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Connected sum constructions in geometry and nonlinear analysis Part I

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Chapter 1

Laplace-Beltrami operator

1.1 Definition

Let (M, g) be a n-dimensional Riemannian manifold. If $x^1, \ldots, x^n$ are local coordinates, we will denote by ∂_{x^i} the associated vector fields. Locally, any vector field can be decomposed as

$$X = \sum_j a^j \, \partial_{x^j} ,$$

and we recall that $X \, u = \sum_j a^j \, \partial_{x^j} u$ for any function $u \in \mathcal{C}^1$.

The coefficients of the metric g are given by

$$g_{ij} := g \left(\partial_{x^i}, \partial_{x^j} \right) ,$$

so that we can also write

$$g = \sum_{i,j} g_{ij} \, dx^i \otimes dx^j .$$

Still using local coordinates, the volume form on M is given by

$$\mathrm{dvol}_g := \sqrt{\det g} \, dx^1 \, \ldots \, dx^n ,$$

so that the integral of a function u (having compact support where the coordinates are defined)

$$\int_M u \, \mathrm{dvol}_g = \int_{\mathbb{R}^n} u \, \sqrt{\det g} \, dx^1 \, \ldots \, dx^n ,$$

does not depend on the choice of the coordinates.

The differential $\mathrm{d}u$ of a function $u \in \mathcal{C}^1$ is given in local coordinates by

$$\mathrm{d}u = \sum_j \partial_{x^j} u \, dx^j .$$

The gradient of u, denoted $\operatorname{grad} u$, is by definition the vector field dual to $\mathrm{d}u$. It is characterized by the validity of the identity

$$g(\operatorname{grad} u, X) = \mathrm{d}\, u(X) = X u,$$

for all vector field $X \in TM$. In local coordinates, $\operatorname{grad} u$ is given by the formula

$$\operatorname{grad} u = \sum_{i,j} \partial_{x^i} u\, g^{ij}\, \partial_{x^j}\, .$$

Hence, we have in local coordinates

$$g(\operatorname{grad} u, \operatorname{grad} v) = \sum_{i,j} g^{ij} \partial_{x^i} u\, \partial_{x^j} v\, ,$$

for any functions $u, v \in \mathcal{C}^1$.

We also define the divergence of a vector field X by the identity

$$\int_M g(X, \operatorname{grad} f)\, \mathrm{dvol}_g = - \int_M f \operatorname{div} X\, \mathrm{dvol}_g$$

for all smooth function f with compact support. In local coordinates we fin

$$\operatorname{div} X = \frac{1}{\sqrt{\det g}} \sum_{i,j} \partial_{x^i} \left(\sqrt{\det g}\, g^{ij}\, g(X, \partial_{x^j}) \right) .$$

We denote by Γ_{ij}^k the Christoffel symbols associated to the Levi-Civita connection ∇ on (M, g). Recall that

$$\nabla_{\partial_{x^i}} \partial_{x^j} = \sum_k \Gamma_{ij}^k\, \partial_{x^k}\, ,$$

and also that

$$\nabla_X (u\, Y) = (X\, u)\, Y + u\, \nabla_X Y\, ,$$

for any function $u \in \mathcal{C}^1$ and any vector fields $X, Y \in TM$. Recall that the Christoffel symbols are given in terms of the metric coefficients by

$$\Gamma_{ij}^k = \tfrac{1}{2} \sum_\ell g^{k\ell}\, \left(\partial_{x^i} g_{\ell j} + \partial_{x^j} g_{\ell i} - \partial_{x^\ell} g_{ij}\right).$$

where $(g^{ij})_{ij}$ are the coefficients of the inverse of the matrix $(g_{ij})_{ij}$ defined by the coefficients of the metric.

The Hessian of a function u is defined by

$$\operatorname{Hess} u\, (X, Y) := X(Y\, u) - (\nabla_X Y)\, u\, .$$

In local coordinates, we have

$$\mathrm{Hess}\, u\ (\partial_{x^i}, \partial_{x^j}) = \partial_{x^i}\partial_{x^j} u - \Gamma_{ij}^k\, \partial_{x^k} u\,.$$

The Laplace-Beltrami operator $\Delta_g u$ is defined to be the trace of $\mathrm{Hess}\, u$ with respect to the metric g,

$$\Delta_g\, u := \mathrm{tr}^g(\mathrm{Hess}\, u) = \sum_{i,j} g^{ij}\ \left(\partial_{x^i}\partial_{x^j} - \Gamma_{ij}^k\, \partial_{x^k}\right)\ u\,.$$

