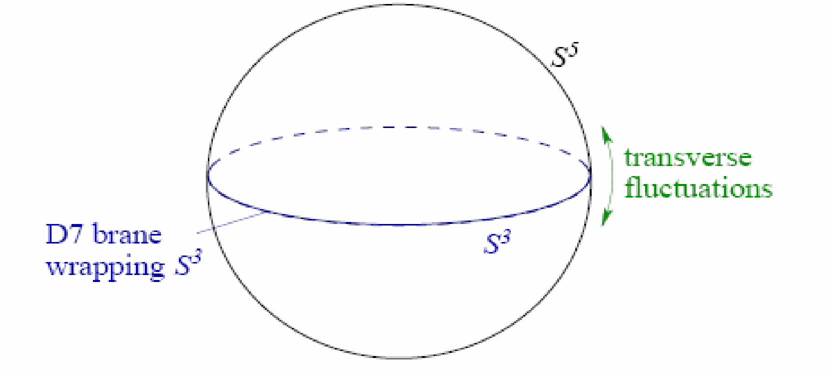


In the brane picture the original D3 and flavor D7 branes are along

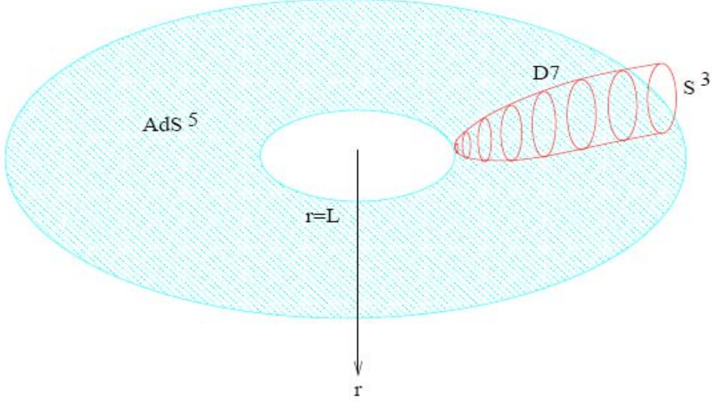


	0	1	2	3	4	5	6	7	8	9
D3	X	X	X	X						
D7	X	X	X	X	X	X	X	X		

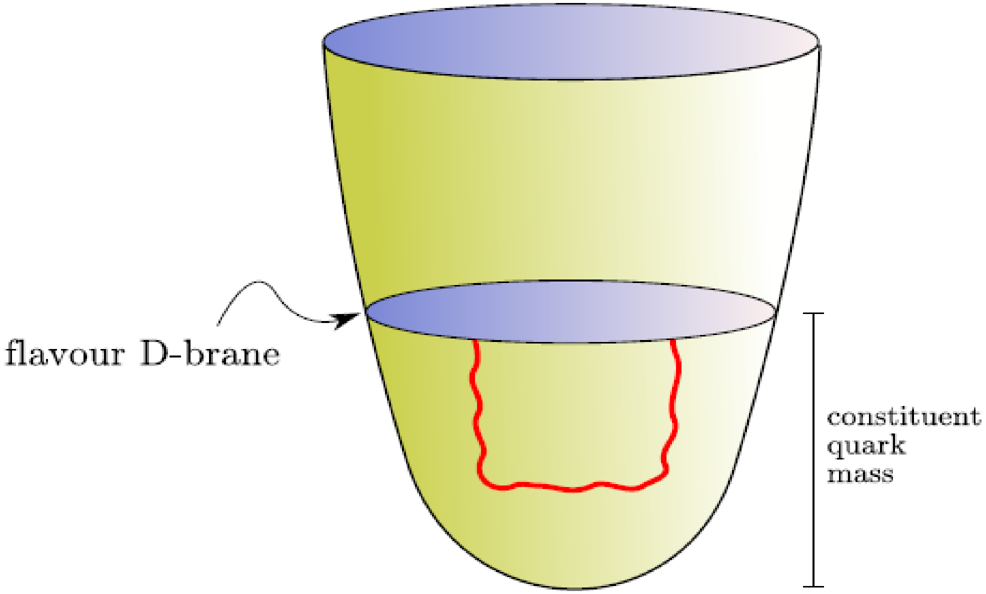
The D7 brane wraps an S3 in S5



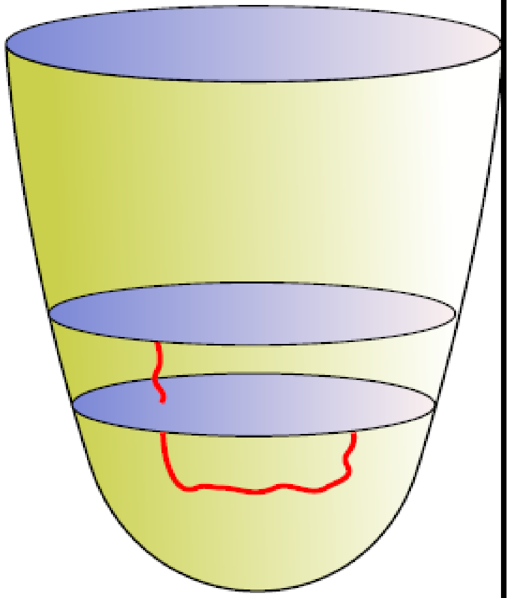
Mass can be introduced by letting the D7 span only part of the Ads5xs5



This can also be viewed as



equal-mass quarks



different mass quarks

The meaning of this mass will be addressed below

- We would like to introduce probe flavor branes to a non-supersymmetric confining background. A natural candidate is therefore *Witten's model.*

- What type of Dp branes should we add D4, D6 or D8 branes?

