The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific and
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1943-4

Introductory School on Gauge Theory/Gravity Correspondence

19 - 30 May 2008

Non-AdS/non-CFT correspondence

Lecture 2

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1.2 The supergravity solution describing (the near-horizon region of) N D5 wrapped on $S^2 \subset CY_3$

Aim: find the SUGRA-solution describing the near horizon region of N D5-branes wrapped on a $S^2 \subset CY_3$ (CY_3 non-compact).

As suggested by MN a useful thing to do is to use lower dimensional GAUGED supergravities.

Let us consider a Dp-brane in 10d:

- the R-symmetry group is the isometry group of S^{8-p}, $SO(9-p)$ ($p=5$, $S^3 \to SO(4)$ in our case)
- the 10-dimensional supergravity fields coupling to the Dp-brane belong to the multiplet which is massless upon compactification of type II SUGRA on S^{8-p} $\longrightarrow$ enough to find a solution in the corresponding lower dimensional p+2-SUGRA and then lift it up back in 10d. ||| gauged supergravity

× Bosonic fields in a gauged SUGRA: metric, scalars, $SO(9-p)$ gauge fields (!) + n-forms possibly

• the solution is a <u>domain wall</u>, respecting some boundary conditions:

• the metric should look as D-brane WV at the boundary

• gauge fields must approach their FT values needed to perform the twist (need to gauge only $H \subset SO(9-p)$ entering the twist)

Flat D3's $\longrightarrow$ 5d GS: metric

Flat Dp($\neq 3$)'s $\longrightarrow$ (p+2)d GS: metric + scalars

Wrapped Dp's $\longrightarrow$ (p+2)d GS: metric + scalars + gauge fields

× <u>Our case</u> is 7d gauge SUGRA:

$\begin{cases} \text{metric}, \ 6 \text{ gauge fields } A_M \text{ (Adj of } SO(4)), \\ 10 \text{ scalars } T_{ij} \ (\boxed{} \text{ of } SO(4)), \ 2\text{-form potential} \end{cases}$

<u>Note</u>: the ansatz for A is dictated by the twist!

The 7d solution looks like

$$\begin{cases} ds^2 \sim b(r)\,(dx_{1,3}^2 + dr^2) + c(r)\,(d\theta_1^2 + \sin^2\theta_1\, d\phi_1^2) \\ T_{ij} \sim d(r)\,\delta_{ij} \\ A^3 \sim \cos\theta_1\, d\phi_1 \qquad (A^3 \leftrightarrow U(1) \subset SU(2)_L) \end{cases}$$

<u>Problem</u> : this solution has a <u>singularity</u> at $r=0$.

It turns out that a more general ansatz for the gauge connection smooths out the singularity :

$$A^1 \sim a(r)\, d\theta_1 \,,\quad A^2 \sim a(r) \sin\theta_1\, d\phi_1 \,,\quad A_3 \sim \cos\theta_1\, d\phi_1$$

(boundary condition : $a(r) \to 0$ when $r \to \infty$)

Plugging this ansatz in 7d SUGRA EoM one gets back a singularity free solution.

The solution should then be up-lifted in 10d on a S^3 (coordinates ψ, θ_2, ϕ_2).

<u>Note</u> : the twist implies the 7d geometry is non-trivially embedded in 10d one $\to$ <u>mix</u> between S^2 & S^3

The 10d regular solution reads:

$$\begin{cases} ds^2 = e^{\phi} dx_4^2 + e^{\phi} \alpha' g_s N \left[e^{2h} \left(d\theta_1^2 + \sin^2\theta_1 \, d\phi_1^2 \right) + d\rho^2 + \sum_{a=1}^{3} (\omega^a - A^a)^2 \right] \\ e^{2\phi} = \dfrac{\sinh 2\rho}{2 e^{2h(\rho)}} \\ F_{(3)} = 2\alpha' g_s N \prod_{a=1}^{3} (\omega^a - A^a) - \alpha' g_s N \sum_{a=1}^{3} F^a \wedge \omega^a \end{cases}$$

