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Introductory School on Gauge Theory/Gravity Correspondence

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Non-AdS/non-CFT correspondence
Lecture 4

M. Bertolini
SISSA
Trieste
Italy

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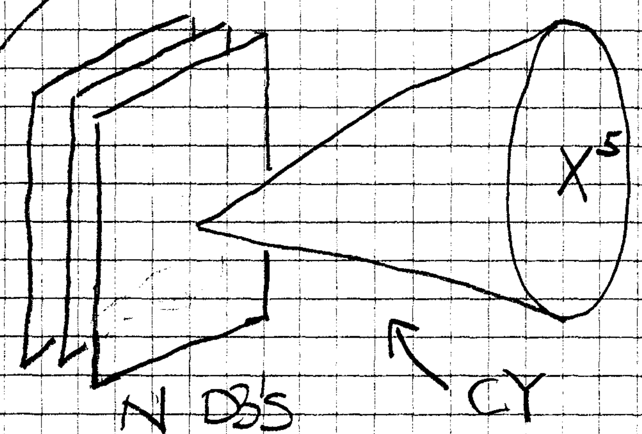
2. D-branes at CY singularities

The original AdS/CFT duality originates by considering D3-branes in $\mathbb{R}^6$, which is a (non-singular) real cone over S^5. The idea is to replace $\mathbb{R}^6$ with a CY with a conical singularity where we place the D3-brane



$S^5 \to X^5$

The CY condition implies that X^5 is a Sasaki-Einstein manifold:

Einstein: $R_{\alpha\beta} = \Lambda g_{\alpha\beta}$, from CY Ricci-flatness

Sasaki: from CY being Kähler, $g_{i\bar{j}} \sim \partial_i \partial_{\bar{j}} K(X, \bar{X})$

$\to$ locally the metric is: $ds^2 = dr^2 + r^2 ds^2(X^5)$

<u>**Remark**</u> : whenever $X^5 \neq S^5$ the 6d conical singularity is a true metric singularity. This means that typically there are <u>topologically non-trivial</u> 2 & 3 cycles collapsing at the tip.

The possibility for string theory to be well defined (and quantizable, in principle) in such singular backgrounds relies on extra non-geometric moduli one can play with. In particular:

$$S^2 : \; G_{(ij)} \quad + \quad B_{[ij]} \quad \text{(NSNS 2-form field)}$$

$$\int_{S^2} \sqrt{G} \quad \longrightarrow \quad \int_{S^2} \sqrt{G+B}$$

When discussing D-branes at such singularities, one should think of a <u>background</u> which, besides $G^0_{\mu\nu}$ (and ϕ_0), includes also constant NSNS fluxes:

$$b^0_I = \int_{S^2_I} B_2 \qquad \text{non-trivial collapsing 2-cycles}$$

The SUGRA solution has a metric & RR F_5:

$$\begin{cases} ds^2 = h^{-1/2}(r)\, dx^2_{1,3} + h^{1/2}(r)\left[dr^2 + r^2 ds^2(X^5)\right] \\ F_5 = \mathcal{F}_5 + {}^*\mathcal{F}_5 \ , \quad \mathcal{F}_5 = \dfrac{16 N \pi^4 \alpha'^2}{V(X_5)}\, v(X^5) \end{cases}$$

where:

$$h(r) = 1 + \frac{L^4}{r^4} \ , \quad L^4 = 4\pi g_s N \frac{V(S^5)}{V(X^5)} \alpha'^2 \qquad V(S^5) = \pi^3$$

$$\frac{1}{(4\pi\alpha')^2} \int_{X_5} F_5 = N \ , \quad v(X^5) = \text{volume form on } X^5$$

The decoupling limit is obtained as usual taking $\alpha', r \to 0$ with $u = r/\alpha'$ fixed

$$ds^2_{nh} = \frac{\alpha'^2 u^2}{L^2} dx_\mu dx^\mu + L^2 \frac{du^2}{u^2} + L^2 ds^2(X^5)$$

This metric is the direct product of two spaces with constant curvature, $AdS_5 \times X^5$, with same radius L. Notice the AdS factor!

The conjecture is then:

IIB string theory on $AdS_5 \times X^5$ with N units of F_5-flux $\Longleftrightarrow$ (some) $\mathcal{N}=1$ 4d SCFT

<u>Question</u>: what is the dual $\mathcal{N}=1$ SCFT?