- How do we incorporate a full *chiral flavor global* symmetry of the form U(Nf)xU(Nf), with left and right handed chiral quarks?

- We place the two endpoints of the probe branes on the compactified circle. If there are additional transverse directions to the probe branes then one can move them along those directions and by that the strings will aquire length and the corresponding fields *mass*. Thus this will contradict the *chiral symmetry* which prevents a mass term.

- Thus we are forced to use D8 branes that do not have additional transverse directions.

- The fact that the strings are indeed chiral follows also from analyzing the representation of the strings under the *Lorentz group*

- *U(Nf)xU(Nf) global flavor symmetry* in the UV calls for two separate stacks of branes.

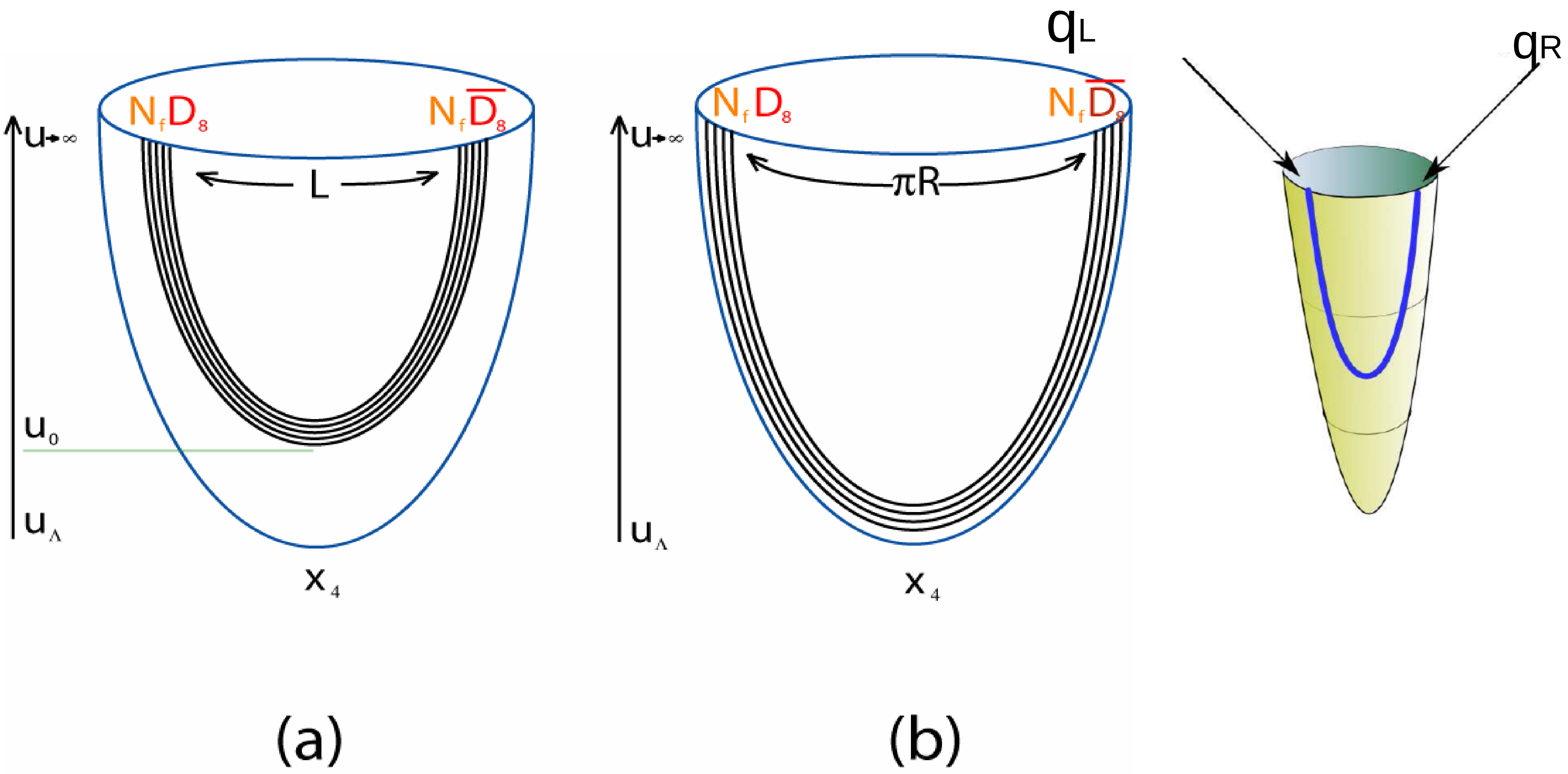
- To have a breakdown of this chiral symmetry to the diagonal U(Nf)D we need the two stacks of branes to *merge* one into the other.

- This requires a *U shape* profile of the probe branes.

