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Uncertainties solution
Part 2

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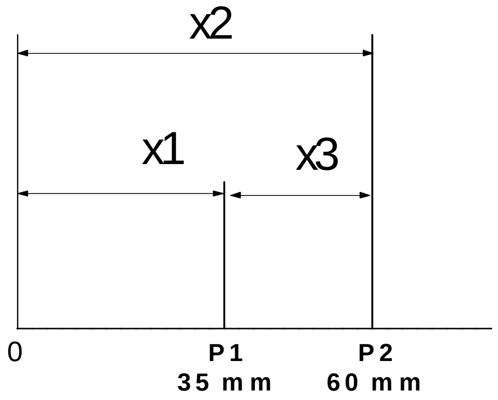
Part II : Uncertainties

Exercise II.1 (Ref. Mannhart, PTB-FMRB-84)

To determine the distance between points three gauge blocks are available as reference material. The characteristics of these blocks are given in the table. The uncertainties on the length of the blocks are not correlated.

Gauge block	Length	Standard deviation	Variance
L1	50 mm	0.05 μm	0.0025 $(\mu m)^2$
L2	15 mm	0.03 μm	0.0009 $(\mu m)^2$
L3	10 mm	0.02 μm	0.0004 $(\mu m)^2$

The blocks can be used to determine the distance x1 and x2 between the zero point and point P1 and P2, respectively (see figure).



(a) Determine the distances x1 and x2 by x1 = L1 – L2 and x2 = L1 + L3 and their uncertainties
(b) Determine the distance between point P1 and P2 together with the uncertainty using two methods:
- x3 = L2 + L3
- x3 = x2 – x1

Solution:
(a) For the first measurement we obtain:

x1 = 35 mm $\qquad u_{x1}^2 = u_{L1}^2 + u_{L2}^2 = 0.0034\ (\mu m)^2 \qquad u_{x1} = \sqrt{u_{L1}^2 + u_{L2}^2} = 0.058 \mu m$

x2 = 60 mm $\qquad u_{x2}^2 = u_{L1}^2 + u_{L3}^2 = 0.0029\ (\mu m)^2 \qquad u_{x2} = \sqrt{u_{L1}^2 + u_{L3}^2} = 0.054 \mu m$

(b) Distance between point P1 and P2
- x3 = L2 + L3 :

x3 = 25 mm $\qquad u_{x3}^2 = u_{L2}^2 + u_{L3}^2 = 0.0013\ (\mu m)^2 \qquad u_{x3} = \sqrt{u_{L2}^2 + u_{L2}^2} = 0.036\ \mu m$

- x3 = x2 – x1= 25 mm
For the calculation of the uncertainty we need the full covariance matrix $V_{x1,x2}$ and not only the diagonal terms. Let us first proceed as part of the exercise (but just as an illustration) with only the diagonal terms:

- x3 = x2 – x1= 25 mm $\qquad u_{x3}^2 = u_{x1}^2 + u_{x2}^2 = 0.0063\ (\mu m)^2 \qquad u_{x3} = \sqrt{u_{x1}^2 + u_{x2}^2} = 0.079\ \mu m$

This uncertainty differs strongly from the first solution. This is due to the fact that we determine x3 from two values which have uncertainties that are fully correlated.

Therefore if we quote two numbers as in exercise (a) we should in principle quote them with its full covariance matrix such that the values can be used for further calculations.
This covariance matrix is:

$$V_{x1,x2} = \begin{bmatrix} u_{x1}^2 & u_2^2 \\ u_2^2 & u_{x3}^2 \end{bmatrix} = \begin{bmatrix} 0.0034 & 0.0025 \\ 0.0025 & 0.0029 \end{bmatrix} \text{ in units of } (\mu m)^2$$

Using this covariance matrix the uncertainty on x3 becomes:

$$u_{x3}^2 = u_{x1}^2 + u_{x2}^2 - 2u_{x,y}^2 = (0.0034 + 0.0034 - 2x0.0025)\ (\mu m)^2 = 0.0013\ \ (\mu m)^2 \ \ u_{x3} = 0.036\ \mu m$$

This is the same as the one obtained before.

Exercise II.2

For the determination of the $^{197}Au(n,\gamma)$ cross section the sample described in exercise I.6 was used. The cross section is deduced from the reaction rate using the relationship:

$$C = \varepsilon\ A\ n\ \sigma_r\ \varphi$$

where C is the observed count rate, ε is the detection efficiency, A the effective area, n the total number of nuclei per area (or target thickness in atoms/barn), σ_r is the cross section and φ the neutron flux.

a) What is the relative uncertainty on the cross section due to the uncertainty on the target characteristics? (the uncertainty on the mass and the area are not correlated)

b) What is the minimum relative uncertainty on the cross section?

Solution

The uncertainty is defined by the uncertainty on the radius and the mass.

The cross section is inversely proportional with $\sigma_r \propto \frac{1}{n}$. Therefore the relative uncertainty on the cross section is defined by the relative uncertainty on the number of atoms per unit area.

The relative uncertainty on the total number of atoms N is : $\frac{u_N}{N} = 4x10^{-5}$

The relative uncertainty on the area S is : $\frac{u_S}{S} = 0.005$

Since they are not correlated the total uncertainty on the cross section is due to the target characteristics is:

$$\frac{u_{\sigma_r}}{\sigma_r} = \sqrt{\left(\frac{u_S}{S}\right)^2 + \left(\frac{u_N}{N}\right)^2} = 0.005$$

and is completely dominated by the uncertainty on the area. This is also the minimum uncertainty on the cross section.