The answer is not known in general, but for a (large & interesting enough) subset of CY singularities:

- orbifolds of $\mathbb{R}^6$: $\mathbb{R}^6/\Gamma$, $\Gamma \subset SU(4)$
- Toric CY singularities (the isometry group has <u>at least</u> a $U(1)^3$ factor)

<u>Orbifolds</u> are the simplest, since string theory can be quantized in such backgrounds and the D-brane open string spectrum computed. Typically one gets Γ-projections of $\mathcal{N}=4$ theory which "lives" in the covering space.

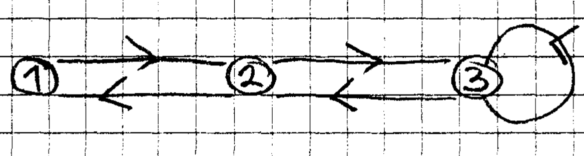
$\Gamma \in SU(2)$: $\mathcal{N}=2$ supersymmetry

$\Gamma \subset SU(3)$: $\mathcal{N}=1$ supersymmetry

- Recently, a new all machinery, DIMER techniques, has been developed to perform the above program for any Toric CY singularity.

Basic reason : Toric CY's can be obtained by "partial resolutions" of $\mathbb{C}^3/\mathbb{Z}_N \times \mathbb{Z}_M$ whose dual effect corresponds to give MASS or turning on VEV's for some fields.

- The 4d $\mathcal{N}=1$ SCFT one gets is a so-called QUIVER GAUGE THEORY :



$+ \quad W = W(X_{ij})$

each node : $SU(N)$ factor

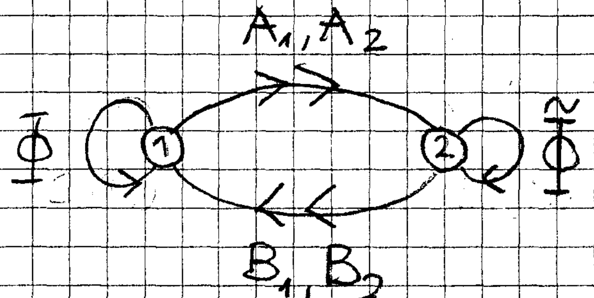
each arrow : a bi-fundamental χ_{SF}, X_{ij}

W : superpotential made of X_{ij} gauge-inv. $\mathcal{O}$

<u>Example</u> : from $X^5 = S^5/\mathbb{Z}_2$ to $X^5 = T^{1,1}$

The QGT living on D3 at $\mathbb{R}^6/\mathbb{Z}_2$ ($z_1 \to -z_1$, $z_2 \to -z_2$, $z_3 \to z_3$) is a $\underline{\mathcal{N}=2}$ theory

$\nwarrow$ $\mathbb{Z}_2 \subset SU(2)$



$\mathcal{N}=1$ language!

$$+ \quad W \sim g \,\mathrm{Tr}\, \Phi (A_1 B_1 + A_2 B_2) + g \,\mathrm{Tr}\, \tilde{\Phi} (B_1 A_1 + B_2 A_2)$$

One can flow to the QGT dual to $T^{1,1}$ by adding a mass deformation to the above :

$$\frac{m}{2} \left(\mathrm{Tr}\, \Phi^2 - \mathrm{Tr}\, \tilde{\Phi}^2 \right)$$

<u>Note</u> : this breaks $\mathcal{N}=2 \to \mathcal{N}=1$ explicitely.

Integrating the adjoints $\Phi, \tilde{\Phi}$ out one gets a W at low energy

$$W = \frac{g^2}{2m} \left[\mathrm{Tr}\, (A_1 B_1 A_2 B_2) - \mathrm{Tr}\, (B_1 A_1 B_2 A_2) \right]$$

$$\sim \varepsilon^{ij} \varepsilon^{kl}\, \mathrm{Tr}\, A_i B_k A_j B_l$$

$$\begin{tikzcd} ① \arrow[r, bend left, "A_1, A_2"] & ② \arrow[l, bend left, "B_1, B_2"] \end{tikzcd} \quad , \quad W = \lambda\, \varepsilon^{ij} \varepsilon^{kl}\, \mathrm{Tr}\, A_i B_k A_j B_l$$

$R[W] = 2 \qquad R[A] = R[B] = \frac{1}{2}$

$\Delta = \frac{3}{2} R$

$\Delta[W] = 3 \qquad \Delta[A] = \Delta[B] = \frac{3}{4}$

marginal operator

non-trivial RG-flow

The geometric counterpart of the above operation is a blow-up of the $S^5/\mathbb{Z}_2$ singularity into the smooth $T^{1,1}$.