We claim that we also have the formula

$$\Delta_g = \mathrm{div}\,(\mathrm{grad}\,\cdot\,) = \sum_{i,j} \frac{1}{\sqrt{\det g}}\, \partial_{x^i}\left(\sqrt{\det g}\, g^{ij}\, \partial_{x^j}\right)\,.$$

Proof of the claim : We start with the last expression of $\Delta_g u$ which we develop

$$\Delta_g = \sum_{i,j} g^{ij} \partial_{x^i}\partial_{x^j} + \sum_i \frac{1}{\sqrt{\det g}}\, \partial_{x^i}(\sqrt{\det g} g^{ij})\, \partial_{x^j}.$$

The identity between the two expression then follows from the fact that

$$\sum_i \frac{1}{\sqrt{\det g}}\, \partial_{x^i}(\sqrt{\det g}\, g^{ij}) = -\sum_{i,k} g^{ik}\, \Gamma_{ik}^j \tag{1.1}$$

To prove this last equality, we use the fact that the differential of $B \longmapsto \det(A+B)$ is given by $B \longmapsto \det A\, \mathrm{tr}(A^{-1}\, B)$ in the case where A is invertible, and hence

$$\sum_i g^{ij}\, \partial_{x^i} \sqrt{\det g} = \sum_i g^{ij}\, \mathrm{tr}^g(\partial_{x^i} g) = \sum_{i,k,\ell} g^{ij}\, g^{k\ell}\, \partial_{x^i} g_{k\ell} = \sum_{i,k,\ell} g^{j\ell}\, g^{ik}\, \partial_{x^\ell} g_{ik}.$$

Now, we have

$$\sum_{i,k,\ell} g^{j\ell}\, g^{ik}\, (\partial_{x^i} g_{\ell k} + \partial_{x^k} g_{\ell i}) = -\sum_{i,k,\ell} g^{j\ell}\, (g_{\ell k}\, \partial_{x^i} g^{ik} + g_{\ell i} \partial_{x^k} g^{ik}) = -2 \sum_i \partial_{x^i} g^{ij}.$$

Using these two fact, we compute

$$\begin{aligned} \sum_i \frac{1}{\sqrt{\det g}}\, \partial_{x^i}(\sqrt{\det g}\, g^{ij}) &= \sum_i \partial_{x^i} g^{ij} + \sum_{i,k,\ell} \frac{1}{2} g^{j\ell}\, g^{ik}\, \partial_{x^\ell} g_{ik} \\ &= -\sum_{i,k} g^{ik} \sum_\ell \frac{1}{2}\, g^{j\ell}\ (\partial_{x^i} g_{\ell k} + \partial_{x^k} g_{\ell i} - \partial_{x^\ell} g_{ik}) \end{aligned}$$

and the equality follows at once from the expression of Γ_{ik}^j. This completes the proof of the claim. $\square$

We claim that (1.1) together with the expression of the divergence of a vector field X in local coordinates implies that

$$\operatorname{div} X = \sum_{ij} g^{ij}\, g(\partial_{x^i}, \nabla_{\partial_{x^j}} X)\,. \tag{1.2}$$

Indeed,

$$\begin{aligned}
\operatorname{div} X &= \frac{1}{\sqrt{\det g}} \sum_{i,j} \partial_{x^i} \left(\sqrt{\det g}\, g^{ij}\, g(\partial_{x^j}, X) \right) \\
&= \frac{1}{\sqrt{\det g}} \sum_{i,j} g(\nabla_{\partial_{x^i}} (\sqrt{\det g}\, g^{ij}\, \partial_{x^j}), X) \\
&+ \sum_{i,j} g^{ij}\ g(\partial_{x^j}, \nabla_{\partial_{x^i}} X))
\end{aligned}$$

But,

$$\begin{aligned}
\sum_i \nabla_{\partial_{x^i}} (g^{ij} \sqrt{\det g}\, \partial_{x^j}) &= \sum_i \left(\partial_{x^i} \left(g^{ij} \sqrt{\det g} \right) \partial_{x^j} + g^{ij} \sqrt{\det g} \nabla_{\partial_{x^i}} \partial_{x^j} \right) \\
&= \sum_i \partial_{x^i} \left(g^{ij} \sqrt{\det g} \right) \partial_{x^j} + \sum_{i,k} g^{ij} \sqrt{\det g}\, \Gamma^k_{ij}\, \partial_{x^k}
\end{aligned}$$

and the claim follows at once from (1.1).