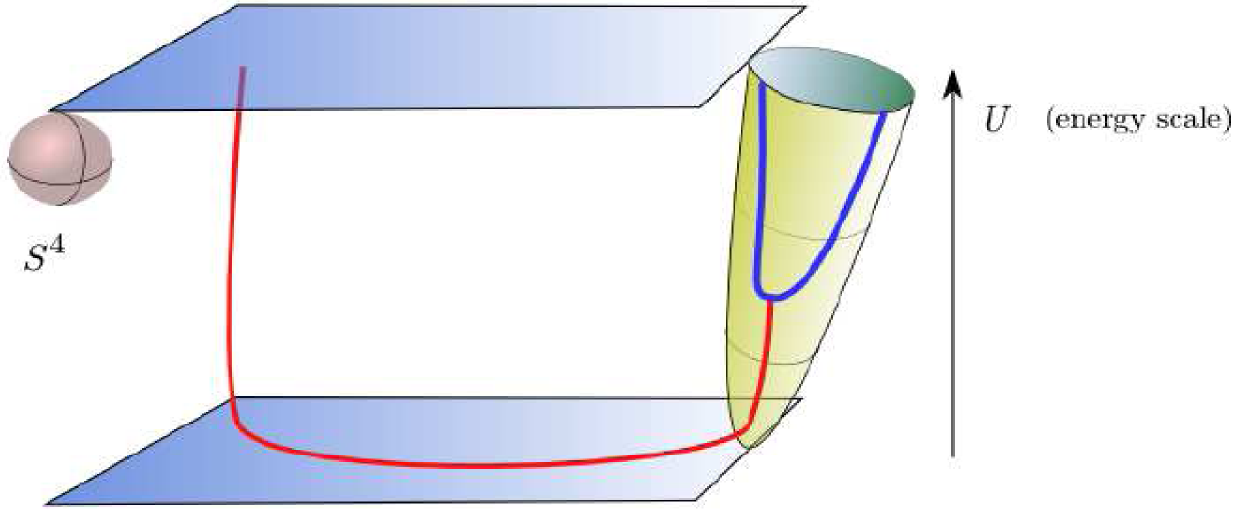
- The opposite orientations of the probe brane at their two ends implies that infact these are stacks of *Nf D8 branes and a stack of Nf anti D8 branes*. (Thus there is no net D8 brane charge)

- This is the *Sakai Sugimoto* model.

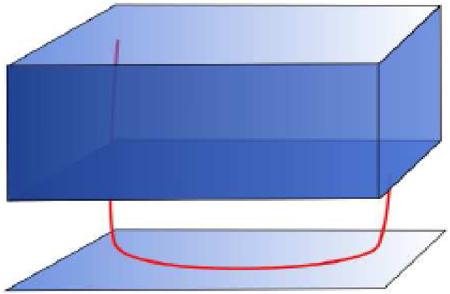
A picture is better than 1000 words. We "see" that the model admits chiral symmetry U(Nf)xU(Nf) in the UV which is broken to a diagonal one U(Nf)D in the IR.



Incorporating the probe branes in Witten’s model yields the following congiguration

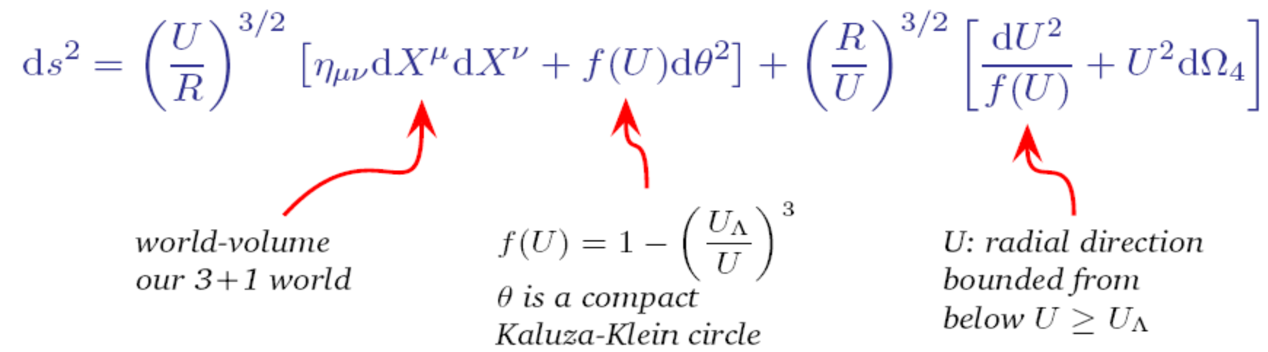


suppressing everything but U
and our 3+1d world:



The Sakai Sugimoto model

- Let us now zoom in and study in details the properties of this model.
- The ba
  branes



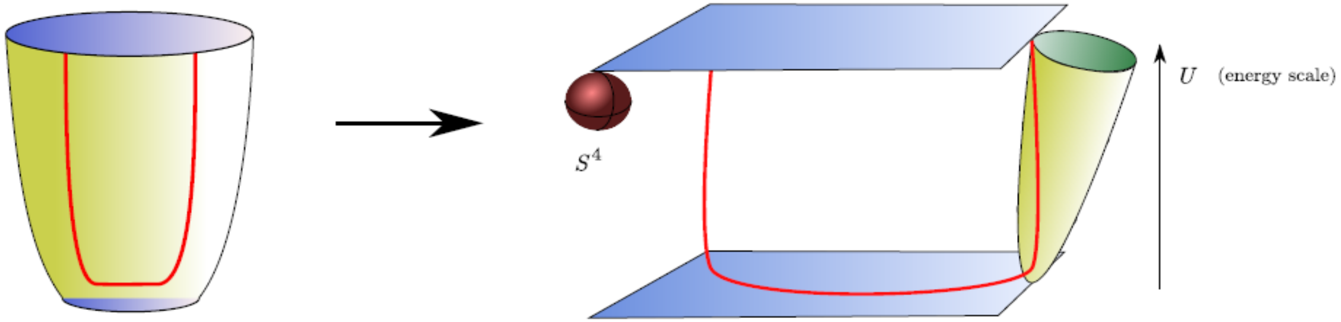
$$ds^2 = \left(\frac{U}{R}\right)^{3/2}\left[\eta_{\mu\nu}dX^\mu dX^\nu + f(U)d\theta^2\right] + \left(\frac{R}{U}\right)^{3/2}\left[\frac{dU^2}{f(U)} + U^2 d\Omega_4\right]$$

world-volume our 3+1 world

$f(U) = 1 - \left(\frac{U_\Lambda}{U}\right)^3$

θ is a compact Kaluza-Klein circle

U: radial direction bounded from below $U \geq U_\Lambda$



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5d coupling

4d coupling

glueball mass

String tension

- The gravity picture is **valid** only provided that $\lambda_5 >> R$

- At energies $E << 1/R$ the theory is **effectively 4d**.