A similar result was obtained in a different way, starting from $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ and the corresponding QGT: by partial resolution (which in this case corresponds to give a VEV to some θ) the above orbifold gets deformed to $C(T^{1,1}) \equiv$ conifold and the QGT to the conifold one.

Let me make a few more comments on the QGT dual to $T^{1,1}$:

- One can check that given the anomalous dimensions on the χ_{SF}

$$\Delta[A,B] = 1 + \frac{1}{2}\gamma_{A,B} = \frac{3}{4} \longrightarrow \gamma_{A,B} = -\frac{1}{2}$$

the β-functions vanish

$$(3-2\Delta_i) \overset{SCFT}{=} (3-3R_i)$$

$$\beta\left(\frac{8\pi^2}{g^2}\right) = \underbrace{3T(G)}_{\equiv N} - \sum_i \underbrace{T(R_i)}_{\equiv 1/2}\underbrace{(1-\gamma_i)}_{\equiv (3-2\Delta_i)}$$

$$\begin{cases} \beta_1 = 3N - \frac{1}{2}4N\left(1+\frac{1}{2}\right) = 3N - 2N\cdot\frac{3}{2} = 0 \\ \beta_2 = \quad '' \quad = \quad '' \quad = 0 \end{cases}$$

$\Rightarrow$ one can proceed in the opposite way: observe that the theory must flow to a non-trivial IR fixed point because of the AdS-dual background, impose $\beta_i = 0$ and get the anomalous dimensions

• One might ask about the W and its meaning. From each $SU(N)$ factor p.o.v. the theory without W is SQCD with $N_F = 2N_C$. This theory does flow to an IR-fixed point with an anomaly-free $U(1)_R$ for which

$$\Delta_{\chi_{sf}} = \frac{3}{2} R_{\chi_{sf}}$$

The symmetries of the problem suggest that $R[A_1] = R[A_2] = R[B_1] = R[B_2]$ and imposing the vanishing of the β-functions one gets

$$\gamma_A = \gamma_B = -\frac{1}{2}$$

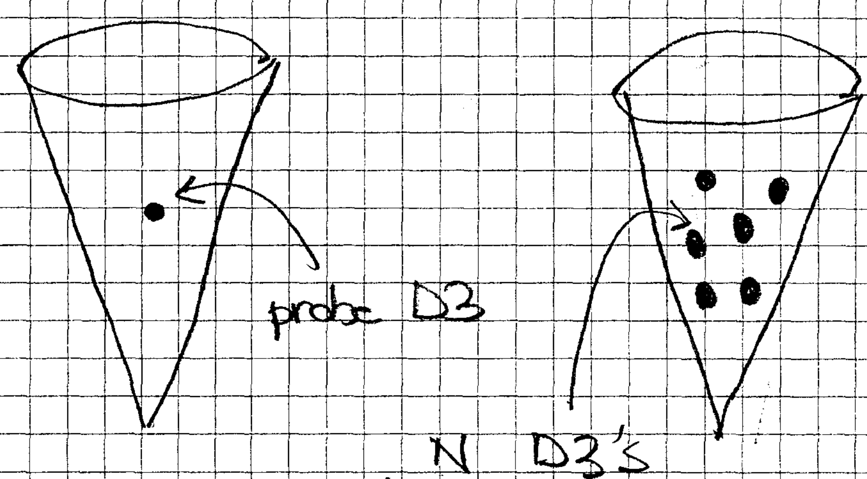
The quartic W has hence $R=2$ and $\Delta=3$ so it is a MARGINAL perturbation of the IR fixed point.

In fact, there are two marginal operators, the superpotential and the difference of the kinetic energies of the two gauge factors.