$\rho = \dfrac{r}{\sqrt{\alpha' g_s N}}$, $0 \le \theta_{1,2} \le \pi$, $0 \le \phi_{1,2} \le 2\pi$, $0 \le \psi \le 4\pi$

$A^1 = -\frac{1}{2} a(\rho)$, $A^2 = \frac{1}{2} a(\rho) \sin\theta_1 \, d\phi_1$, $A^3 = -\frac{1}{2} \cos\theta_1 \, d\phi_1$

$$e^{2h(\rho)} = \rho \coth 2\rho - \frac{\rho^2}{\sinh^2 2\rho} - \frac{1}{4}, \quad a(\rho) = \frac{2\rho}{\sinh 2\rho}$$

$F^a = \nabla A^a \equiv dA^a + \epsilon^{abc} A^b \wedge A^c$

ω^a are left-invariant one-forms of S^3:

$\omega^1 = \frac{1}{2} \left(\cos\psi \, d\theta_2 + \sin\psi \sin\theta_2 \, d\phi_2 \right)$

$\omega^2 = -\frac{1}{2} \left(\sin\psi \, d\theta_2 - \cos\psi \sin\theta_2 \, d\phi_2 \right)$

$\omega^3 = \frac{1}{2} \left(d\psi + \cos\theta_2 \, d\phi_2 \right)$

For later convenience let us also write down the form of the RR 2-form potential:

$$C_{(2)} = \frac{1}{4}\alpha' g_s N \Big[(\psi + \psi_0)(\sin\theta_2\, d\theta_2 \wedge d\phi_2 - \sin\theta_1\, d\theta_1 \wedge d\phi_1) \cdot$$

$$+ \cos\theta_1 \cos\theta_2\, d\phi_1 \wedge d\phi_2 \Big] + \frac{1}{2}\alpha' g_s N a(\rho) \cdot$$

$$\cdot \left[d\theta_1 \wedge \omega^1 - \sin\theta_1\, d\phi_1 \wedge \omega^2 \right]$$

as well as the asymptotic behavior of the non-trivial functions $a(\rho)$, $h(\rho)$ and the dilaton

$$\begin{cases} a(\rho) \underset{\rho\to\infty}{\sim} \rho\, e^{-2\rho} \to 0 & a(\rho) \underset{\rho\to 0}{\sim} 1 \\ e^{2h(\rho)} \underset{\rho\to\infty}{\sim} \rho & e^{2h(\rho)} \underset{\rho\to 0}{\sim} \rho^2 \\ e^{2\phi} \underset{\rho\to\infty}{\sim} \rho^{-\frac{1}{2}}\, e^{2\rho} & e^{2\phi} \underset{\rho\to 0}{\sim} 1 \end{cases}$$

Comments :

1. The function $a(\rho)$ plays a rôle at short distances only : at large ρ $A_1, A_2 \to 0$ and the solution becomes essentially the same of the singular one. Note that $a(\rho)$ is the function de-singularizing the solution.

2. The function $a(\rho)$ does NOT preserve the twist. This is ok, though. The region $\rho \to \infty$ corresponds, holographically, to the UV of the dual gauge theory (see later) : this is where the D5 spectrum has been computed, and the twist performed. In the IR, the theory becomes strongly coupled and the relevant d.o.f. do not coincide with the UV ones.

3. For later pourposes we also need to pint-point the S^2 the D5 "wraps" in the 10d metric (recall the mixing between S^3 and the obvious 4d S^2 parametrized by θ_1, ϕ_1 induced by the twist : this is manifest by the presence of the gauge connection A^a in the 10d metric).

At large ρ the angular part of the metric is that of $T^{1,1} = [SU(2) \times SU(2)]/U(1)$ for which the parametrization of the non-trivial S^2 is known

$$S^2 : \quad \theta_1 = \pm \theta_2 , \quad \phi_1 = -\phi_2 , \quad \psi = \text{const.} \qquad (22.1)$$

any!