In particular, if u, v are smooth functions of M, we have

$$\int_M u\, \Delta_g v \operatorname{dvol}_g = - \int_M g(\operatorname{grad} u, \operatorname{grad} v) \operatorname{dvol}_g$$

as can be seen from the following computation in local coordinates

$$\int_M u\, \Delta_g v \operatorname{dvol}_g = - \int_M g^{ij} \partial_{x^i} u\, \partial_{x^j} v \operatorname{dvol}_g = - \int_M g(\operatorname{grad} u, \operatorname{grad} v) \operatorname{dvol}_g$$

provided the functions u and v have support in the local chart where the coordinates are defined.

Example : Cone metrics. Assume that (Σ, h) is a compact $(n-1)$-dimensional Riemannian manifold without boundary. Let $\Gamma(\Sigma) := (0, \infty) \times \Sigma$ be the cone over Σ which is endowed with the metric

$$g_{cone} = dr^2 + r^2\, h\,,$$

where $r \in (0, \infty)$. The Laplace-Beltrami operator on Γ is given by

$$\Delta_{g_{cone}} = \partial_r^2 + \frac{n-1}{r}\, \partial_r + \frac{1}{r^2}\, \Delta_h. \tag{1.3}$$

Example : In the special case where $\Sigma = S^{n-1}$ is the unit sphere with $g_{S^{n-1}}$ the canonical metric induced by the embedding $S^{n-1} \hookrightarrow \mathbb{R}^n$, we recover the expression of the Laplacian in polar coordinates

$$\Delta = \partial_r^2 + \frac{n-1}{r}\,\partial_r + \frac{1}{r^2}\,\Delta_{g_{S^{n-1}}}\,. \tag{1.4}$$

Example : Cylindrical metrics. Assume that (Σ, h) is a compact $(n-1)$-dimensional Riemannian manifold without boundary. Let $C := \mathbb{R} \times \Sigma$ be the cylinder with base Σ which is endowed with the product metric

$$g_{cyl} = dt^2 + h\,.$$

The Laplace-Beltrami operator on C is given by

$$\Delta_{g_{cyl}} = \partial_t^2 + \Delta_h. \tag{1.5}$$

1.2 Spectrum of the Laplace-Beltrami operator

Here we assume that (M, g) is a compact manifold without boundary. We will say that λ is an eigenvalue of $-\Delta_g$ if there exists a function φ such that

$$-\Delta_g\,\varphi = \lambda\,\varphi\,.$$

The spectrum of the Laplace Beltrami operator is well understood and we have the

Theorem 1.2.1. *The eigenvalues of $-\Delta_g$ form an increasing sequence of numbers*

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots$$

tending to ∞. The dimension of the eigenspace E_j associated to λ_j is finite. Moreover, the union of L^2-orthonormal basis of the E_j form a Hilbert basis of $L^2(M)$.

In the special case where $(M, g) = (S^n, g_{S^n})$, more is known and we have the

Proposition 1.2.1. *The eigenvalues of $-\Delta_{S^n}$ are given by*

$$\lambda_j = j\,(n-1+j)\,,$$

where $j \in \mathbb{N}$. The corresponding eigenspace, which will be denoted by E_j, is spanned by the restrictions to S^n of the homogeneous harmonic polynomials in $\mathbb{R}^{n+1}$.

One easy computation is the following : If P is a homogeneous harmonic polynomial of degree j defined in $\mathbb{R}^{n+1}$, then $P(x) = |x|^j\,P(x/|x|)$ and hence

$$r\,\partial_r P = j\,P$$

Using the expression of the Laplacian in polar coordinates, we find that

$$r^2\,\Delta P = j\,(n-1+j)\,P + \Delta_{g_{S^n}} P\,.$$

Since P is assumed to be harmonic, when restricted to the unit sphere this equality leads to

$$\Delta_{g_{S^n}} P = -j\,(n-1+j)\,P\,.$$

This (at least) shows that the restrictions to S^n of the homogeneous harmonic polynomials of degree j on $\mathbb{R}^{n+1}$ belong to E_j.

Exercise : What is the dimension of the j-th eigenspace E_j in the case of the sphere ?