- However it is not really QCD since $M_{gb} \sim M_{KK}$

- In the opposite limit of $\lambda_5 << R$ **we approach QCD**

- The profile of the D8 probe branes is determined by the solution of the e.o.m derived from the *DBI action* (note that the CS term does not contribute)

$$S_{D8} \propto \int d^4x d\tau\, \epsilon_4\, e^{-\phi}\sqrt{-\det(g_{D8})} \propto \int d^4x d\tau\, U^4 \sqrt{f(U) + \left(\frac{R}{U}\right)^3 \frac{U'^2}{f(U)}}\,.$$

- Just for the Wilson loop we use the Hamiltonian e.o.m

- The solution of this equation is

$$\frac{d}{d\tau}\left(\frac{U^4 f(U)}{\sqrt{f(U) + \left(\frac{R}{U}\right)^3 \frac{U'^2}{f(U)}}}\right) = 0\,.$$

$$\tau(U) = U_0^4 f(U_0)^{1/2} \int_{U_0}^{U} \frac{dU}{\left(\frac{U}{R}\right)^{3/2} f(U)\sqrt{U^8 f(U) - U_0^8 f(U_0)}}\,.$$

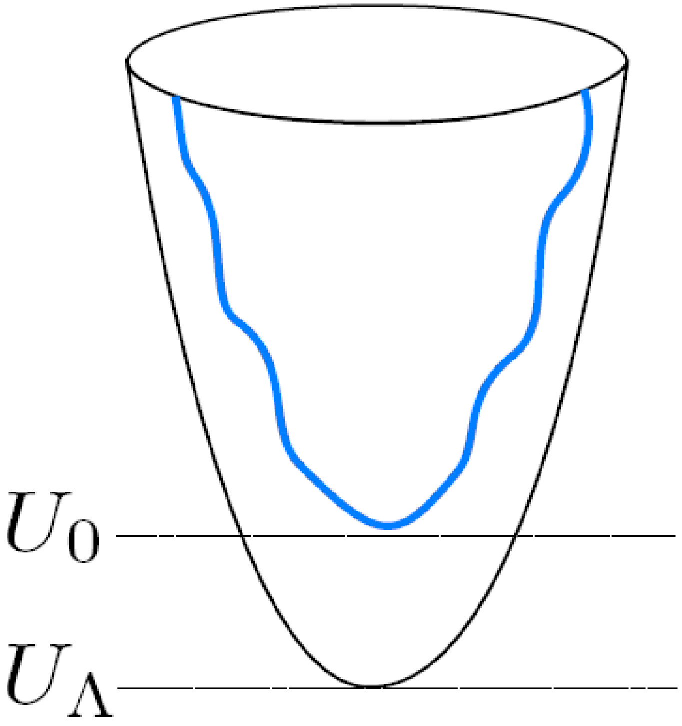
- For the special case of $u_0 = u_{KK}$ or ($u_0 = u_\Lambda$) the configuration is anti-podal. In general we find a *family of solutions* which is characterized by u0 or the separation distance L.

- The SS model deals only with the anti-podal case. We will see below the physical meaning of u0.

- It is useful to introduce a new set of coordinates (y,z) where the classical trajectory is along y=0

$$y = r\cos\theta \ , \qquad z = r\sin\theta \ ,$$

$$U^3 = U_{\rm KK}^3 + U_{\rm KK} r^2 , \quad \theta \equiv \frac{2\pi}{\delta\tau}\tau = \frac{3}{2}\frac{U_{\rm KK}^{1/2}}{R^{3/2}}\tau$$


U_0
U_Λ

Fluctuations of the branes and mesons

- The branes can fluctuate both in the *embedding*, namely along y(x,z) (recall y=0 is the classical configuration) as well as with the U(Nf) *gauge fields* that reside on the branes.
- Strings that start and end on the flavor branes correspond obviously to mesons (in the adjoint representation of the flavor group)

- Such mesons will carry the spin struture of the fluctuations of their enepoints and their spectra are determined by the fluctuations.

Fluctuations of *gauge fields* $\longrightarrow$ *vector* mesons

Fluctuations of the *embedding* $\rightarrow$ *scalar* mesons

Flavor gauge fields and vector mesons

- The gauge fields have in principle legs on nine directions (x0….x3,z, x5,…x8). Since we are intersted only in the fields which are singlets of the KK SO(5) symmetry we take only

$$A_\mu(x^\mu, z) = \sum_n B_\mu^{(n)}(x^\mu)\psi_n(z) \ ,$$

$$A_z(x^\mu, z) = \sum_n \varphi^{(n)}(x^\mu)\phi_n(z) \ .$$

- Expanding the DBI action we find

$$S_{D8} = -T\int d^9x\, e^{-\phi}\sqrt{-\det(g_{MN} + 2\pi\alpha' F_{MN})} + S_{CS}$$

$$= -\widetilde{T}(2\pi\alpha')^2\int d^4x dz \left[\frac{R^3}{4U_z}\eta^{\mu\nu}\eta^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} + \frac{9}{8}\frac{U_z^3}{U_{\rm KK}}\eta^{\mu\nu}F_{\mu z}F_{\nu z}\right] + \mathcal{O}(F^3)$$