The SUSY cycle is the one with minimal volume within this class. This gives :

not the period which is 4π

$$S^2 : \quad \theta_1 = -\theta_2 , \quad \phi_1 = -\phi_2 , \quad \psi = 0 \mod 2\pi$$

$$S^2 : \quad \theta_1 = \theta_2 , \quad \phi_1 = -\phi_2 , \quad \psi = \pi \mod 2\pi$$

$$\text{Vol}(S^2) = \int_{S^2} e^{-\phi} \sqrt{\det G} \sim 4\, e^{2h(\rho)} + (a(\rho)-1)^2$$

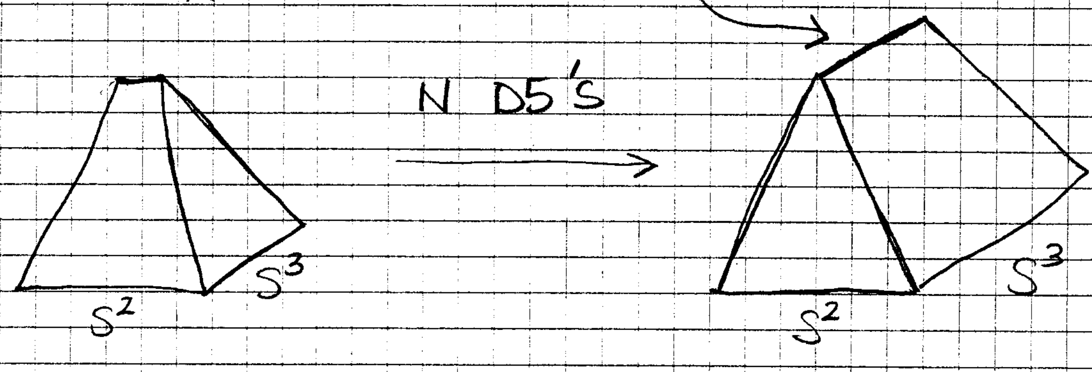
4) Let us check the claim that there is no singularity at $\rho = 0$. The dilaton is fine ($e^{2\phi} \to 1$), so problems may come from the angular part of the metric:

$$ds_5^2 = \alpha' g_s N \Big[(\cos\psi \sin\theta_2 \, d\phi_2 - \sin\psi \, d\theta_2 - \sin\theta_1 \, d\phi_1)^2 +$$
$$+ (\sin\psi \sin\theta_2 \, d\phi_2 + \cos\psi \, d\theta_2 + d\theta_1)^2 +$$
$$+ (d\psi + \cos\theta_1 \, d\phi_1 + \cos\theta_2 \, d\phi_2)^2 \Big]$$

This is finite and topologically a S^3!

Exercise: using the S^2 and S^3 parametrization given previously one finds $R(S^2) = 0$ and $R(S^3) = \sqrt{\alpha' g_s N}$.

because $a(\rho) \neq 0$ at $\rho = 0$



geometric TRANSITION

Note: the 10d metric is $\sim \mathbb{R}^{1,6} \times S^3$ at $\rho \sim 0$

1.3 Gauge / gravity duality checks

The D-brane dynamics is described at low energy by the DBI + WZ action : it describes the dynamics of open string fields (A_μ, λ, X^i, ...) in the background of closed string field (G_{MN}, B_{MN}, A_{p+1}, ...) .

The DBI + WZ action of a wrapped D5 is a SYM theory at low enough energy : take the action, choose a static gauge $X^\mu = x^\mu$ $\mu = 0, \ldots, 5$, $X^i = X^i(x^\mu)$ $i = 6, \ldots, 9$ & expand the action up to terms <u>quadratic</u> in the derivatives of the WV fields .