Remark : In the case of the unit circle $S^1 \subset \mathbb{R}^2$, which we identify with $\mathbb{R}/2\pi\mathbb{Z}$, the Laplace-Beltrami operator identifies with the second order operator ∂_x^2 acting on 2π-periodic functions. The eigenvalues are given by $\lambda_j = j^2$ and the corresponding eigenspace is spanned by the functions $x \longmapsto \cos(jx)$ and $x \longmapsto \sin(jx)$. The fact that these constitute a Hilbert basis of $L^2(S^1)$ is nothing but the Fourier decomposition of a 2π-periodic functions.

1.3 Bibliography

A clear exposition of the above material can be found in the first chapter of the recent book by J.W. Morgan and G. Tian, *Ricci Flow and the Poincaré Conjecture.* An electronic version can be found at
http://arxiv.org/abs/math.DG/0607607

Classical texts are M. P. do Carmo, *Riemannian Geometry.* Birkhäuser, Boston, 1993. or P. Petersen *Riemannian geometry*, Vol 171, GTM Springer-Verlag, New York, 2006.

Let us also mention the book of R. Schoen and S.-T. Yau *Lectures on differential geometry*, Conference Proceedings and Lecture Notes in Geometry and Topology, I. International Press, Cambridge, MA, 1994.

The analysis of the spectrum of the Laplace-Beltrami operator can be found in M. Berger, P. Gauduchon and E. Mazet, *Le spectre d'une variété Riemannienne* Lecture Notes in Math. (1971). See also C. Müller *Spherical harmonics*, ETH Lectures, Springer for the spectrum of the Laplacian on the sphere.

Chapter 2

Function spaces

2.1 Definition

Assume that (M, g) is a Riemannian manifold with or without boundary. We define the Lebesgue spaces $L^p(M, g)$ (or simply $L^p(M)$) to be the space of integrable functions u defined on M such that

$$\|u\|_{L^p} := \left(\int_M |u|^p \, \mathrm{dvol}_g \right)^{1/p}$$

is finite.

For all $k \in \mathbb{N}$, we define

$$|\nabla^k u|_g^2 := \sum_{\alpha_j, \beta_j} g(\nabla_{e_{\alpha_1}} \ldots \nabla_{e_{\alpha_k}} u, \nabla_{e_{\beta_1}} \ldots \nabla_{e_{\beta_k}} u),$$

where $e_1, \ldots, e_n$ is an orthonormal basis of T_pM at the point p where the computation is performed. Given $k \in \mathbb{N}$ and $p \geq 1$, we define the Sobolev space $W^{k,p}(M, g)$ (or simply $W^{k,p}(M)$) to be the space of k times weakly differentiable functions u such that

$$\|u\|_{W^{k,p}} := \sum_{j=0}^{k} \left(\int_M |\nabla^j u|_g^p \, \mathrm{dvol}_g \right)^{1/p},$$

is finite. We also define $W_0^{k,p}(M, g)$ to be the closure in $W^{k,p}(M, g)$ of $\mathcal{C}_0^\infty(\mathring{M})$, the space of smooth functions with compact support in M.

2.2 Embedding results

Assume that (M, g) is a compact n-dimensional Riemannian manifold.

Proposition 2.2.1. *Let $k \in \mathbb{N}$ and $p \geq 1$ be given. Assume that $k\,p < n$, then the embedding*

$$W^{k,p}(M) \hookrightarrow L^{\frac{np}{n-kp}}(M)\,,$$

is continuous. If $k\,p > n$ and if $m \in \mathbb{N}$ is chosen so that $m < k - \frac{n}{p} < m+1$, the embedding

$$W^{k,p}(M) \hookrightarrow \mathcal{C}^{m,\alpha}(M)\,,$$

is continuous, provided $\alpha = k - \frac{n}{p} - m$.

2.3 Compactness results

Assume that (M, g) is a compact n-dimensional Riemannian manifold. We provide two important compactness results which will be used in the lectures.

Proposition 2.3.1. *Given $k, \ell \in \mathbb{N}$ and $\alpha, \beta \in (0,1)$, the embedding*

$$\mathcal{C}^{k,\alpha}(M) \hookrightarrow \mathcal{C}^{\ell,\beta}(M)\,,$$

is compact, provided $\ell + \beta < k + \alpha$.

The following is a classical compactness result for Sobolev embeddings.