- Upon substituting the mode expansion the action reads

$$S_{D8} = -\widetilde{T}(2\pi\alpha')^2 \int d^4x dz \sum_{m,n} \Bigg[\frac{R^3}{4U_z} F^{(m)}_{\mu\nu} F^{\mu\nu(n)} \psi_m \psi_n$$

$$+ \frac{9}{8}\frac{U_z^3}{U_{\mathrm{KK}}} \Big(\partial_\mu \varphi^{(m)} \partial^\mu \varphi^{(n)} \phi_m \phi_n + B^{(m)}_\mu B^{\mu(n)} \dot{\psi}_m \dot{\psi}_n - 2\partial_\mu \varphi^{(m)} B^{\mu(n)} \phi_m \dot{\psi}_n \Big) \Bigg]$$

- The $B^{(n)}_\mu$ sector takes the form

$$S_{D8} = -\widetilde{T}(2\pi\alpha')^2 \int d^4x dz \sum_{m,n} \Bigg[\frac{R^3}{4U_z} F^{(m)}_{\mu\nu} F^{\mu\nu(n)} \psi_m \psi_n + \frac{9}{8}\frac{U_z^3}{U_{\mathrm{KK}}} B^{(m)}_\mu B^{\mu(n)} \dot{\psi}_m \dot{\psi}_n \Bigg]$$

using the

$$Z \equiv \frac{z}{U_{\mathrm{KK}}} \,, \qquad K(Z) \equiv 1 + Z^2 = \left(\frac{U_z}{U_{\mathrm{KK}}} \right)^3$$

- We take ψ_n as an eigenfuction satisfying

$$-K^{1/3}\,\partial_Z\left(K\,\partial_Z\psi_n\right)=\lambda_n\psi_n$$

- With the normalization condition

$$\widetilde{T}(2\pi\alpha')^2R^3\int dZ\,K^{-1/3}\,\psi_n\psi_m=\delta_{nm}\ .$$

- We obtain

$$\widetilde{T}(2\pi\alpha')^2R^3\int dZ\,K\,\partial_Z\psi_m\,\partial_Z\psi_n=\lambda_n\delta_{nm}$$

- So that the 4d action is

$$S_{D8}=\int d^4x\sum_{n=1}^{\infty}\left[\frac{1}{4}F_{\mu\nu}^{(n)}F^{\mu\nu(n)}+\frac{1}{2}m_n^2\,B_\mu^{(n)}B^{\mu(n)}\right]$$

- Thus we have found that the $B_\mu^{(n)}$ are 4d massive vector fields with masses that are determined by the eigenvalue equation

- Next we consider the $\varphi^{(n)}$ fileds. Impose the normalization

$$(\phi_m, \phi_n) \equiv \frac{9}{4} \widetilde{T} (2\pi\alpha')^2 U_{\mathrm{KK}}^3 \int dZ \, K \, \phi_m \phi_n = \delta_{mn}$$

- We can choose

$$\phi_n = m_n^{-1} \, \dot{\psi}_n \ \ (n \geq 1) \quad \phi_0 = C/K$$

- For ϕ_0 which should be orthonormal to $\dot{\psi}_n$ we take so that

- Then we get

$$F_{\mu z} = \partial_\mu \varphi^{(0)} \phi_0 + \sum_{n \geq 1} \left(m_n^{-1} \partial_\mu \varphi^{(n)} - B_\mu^{(n)} \right) \dot{\psi}_n$$

- We can absorb into by using the gauge transformation

$$B_\mu^{(n)} \longrightarrow B_\mu^{(n)} + m_n^{-1} \partial_\mu \varphi^{(n)}$$

- Then the action is

$$S_{D8} = -\int d^4x \left[\frac{1}{2} \partial_\mu \varphi^{(0)} \partial^\mu \varphi^{(0)} + \sum_{n \geq 1} \left(\frac{1}{4} F_{\mu\nu}^{(n)} F^{\mu\nu(n)} + \frac{1}{2} m_n^2 B_\mu^{(n)} B^{\mu(n)} \right) \right]$$

- Thus the $\varphi^{(0)}$ are massless. They are the *Pions or Goldstone bosons* associated with the *spontaneous chiral symmetry breaking.*

- How comes we found a massless GB associated with the breaking of the ableina part of the global symmetry. This is the which is massive due to the U(1)A anomaly. However in the large Nc limit there is no anomaly and it should be masless

- It is important to note that the GB mode is massless also for the non antipotal case where u0 is not uL

Determining the spectrum

- We now solve the eigenvalue problem

$$-K^{1/3}\,\partial_Z\left(K\,\partial_Z\psi_n\right)=\lambda_n\psi_n\ ,$$

- The asymptotic behavior of

$$\psi_n(z)\sim\mathcal{O}(1)\quad\text{or}\quad\mathcal{O}(z^{-1})\qquad(\text{for}\quad z\longrightarrow\infty)$$

- We redefine the wave function

$$\widetilde{\psi}_n(Z)\equiv Z\,\psi_n(U_{\mathrm{KK}}Z)$$

- In term of which the equation is

$$K\partial_Z^2\widetilde{\psi}_n-\frac{2}{Z}\partial_Z\widetilde{\psi}_n+\left(\frac{2}{Z^2}+\lambda_nK^{-1/3}\right)\widetilde{\psi}_n=0$$

- We solve for the eigenvalues using the shooting method.