$$\int d^6 x \, e^{-\phi} \sqrt{G + F + B} = \left(\overset{\text{gauge field}}{F} = F(x^0, \ldots, x^3) \right)$$

$$= \int d^4 x \, \underbrace{\left[e^{-\phi} \int_{S^2} dA_2 \sqrt{(G+B)_{S^2}} \right]}_{\equiv \frac{1}{g^2_{YM}}} \sqrt{(G+F+B)_{\mathbb{R}^{1,3}}}$$

$$\frac{\pi}{g_{YM}^2} = \frac{1}{2(2\pi)^3 \alpha'^{\,} g_s} \int_{S^2} e^{-\phi} \sqrt{G} = \frac{N}{16\pi^2} Y(\rho)$$

$$\theta_{YM} = -\frac{1}{2\pi\alpha' g_s} \int_{S^2} C_2 = -N \psi_0$$

$$Y(\rho) = 4 e^{2h(\rho)} + (a(\rho)-1)^2 = 4\rho \tanh \rho$$

The eq. for the gauge coupling shows the typical UV/IR gauge/gravity relation:

$$\frac{1}{g_{YM}^2 N} \sim \frac{\rho}{4\pi^2} \quad \text{for} \quad \rho \to \infty \qquad \text{UV}$$

$$\frac{1}{g_{YM}^2 N} \sim 0 \quad \text{for} \quad \rho \to 0 \qquad \text{IR}$$

- The $U(1)_R$ symmetry breaking pattern

From above formula it is clear that $U(1)_R$ transformations are realized as shifts in ψ in the dual supergravity background.

• The $U(1)_R \to \mathbb{Z}_{2N}$ breaking is triggered by instantons and should be visible already in the UV, $\rho \to \infty$. The MN solution is NOT invariant under shifts in ψ. However, for $\rho \to \infty$ $A_1, A_2 \to 0$ and the metric IS invariant while C_2 is NOT.

$$C_2\Big|_{S^2_{UV}} \sim (\psi + \psi_0) \sin\theta \, d\theta \wedge d\phi$$

NOT fixed anymore!

$$\frac{1}{2\pi\alpha g_s} \int_{S^2_{UV}} C_2 = N(\psi + \psi_0)$$

This flux is periodic of period 2π so shifts in ψ such that

$$\psi \to \psi + \frac{2\pi}{N} k$$

are symmetries of the (UV!) solution. Under a $U(1)_R$ transformation with parameter ε $\theta_{YM} \to \theta_{YM} - 2N\varepsilon$, so $\psi \to \psi + 2\varepsilon$. This shows that the UV background encodes the $U(1)_R \to \mathbb{Z}_{2N}$ anomalous breaking!

• **Note**: in QFT the $U(1)_R$ is broken to $\mathbb{Z}_{2N}$ by instantonic effects: these non-perturbative effects are already captured at the classical supergravity level.

Comment: we can see explicitly the rôle of instantons in the anomaly by considering an instantonic probe, i.e. a ED1-brane wrapped on S^2

$$S_{ED1} = \frac{1}{2\pi\alpha' g_s}\left[\int_{S^2} e^{-\phi}\sqrt{G} - i\int_{S^2} C_2\right]$$

$$= \frac{8\pi^2}{g_{YM}^2} - i\,\theta_{YM}$$

shifts $\theta_{YM} \to \theta_{YM} + 2\pi$ leave the path integral invariant

• What about the $\mathbb{Z}_{2N} \to \mathbb{Z}_2$ breaking and the gaugino condensation? This is also encoded in the supergravity background.

The full solution is no more invariant under $\psi \to \psi + \frac{2\pi}{N} k$ but, due to $a(\rho)$, which multiplies $\cos\psi$ and $\sin\psi$ terms, only under

$$\psi \to \psi + 2\pi k$$

which is nothing but a $\mathbb{Z}_2$ θ_{YM}-angle transformations!

<u>Comment</u>: the MN solution describes SYM in a given vacuum. What about the others, physically equivalent, N-1 vacua?

• From gauge theory side they are described by different (allowed) values of θ_{YM} or, equivalently, by the same θ_{YM}-angle but with the gaugino condensate picking-up a phase.