Proposition 2.3.2. *Let $k \in \mathbb{N}$ and $p \geq 1$ be given. Assume that $k\,p < n$ and $q < \frac{np}{n-kp}$, then the embedding*

$$W^{k,p}(M) \hookrightarrow L^{q}(M)\,,$$

is compact. Assume that $k\,p > n$ and $m \in \mathbb{N}$ is chosen so that

$$m < k - \tfrac{n}{p} < m+1$$

and $0 < \alpha < k - \frac{n}{p} - m$, then the embedding

$$W^{k,p}(M) \hookrightarrow \mathcal{C}^{m,\alpha}(M)\,,$$

is compact.

2.4 Bibliography

A classical text on Sobolev spaces is the book of R. Adams, *Sobolev Spaces*, Academic Press, 1975.

The extension of the theory to the Riemannian setting can be found in the book of T. Aubin *Non-linear Analysis on Manifolds, Monge-Ampère equations* Ed. Springer (1982) and also in E. Hebey *Nonlinear analysis on manifolds : Sobolev spaces and inequalities*, CIMS Lecture Notes, Courant Institute of Mathematical Sciences, Volume 5, (1999).

Chapter 3

Second order elliptic operators

In this chapter we recall a few well known results concerning the existence and regularity of solutions of linear second order elliptic equations.

An operator $L : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ is said to be strictly elliptic if it can be written in local coordinates as

$$L = \sum_{i,j} a_{ij} \partial_{x^i} \partial_{x^j} + \sum_j b_j \, \partial_{x^j} + d \, ,$$

where the coefficients a_{ij}, b_j and the function d are smooth functions and if the matrix $(a_{ij})_{i,j}$ is definite positive, i.e.

$$\sum_{i,j} a_{ij} \xi^i \xi^j \geq A \, |\xi|^2 \, , \tag{3.1}$$

for some constant $A > 0$.

3.1 Existence result in L^p spaces

Assume that (M, g) is a compact n-dimensional Riemannian manifold with boundary. We recall the following classical result concerned with the existence of a solution of some elliptic problem in Lebesgue spaces.

Proposition 3.1.1. *Assume that $d \leq 0$. Given $p \in (1, \infty)$ and $f \in L^p(M)$, there exists a unique solution of*

$$\left\{ \begin{array}{rclcl} L\, v & = & f & \text{in} & M \\ v & = & 0 & \text{on} & \partial M \, , \end{array} \right. \tag{3.2}$$

which belongs to $W^{2,p}(M) \cap W_0^{1,p}(M)$.

Hence, when the function on the right hand side f belongs to L^p, there is a unique (weak) solution to the equation (3.2) in the sense that the identity $L\,v = f$ is satisfied almost everywhere in M, in particular

$$\int_M \left(\sum_{i,j} v\,\partial_{x^i}\partial_{x^j}(a_{ij}\,w) + \sum_j v\,\partial_{x^j}(b_j\,w) - v\,d\,w \right) \mathrm{dvol}_g = 0,$$

for all smooth function w with compact support in $\mathring{M}$. The boundary condition $v = 0$ on ∂M has to be understood in the sense that v is the limit, in $W^{1,p}$ of a sequence of functions in $\mathcal{C}_0^\infty(\mathring{M})$.

This proposition extends to the case where the manifolds M has no boundary under some slightly more restrictive assumption on the function d.

Proposition 3.1.2. *Assume that* $d \leq 0$ *is not identically equal to* 0. *Given* $p \in (1,\infty)$ *and* $f \in L^p(M)$, *there exists a unique solution of*

$$L\,v = f \qquad \text{in} \qquad M,$$

which belongs to $W^{2,p}(M)$.

3.2 L^p-Elliptic estimates

Assume that M' is compact included in the interior of M, we will write $M' \subset\subset \mathring{M}$. The next proposition gives an estimate in L^p-spaces of the first and second derivatives of a solution of $L\,v = f$ in terms of the L^p-estimate of the functions v and f on a larger domain.

Proposition 3.2.1. *Given* $p \in (1,\infty)$, *there exists a constant* $c > 0$ *such that, if* $g \in L^p(M)$, $v \in L^p(M)$ *satisfy*

$$L\,v = g \qquad \text{in} \qquad M\,,$$

in the sense of distributions, then

$$\|v\|_{W^{2,p}(M')} \leq c\,\left(\|v\|_{L^p(M)} + \|g\|_{L^p(M)}\right)\,.$$

This result essentially states that the norm of the derivatives of the function v up to order two are controlled by the right hand side of the equation as well as the norm of the solution on some strictly larger set. The constant c depends on the distance between M' and ∂M.