- Rewrite the equation as $\partial_\eta^2 \widetilde{\psi}_n + A\partial_\eta \widetilde{\psi}_n + B\widetilde{\psi}_n = 0$

Where $Z = e^\eta$

- Then we expand

$$A = -\frac{1+3e^{-2\eta}}{1+e^{-2\eta}} = \sum_{l=0}^{\infty} A_l\, e^{-\frac{2l}{3}\eta} \,,$$

$$B = \frac{2e^{-2\eta}}{1+e^{-2\eta}} + \lambda_n e^{-\frac{2}{3}\eta}(1+e^{-2\eta})^{-4/3} = \sum_{l=0}^{\infty} B_l\, e^{-\frac{2l}{3}\eta} \,.$$

- We get the following recursion relation

$$\frac{4l^2}{9}\alpha_l - \frac{2}{3}\sum_{m=1}^{l} m A_{l-m}\alpha_m + \sum_{m=0}^{l-1} B_{l-m}\alpha_m = 0$$

- This yields

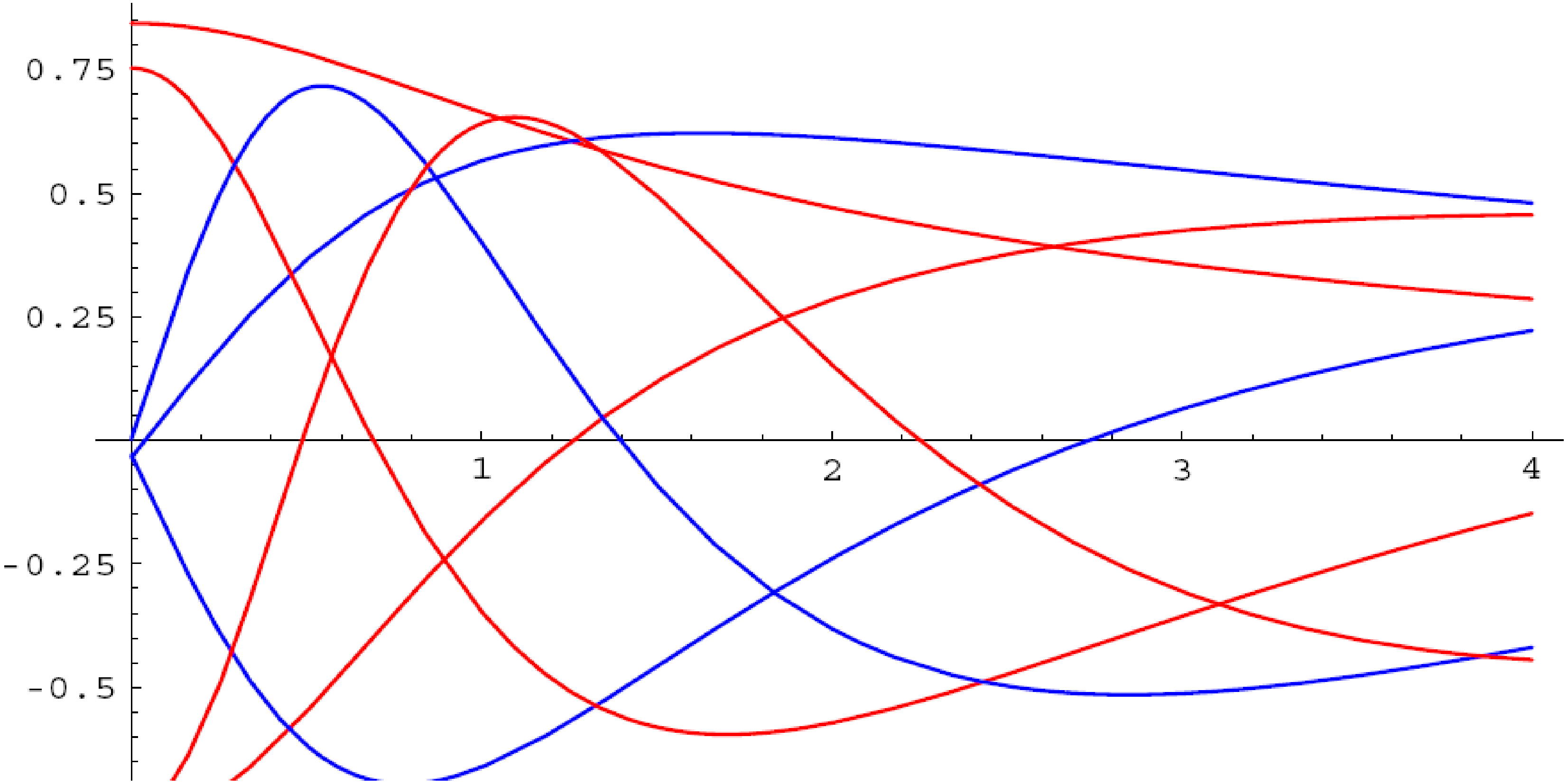
$$\alpha_1 = -\frac{9}{10}\lambda_n \ , \quad \alpha_2 = \frac{81}{280}\lambda_n^2 \ , \quad \alpha_3 = -\frac{1}{3} - \frac{27}{560}\lambda_n^3$$

- We use these data to shoot to z=0
- Since the equation is invariant under z<--> -z we impose even or odd boundary conditions
- These translate to even and odd values of *charge conjugation and parity*

$$\partial_Z \psi_n(0) = 0 \quad \text{or} \quad \psi_n(0) = 0$$

- The values of the *eigenvalues* found are

$$\lambda_n^{CP} = \ 0.67^{--} \ , \quad 1.6^{++} \ , \quad 2.9^{--} \ , \quad 4.5^{++} \ , \quad \cdots$$


0.75
0.5
0.25
-0.25
-0.5
1
2
3
4

	experiment	supergravity
$\frac{m^2(\rho(770))}{m^2(a_1(1260))}$	2.51	2.4
$\frac{m^2(\rho(1450))}{m^2(\rho(770))}$	3.56	4.3
$\frac{4\, g_{\rho\pi\pi}\, f_\pi^2}{m_\rho^2}$	2.03	1.31
$\frac{g_\rho\, g_{\rho\pi\pi}}{m_\rho^2}$	1.20	1.31

The puzzle of the quark mass

- We can also compute the spectrum for u_0 not equal to u_Λ . In fact we define the following mass parameter

- As observed above even in this case the *pions* are *massless*.