There exist in fact N regular, large ρ equivalent, MN-like solutions $\to$ they can be generated in the 7d gauged supergravity by a $SU(2)_L$ gauge transformations (recall $U(1)_L = U(1)_R$)

$$A' = e^{i\varepsilon\sigma_3} A e^{-i\varepsilon\sigma_3}$$

which correspond to different mixing between S^2 and S^3. One gets for the calibrated 2-cycle

$$S^2 : \quad \theta_1 = -\theta_2 , \quad \phi_1 = -\phi_2 , \quad \psi = 2\varepsilon \mod 2\pi$$

Nothing changes but the θ_{YM} angle which is now

$$\theta_{YM} = -N(\psi_0 + 2\varepsilon)$$

For $\varepsilon = \pi k/N$ $k = 0, \dots, N-1$ one describes the N different vacua, each with its own $\mathbb{Z}_2$ symmetry

The supergravity solution is pretty similar, but with a phase for $a(\rho)$:

$$a(\rho) \rightarrow a(\rho) e^{2i\varepsilon}$$

- The gaugino condensate

All what we have seen seems to suggest that $a(\rho)$ should be dual (in the AdS/CFT sense) to the gaugino condensate $\langle \lambda\lambda \rangle$:

- it is responsible for $\mathbb{Z}_{2N} \to \mathbb{Z}_2$
- it is relevant in the IR only
- it transforms as $\langle\lambda\lambda\rangle$ under $U(1)_R$ transformations
- its phase distinguishes among the various vacua

Using standard AdS/CFT techniques (VEV/operator correspondence) one finds that a couples to the fermionic bilinear

$$A_{ij}\, \overline{\Psi}\, \gamma^{\mu}\, \Gamma^{ij}\, \Psi \longrightarrow a\, \lambda\lambda$$

coupling on the D5 WV

$$a \simeq \frac{Y}{\sqrt{2\rho}} + 2C\rho e^{-2\rho}$$

asymptotic sol. of 2^{nd} order eq. for $a(\rho)$ in 7d SUGRA

VEV of dual $\mathcal{O}$ (norm. sol.)

deformation ($Y=0$ in MN) of dual $\mathcal{O}$ (non-norm. sol.)

The gaugino bilinear is a protected operator since it is the lowest component of $W_\alpha W^\alpha$ whose top component is the trace of the E-M tensor which is a conserved operator so it does not acquire anomalous dimension through renormalization

$$a(\rho) = \frac{N\Lambda^3}{\mu^3} \sim \langle\lambda\lambda\rangle$$

μ: subtraction scale

Note: the S^3 radius at the tip is proportional to $a(\rho)$, hence to Λ; the geometric transition is a IR effect and set the scale of the FT non-perturbative dynamics!

- The β-function

The above identification, being a relation between a ρ-dependent supergravity field and a protected operator, can be exploited to derive an energy / radius relation that can be used in

$$\frac{1}{g^2} \sim \int_{S^2} e^{-\phi} \sqrt{G}$$

to extract the β-function. This reads

$$\beta(g) = \frac{\partial g}{\partial \ln\left[\frac{\mu}{\Lambda}\right]} = \frac{\partial g}{\partial \rho} \frac{\partial \rho}{\partial \ln \frac{\mu}{\Lambda}}$$

Evaluating the two terms by (first) disregarding subleading corrections in $Y(\rho)$ and $a(\rho)$ for $\rho \to \infty$ one gets

$$\beta(g) = -3 \frac{N g^3}{16\pi^2} \left(1 - \frac{N g^2}{8\pi^2}\right)^{-1}$$

which is the exact NSVZ β-function!

Some (preliminary) comments :

1. The operator / field correspondence $a \leftrightarrow \lambda\lambda$ is insensible to a redefinition by means of an analytic function of the gauge coupling. One can show that allowing for $f(g) \neq 1$ would lead to a new β

$$\beta(g) = -3 \frac{Ng^3}{16\pi^2} \left(1 - \frac{Ng^2}{8\pi^2} - \frac{N \,\partial f/\partial g}{2f} \frac{g^3}{8\pi^2} \right)^{-1}$$

which differs from the NSVZ beyond 2-loops only, meaning that the duality respects the universality of the 2-loop coefficient.

2. Taking into account subleading non-perturbative corrections one would get a new β-function where the (physically misterious) pole of the NSVZ β-function would be removed by some fractional instanton contributions.