A powerful hint to check for the validity of the proposed dual gauge theory stems from the observation that

"the space transverse to the branes should describe the MODULI SPACE of the gauge theory"

In other words, the D-brane gauge theory has a moduli space which corresponds to the configuration space of the D-brane on the CY



→ D-branes moduli space consists of N copies of the manifold (up to identifications)

Ex. 1 : 1 D3 on $\mathbb{R}^6$ $\rightarrow$ $U(1)$ $\mathcal{N}=4$ SYM

There are 6 scalars and no potential so the moduli space is precisely $\mathbb{R}^6$

N D3 on $\mathbb{R}^6$ $\rightarrow$ $U(N)$ $\mathcal{N}=4$ SYM

There are 6 $N \times N$ scalar matrices. The D & F-term eqs. force the matrices to be diagonal, hence only 6N eigenvalues can be $\neq 0$ $\rightarrow$ $\mathbb{R}^{6N}$, which is the configuration space of N points in $\mathbb{R}^6$

Ex. 2 : 1 D3 on $C(T^{1,1})$ $\rightarrow$ $U(1) \times U(1)$ $\mathcal{N}=1$ with 4 charged fields

$$A_{1,2} = (1,-1) \quad \& \quad B_{1,2} = (-1,1)$$

No F-term eqs. ($W=0$); the two D-term eqs. give both something like

$$|A_1|^2 + |A_2|^2 = |B_1|^2 + |B_2|^2$$

The diagonal U(1) is decoupled while the other U(1) acts as

$$A_i \to e^{i\alpha} A_i \quad , \quad B_i \to e^{-i\alpha} B_i$$

$\Rightarrow$ 6 d.o.f. in total. The gauge inv. objects which parametrize the moduli space are $Z_{ij} = A_i B_j$; these 4 are not l.i. since

$$(A_1 B_1)(A_2 B_2) = (A_1 B_2)(A_2 B_1)$$

which can be rewritten as

$$\det Z_{ij} = 0$$

conifold eq!

$$Z_1 Z_2 - Z_3 Z_4 = 0$$

$Z_1 = A_1 B_1$, $Z_2 = A_2 B_2$, $Z_3 = A_1 B_2$, $Z_4 = A_2 B_1$

N D3 on $C(T^{1,1})$ → this is a bit more complicated. The presence of W plays a crucial rôle in the matching and making the moduli space as N copies of the conifold.

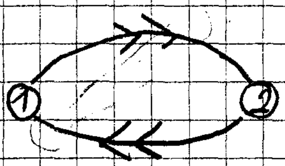
Check: take the N D3 out of the origin and see what happens. One expects $\mathcal{N}=4$ $SU(N)$. The quartic W, upon higgsing of one linear combination of the fields provide the cubic $W \sim \mathrm{Tr}\, X[Y,Z]$ of $N=4$ SYM!

This is all about SCFT with reduced supersymmetry. What about breaking conformal invariance in this set-up?

The singularity provides us an answer: whenever there is a singularity in a CY, there exist a particular kind of D-branes, called FRACTIONAL D-branes, whose low-energy effective dynamics is described by a non-conformal gauge theory.

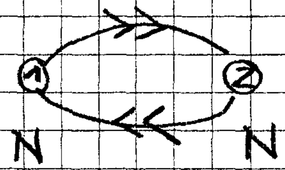
Fractional D3-branes (QGT perspective): independent anomaly-free rank assignments on the quiver (modulo regular branes) $\to$ they correspond to all possible anomaly-free subquivers one can build