- It is known from the *GOR* relation that

$$M_\pi^2 = \frac{2m_q < \bar{q}q >}{f_\pi^2}$$

- Hence this parameter cannot be the QCD mass

Scalar mesons from the fluctuations of the embedding

- Fluctuations of the *embedding* in the general case of $u_0 > u_\Lambda$ yields the spectra of *scalars*.

- The only non singular formulation is in terms of the fluctuations of the u coordinate

$$u(x_4, x^\mu) = u_{cl}(x_4) + \xi(x_4, x^\mu)$$

- The corresponding e.o.m is

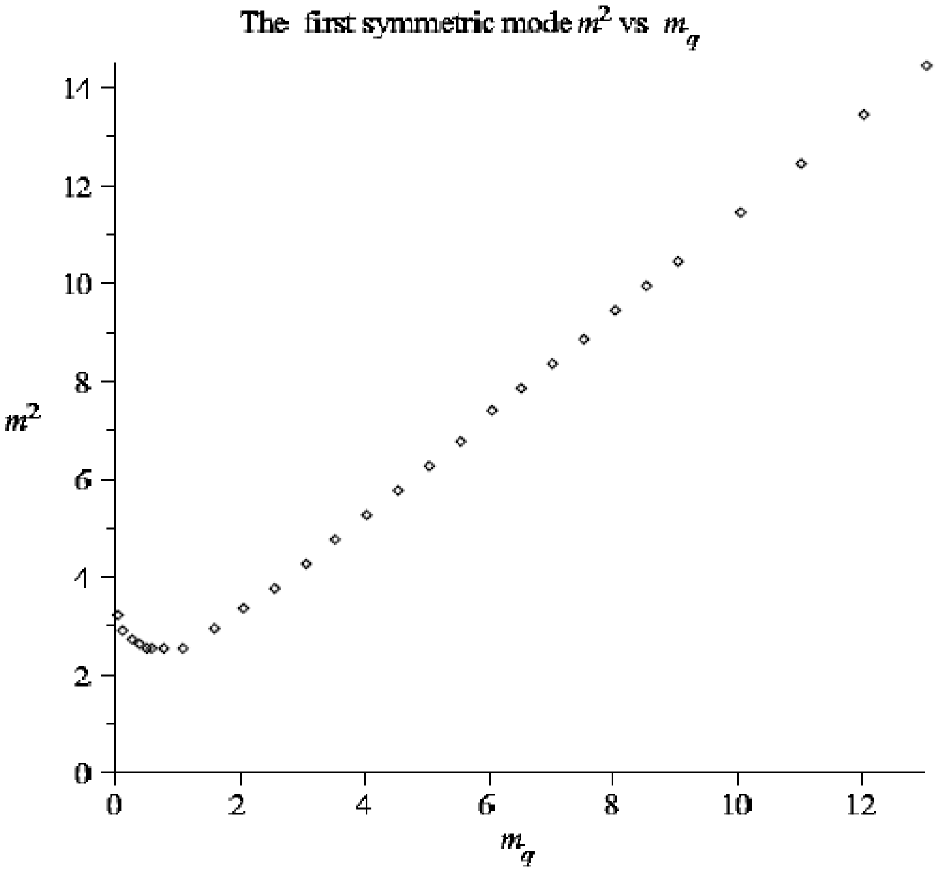
$$\begin{aligned}\partial_x^2 \xi_n &- (\frac{11}{u} + \frac{9}{uf}) u_x \partial_x \xi_n - \frac{f^2 u^8 m_n^2}{a_0} \xi_n \\ &+ \frac{(22u^{14} + 36a_0 + 6u^{11} - 24u^8 - 54a_0(u^3 + u^6) - 4u^5)}{a_0 u^{16} f^3} \xi_n = 0\end{aligned} \qquad (38)$$

- Using again the shooting method the spectra of scalars is computed as a function of both the radial excitation