Exercise II.3

The experiment described in exercise 2 has been repeated several times:

Campaign 1: 100 measurements using the same sample (the one used in exercise II.2)

Campaign 2: 50 measurements with 50 different samples. The characteristics of the 50 samples have been determined with different instruments such that the uncertainties are not fully correlated. The total uncertainty of the target thickness in at/b for the 50 samples was 0.5%. The correlated component was 0.3%

For both measurement campaigns the uncertainty on C due to counting statistics was 0.2% for each individual measurement.

(a) What is the minimum relative uncertainty on the cross section deduced from the data in Campaign 1?

(b) What is the minimum relative uncertainty on the cross section deduced from the data in Campaign 2?

Solution

(a) The relative uncertainty due to the target thickness is the minimum uncertainty on the cross section, which is 0.5%.

This can be seen from the following general scheme of propagation of uncertainties.

Assume that we first have n results x_i i = 1,…,n each with an uncertainty u_{Ci}. The uncertainties are not correlated. From these data points we calculate the quantitis: $y_i = N\, x_i$. Where N is a common correction factor with a uncertainty uN.

The average value is given by: $\bar{y} = \frac{\sum_{i=1}^{n} y_i}{n} = \frac{N\sum_{i=1}^{n} x_i}{n} = N\bar{x}$

Since the uncertainty of the factor N is not correlated with the uncertainty of any x_i, and therefore not with the uncertainty on $\bar{x}$ the uncertainty on $\bar{y}$ becomes:

$$u_{\bar{y}}^2 = u_N^2\,\bar{x}^2 + N^2\,u_{\bar{x}}^2 = u_N^2\,\bar{x}^2 + \Big(\sum_{i=1}^{n} u_{x_i}^2\Big)/n^2$$

Since all measurements x_i have the same uncorrelated uncertainty $u_{xi} = u_x$, the relative total uncertainty on $\bar{y}$ can be written as:

$$\frac{u_{\bar{y}}^2}{\bar{y}^2} = \frac{u_N^2}{N^2} + \frac{1}{n}\frac{u_x^2}{\bar{x}^2} \quad .$$

This equation shows that the minimum uncertainty is the uncertainty on the common factor N independent of the number n. Only the impact of the uncorrelated uncertainty on the data x_i can be reduced by increasing the number of measurement. This solution is identical to the one starting from the values yi and using the covariance matrix in the propagation of uncertainties.

Note that this solution is different if one would calculate the average starting from the variables y_i without accounting for the covariance due to N and only including in the uncertainty propagation the diagonal terms which are $u_{y_i}^2 = x_i^2 u_N^2 + N^2 u_{x_i}^2$. This approach (<u>which one should never apply!</u>) would result in:

$$\frac{u_{\bar{y}}^2}{\bar{y}^2} \rightarrow \frac{\sum_{i=1}^{n} u_{y_i}^2}{n^2} = \frac{u_N^2}{N^2}\frac{\sum_{i=1}^{n} x_i^2}{n^2\bar{x}^2} + \frac{\sum_{i=1}^{n} u_{x_i}^2}{n^2\bar{x}^2} = \frac{u_N^2}{N^2}\frac{1}{n} + \frac{1}{n}\frac{u_x^2}{\bar{x}^2}$$

in the last step we assumed that $x_i = \bar{x}$ and $u_{xi} = u_x$.

(b) Since the correlated component is 0.3% the uncorrelated component on the common factor N is 0.4% :

$\sqrt{0.005^2 - 0.003^2} = 0.004$.

We can use the last equation deduced above where we combine the 0.2 % uncorrelated due to counting statistics with the 0.035% uncorrelated part due to the target thickness:

$$\frac{u_x}{\bar{x}} = \sqrt{0.002^2 + 0.004^2} \approx 0.0045$$

The total uncorrelated uncertainty component for each measurement becomes 0.45%. The correlated uncertainty component is 0.3%, which is the minimum relative uncertainty from the data in Campaign 2.

Exercise II.4

Assume a cross section that is linear dependent on the neutron energy:

$$\sigma(E_n) = a + bE_n$$

The parameters a and b have been determined from a least square fit procedure on a set of experimental data points.

The result of the fit is given in the table.

Parameter	Value	Covariance	
a /barn	400.00	12.430	-0.220
b / (barn/eV)	-5.00	-0.220	0.00413

The cross section is used to determine the integral quantity:

$$I = \frac{\int_{E1}^{E2} \sigma(E_n)dE_n}{E2 - E1}$$

for E1 = 10 eV and E2 = 30 eV.

(a) Determine the integral and uncertainty on the integral by considering only the diagonal elements of the covariance matrix.

(b) Determine the integral and the uncertainty on the integral by uncertainty propagation using the full covariance matrix.

Solution:

The integral is given by:

$$I = a + b\,\frac{(E2 + E1)}{2} = 300 \text{ b}$$

a) The uncertainty without correlated terms is:

$$\sigma_I^2 = \sigma_a^2 + E_M^2\sigma_b^2 \quad \text{with} \quad E_M = \frac{E1 + E2}{2}$$

$\sigma_I^2 = 14.1\,\text{b}^2$ I = (300.0 $\pm$ 3.8)b

b) The uncertainty with correlated terms is:

$$\sigma_I^2 = \sigma_a^2 + 2E_M\sigma_{ab} + E_M^2\sigma_b^2 \quad \text{with} \quad E_M = \frac{E1 + E2}{2}$$

$\sigma_I^2 = 5.3\,\text{b}^2$ I = (300.0 $\pm$ 2.3)b