Ex. 1



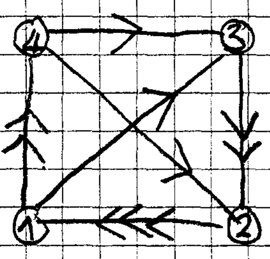
$T^{1,1}$

N D3



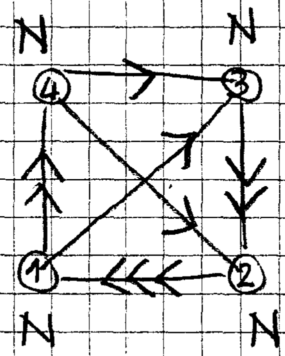
M $D3_f$

Ex. 2

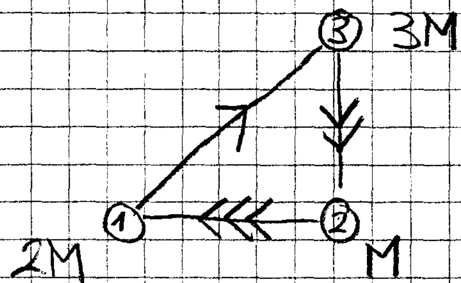


$Y^{2,1}$

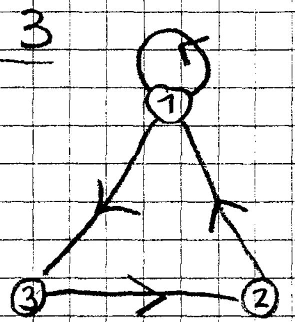
N D3



M $D3_f$

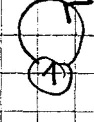


Ex. 3



SPP

M $D3_{f_1}$



M $D3_{f_2}$

Fractional D3-branes (geometric perspective):

D5-branes wrapped on collapsed 2-cycles. There are as many as independent topologically non-trivial collapsing 2-cycles.

The way the S^2 is embedded in the CY implies:

- amount of SUSY preserved on the D3-brane WV (recall the MN model)
- the kind of subquiver they correspond to and hence the specific (IR) dynamics of the gauge theory living on them

The fields a $D3_f$ couples to can be read off the WV action (Einstein frame)

$$D3 \sim \int d^6x \; e^{\phi/2} \sqrt{G + e^{-\phi/2}(B + 2\pi\alpha' F)}$$

$$- \int \left[C_6 + C_4 \wedge (B_2 + 2\pi\alpha' F) \right] = (F=0) =$$

$$= \int d^4x \sqrt{G} \cdot \int_{S^2} B_2 - \int C_6 - \int C_4 \cdot \int_{S^2} B_2$$

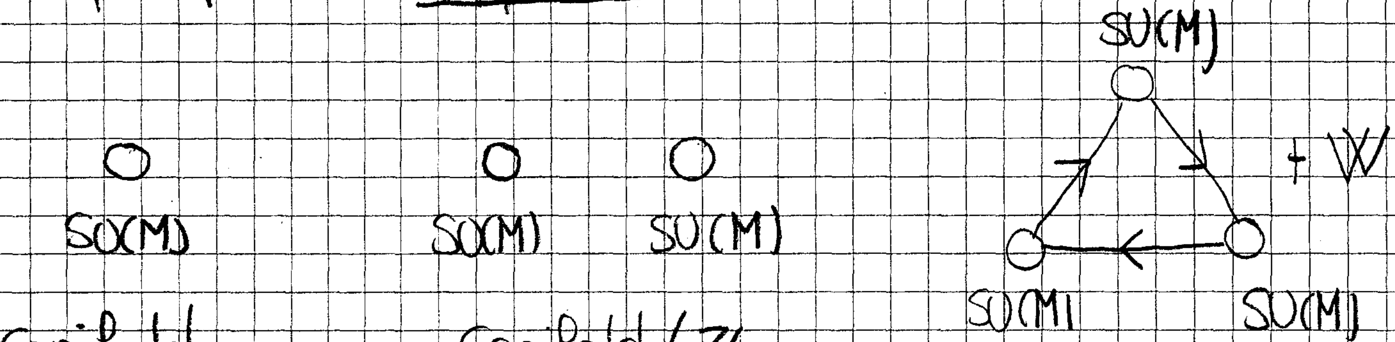
$$\rightarrow G_{\mu\nu}, \; F_5, \; G_3 = F_3 + i H_3 \quad (G_3 : \mathrm{ISD})$$

Given a CY singularity, by considering suitable bound states of regular & fractional D3-branes, one can construct a plethora of (maybe interesting) NON-CONFORMAL $\mathcal{N}=1$ QGT's.

- Generically (e.g. the conifold) the S^2 minimizes its volume at the tip, so fractional branes are STUCK there. In some cases (e.g. $\mathbb{R}^2 \times \mathbb{R}^4/\mathbb{Z}_2$) there is a complex singularity line where the fractional brane can move.