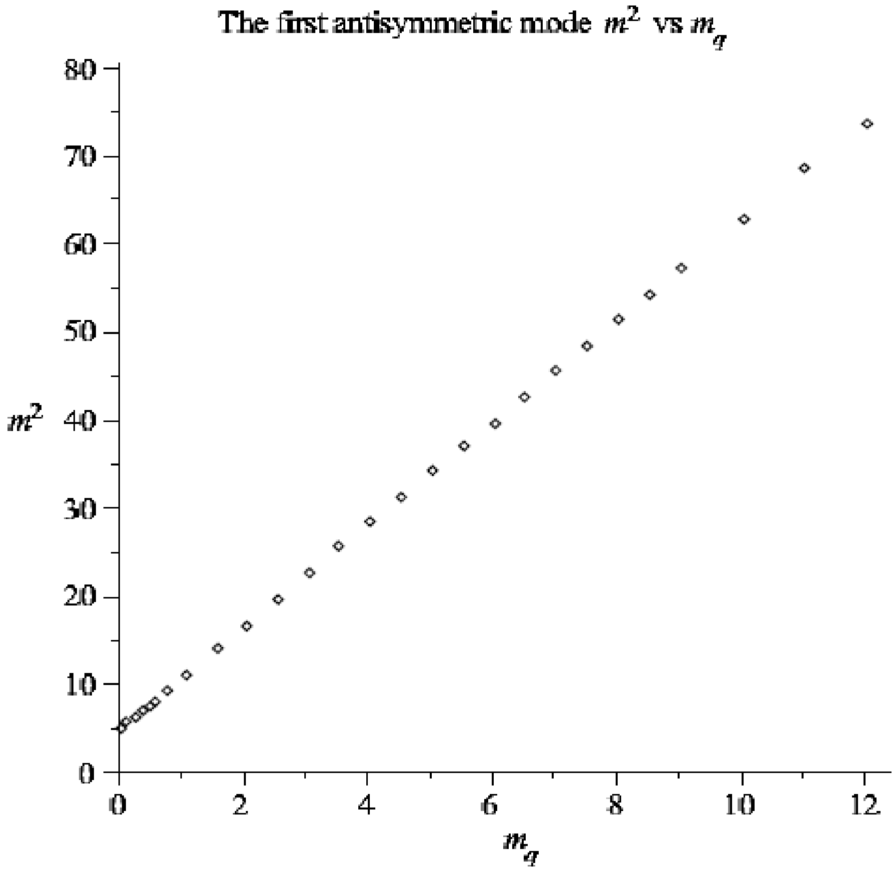
n and the mass parameter

$$m_q = \frac{1}{2\pi\alpha'}\int_{u_\Lambda}^{u_0}\sqrt{-g_{tt}g_{uu}}du = \frac{1}{2\pi\alpha'}\int_{u_\Lambda}^{u_0} f^{-1/2}(u)du$$

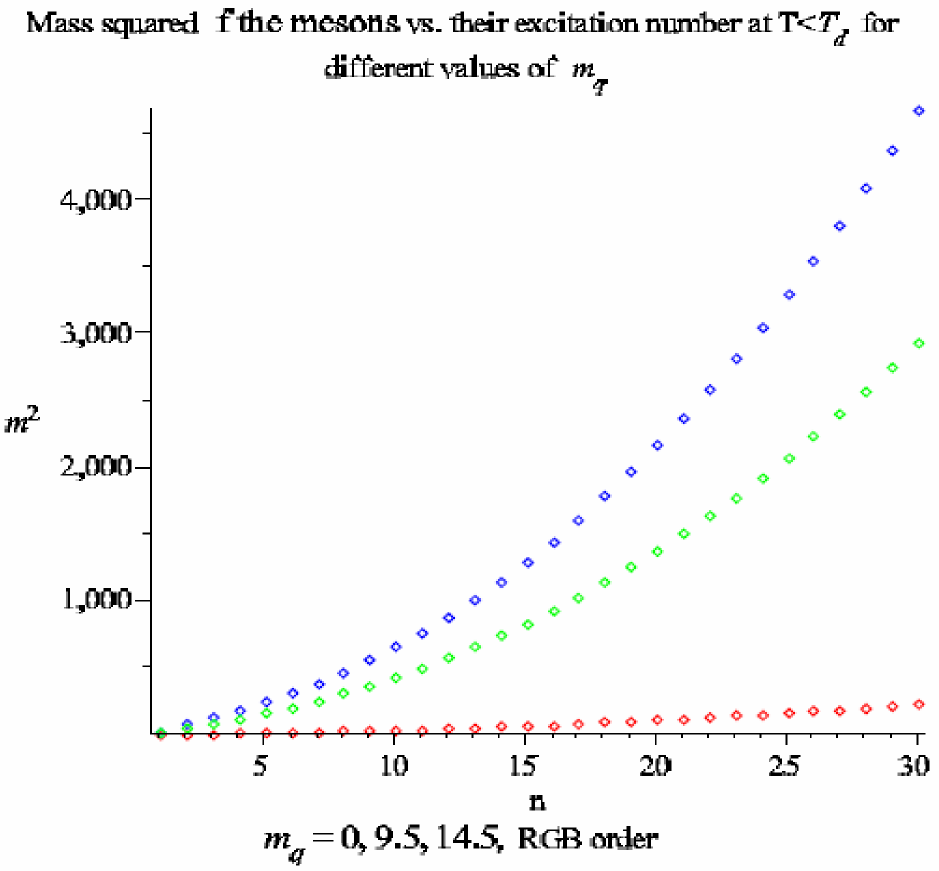
- From the dependence of the meson masses *constituent quark mass* on mq and form the fact that the pions are massless we deduce that mq is related to the and to the QCD mass.



The first symmetric mode m^2 vs m_q



The first antisymmetric mode m^2 vs m_q


Mass squared f the mesons vs. their excitation number at T<T_d for different values of m_q

Parity and charge conjugation of the scalar mesons

- The holographic operation of parity and charge conjugations are

$$(x_i, z) \to (-x_i, -z).$$

- From the demand that the CS is invariant under these discrete operations one finds

- Hence $$\xi(x, z) = \sum_n f_n(x^\mu)\xi_n(z)$$

$$\begin{array}{rccc} symmetric & \xi_n & \to & 0^{++}\ mesons \\ antisymmetic & \xi_n & \to & 0^{--}\ mesons \end{array}$$

- However in *nature there are no* *mesons!*

- In the context of the Sakai Sugimoto model one can address several other characteristics of Hadron physics. In particular the *chiral anomaly*,

decay of vector mesons to pions, baryons and more.

- We will come back to the model to describe the thermal phase diagram of QCD in lecture 4.

Summary of lecture 2.

- To summarize the Sakai Sugimoto model

admits

- Flavor chiral symmetry and a geometrical spontaneous breaking of this symmetry
- It yields resonable spectra of mesons and other hadronic states
- On the other hand it incorporate a zoo of undesired KK particles in the same mass scale as the Hadrons
- It has string coupling divergence in the UV