- The RG-flow of the QGT one obtains is complicated. A generic phenomenon (whose interpretation depends on the specific model at hand) is that the ranks of the gauge groups EFFECTIVELY diminish towards the IR: the theory in the IR behaves as if regular branes have disappeared ($N=0$) and only fractional branes are left → fractional branes trigger the QGT IR dynamics. They can be divided into 3 classes:

I. FB associated to a single node or to several decoupled nodes or to contiguos nodes whose loop operator <u>appears</u> in W

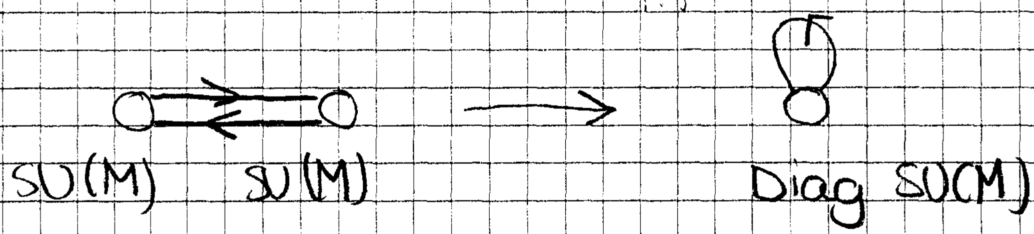


Conifold   Conifold/$\mathbb{Z}_2$ dP3

The gauge theory undergoes CONFINEMENT.
The dual geometric effect is a geometric transition ($\sim$ the MN model): a S^3 blows-up at the singularity, $V(S^3) \sim \Lambda$.

→ <u>DEFORMATION</u> fractional branes

II. FB associated to closed loops in the quiver whose corresponding operator does not appear in W.



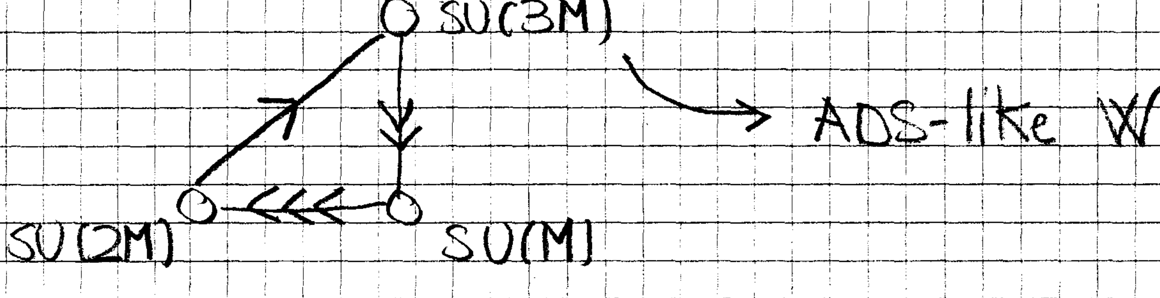
Conifold / $\mathbb{Z}_N$

The gauge theory has a mesonic moduli space which corresponds to the Coulomb branch of an effective $\mathcal{N}=2$ SYM.

Geometrically this means that the 2-cycle is NOT isolated ($\sim$ non-rigid), there is a complex singularity line $\sim \mathbb{C}^*$

$\rightarrow$ $\underline{\mathcal{N}=2}$ fractional branes

III. FB not belonging to classes I & II :) Typically, the occupied nodes have unbalanced ranks



Usually, at least one gauge factor has $N_f < N_c$ → the gauge theory develops an ADS-like effective W which leads to <u>runaway</u>.

The geometric dual is that there is NO complex structure deformation associated to S^2, this "tension" being at the origin of the runaway

→ <u>SUPERSYMMETRY BREAKING</u> fractional branes

<u>Note 1</u>: only QGT made of FB of class I are expected to give back a <u>smooth</u> gravity dual

<u>Note 2</u>: in complicated enough singularity, combining FB of a given class, one can obtain fractional branes of any other → the "bases" are defined in terms of the primitive 2-cycles.

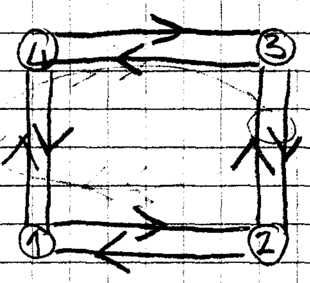
• Since on a D5 wrapped on a shrinking cycles lives a 4d U(1) gauge theory, naively one might make the association:

single quiver node $\longleftrightarrow$ single wrapped D5 (possibly on a non-primitive cycle)

This is WRONG: it works only for non-chiral theories.

<u>Ex. 1</u>

$T^{1,1}/\mathbb{Z}_2$

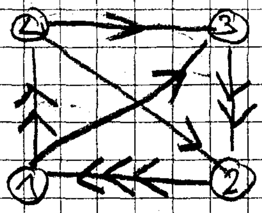


3 S^2's $\rightarrow$ 3 FB which can be associated to node 1, 2, 3 respectively

<u>Note</u>: there's one node more wrt # of FB
$\rightarrow$ the last FB is defined by a D5 wrapped on $-(S^2_1 + S^2_2 + S^3_3) + F$ such that D3 charge is POSITIVE. Geometrically the sum of ALL FB's corresponds to a regular (UNWRAPPED) D3.

Ex. 2

$Y^{2,1}$



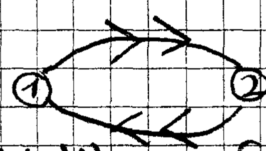
1 $S^2 \to$ 1 (!) FB which is associated to nodes 1, 2 & 3 with ranks 2M, M & 3M, respectively!

• From the definition of FB, it follows that:

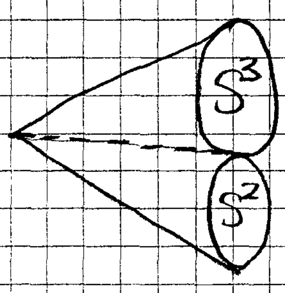
$$\frac{1}{g^2} \sim \int B_2$$

→ gauge coupling of the U(1) living on the D3 FB

Let us consider the conifold example (N D3 + M D3$_f$)



SU(N+M) SU(N)



$$\begin{cases} \dfrac{1}{g_1^2} = \dfrac{1}{4\pi g_s} e^{-\phi} \dfrac{1}{4\pi^2 \alpha'} \displaystyle\int B_2 \\[2ex] \dfrac{1}{g_2^2} = \dfrac{1}{4\pi g_s} e^{-\phi} \left(1 - \dfrac{1}{4\pi^2 \alpha'} \displaystyle\int B_2 \right) \end{cases}$$

The first relation is obvious. The second relation comes from:

$$WZ = \mu_5 \int_{D5} C_4 \wedge (B_2 + 2\pi\alpha' F) = \left(\int_{S^2} F \equiv 2\pi \right)$$

$$= \mu_5 \int_{S^2} (B_2 + 4\pi^2\alpha') \cdot \int C_4 =$$

$$= \mu_5 \, 4\pi^2\alpha' \left[1 - \frac{1}{4\pi^2\alpha'} \int B_2 \right] \cdot \int C_4$$

$$= \mu_3 \left(1 - \frac{1}{4\pi^2\alpha'} \int B_2 \right) \cdot \int C_4$$

By considering the DBI action and expanding with a non-trivial 4d F one gets:

$$DBI = -\mu_5 \int d^4x \, e^{-\phi} \sqrt{G + 2\pi\alpha' F} \cdot 4\pi^2\alpha' \cdot \left(1 - \frac{1}{4\pi^2\alpha'} \int B_2 \right)$$

$$= \ldots = -\frac{1}{g^2} \int d^4x \, \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \ldots$$

$$\frac{1}{g^2} = \frac{1}{4\pi g_s} e^{-\phi} \left(1 - \frac{1}{4\pi^2\alpha'} \int B_2 \right)$$

<u>Note</u>: for $M \neq 0$ the gauge theory is not conformal so one expects the SUGRA solution to have:

$$\int B_2 = f(r) \quad \text{running NSNS flux}$$

Note : the dilaton is expected TO be CONSTANT in the gravity dual. If one sums the g_1, g_2 :

$$\frac{1}{g_1^2} + \frac{1}{g_2^2} = \frac{1}{4\pi g_s} \cancel{e^{-\phi}} \quad (\phi \text{ does NOT run})$$

⇒ Correct : this is the $U(1) \times U(1)$ on a regular D3 which is a SCFT. This is generic in any such model.

• There is another RUNNING flux in the SUGRA dual

$$\int_{X^5} F_5 = g(r)$$

This is due to the B_2-running since :

$$F_5 = dC_4 + B_2 \wedge F_3 \longrightarrow dF_5 = H_3 \wedge F_3$$

This is the SUGRA counterpart of the diminishing of the effective overall rank of the QGT towards the IR .