Assume that M is a smooth manifold with a boundary that has two disjoint closed components T_1 and T_2 such that

$$\partial M = T_1 \cup T_2 \, .$$

We assume that $M' \subset\subset \mathring{M} \cup T_1$ and that χ is a smooth function such that $\chi \equiv 1$ in a collar neighborhood of T_1 and $\chi \equiv 0$ in a collar neighborhood of T_2.

Proposition 3.2.2. *Given $p \in (1, \infty)$, there exists a constant $c > 0$ such that, if $g \in L^p(M)$, $v \in W^{2,p}(M)$ and $\chi\, v \in W_0^{1,p}(M')$, satisfy*

$$\left\{ \begin{array}{rclcl} L\, v & = & g & \text{in} & M \\ v & = & 0 & \text{on} & T_1 \, , \end{array} \right.$$

then

$$\|v\|_{W^{2,p}(M')} \leq c \, \left(\|v\|_{L^p(M)} + \|g\|_{L^p(M)} \right) \, .$$

This result essentially shows that the norm of the solution v, in the natural space, is controlled by the norm of the right hand side and some information about the function v.

3.3 Schauder's estimates

We keep the notations and assumptions of the previous sections and recall some well known results concerning the Hölder regularity of solutions of the equation $L\, v = g$.

Proposition 3.3.1. *Assume that $p \geq 1$. There exists a constant $c > 0$ such that, if $g \in L^p(M)$ and $v \in L^p(M)$ satisfy*

$$L\, v = g \qquad \text{in} \qquad M \, ,$$

and if $g \in \mathcal{C}^{0,\alpha}(M)$ then $u \in \mathcal{C}^{2,\alpha}(M')$ and

$$\|u\|_{\mathcal{C}^{2,\alpha}(M')} \leq c \, \left(\|g\|_{\mathcal{C}^{0,\alpha}(M)} + \|u\|_{L^p(M)} \right) \, .$$

We also have the :

Proposition 3.3.2. *Assume that $p \geq 1$. There exists a constant $c > 0$ such that, if $g \in L^p(M)$, $v \in W^{2,p}(M)$ and $\chi\, v \in W_0^{1,p}(M')$, satisfy*

$$\left\{ \begin{array}{rclcl} L\, v & = & g & \text{in} & M \\ v & = & 0 & \text{on} & T_1 \, , \end{array} \right.$$

and if $g \in \mathcal{C}^{0,\alpha}(M)$ then $u \in \mathcal{C}^{2,\alpha}(M')$ and

$$\|u\|_{\mathcal{C}^{2,\alpha}(M')} \leq c \, \left(\|g\|_{\mathcal{C}^{0,\alpha}(M)} + \|u\|_{L^p(M)} \right) \, .$$

3.4 Regularity issues

We also recall the following well known result whose proof is quite hard to find in the literature. It essentially states that solutions of the homogeneous problem $L\,v = 0$ are smooth.

Proposition 3.4.1. *Assume that $v \in \mathcal{D}'(M)$ is a solution of*

$$L\,v = 0\,,$$

in the sense of distribution. Then $v \in \mathcal{C}^\infty(\mathring{M})$.

In the case where $M \subset \mathbb{R}^n$ is a smooth domain (endowed with the euclidean metric), the coefficients of L are constant and $v \in L^p(M)$, with $p > 1$, the proof of the result follows at once from the result of Proposition 3.3.2 together with a convolution argument. Indeed, consider a smooth function φ which is positive, has compact support in the unit ball and whose integral over $\mathbb{R}^n$ is equal to 1.

We define

$$v_\varepsilon(x) = v * \varphi_\varepsilon(x) := \int_M \varphi_\varepsilon(y)\, v(x-y)\, dy,$$

where

$$\varphi_\varepsilon := \varepsilon^{-n}\, \varphi(\cdot/\varepsilon).$$

Then v_ε is smooth and satisfies $L\,v_\varepsilon = 0$ in $M_\varepsilon := \{p \in M \,:\, \mathrm{dist}(p, \partial M) \geq 2\,\varepsilon\}$. This uses in an essential way the fact that the coefficients of L are constant and hence the translates of a solution of $L\,v = 0$ are also solutions of the same equation.

Fix a compact $M' \subset\subset M$ and apply the result of Proposition 3.3.2 to obtain some estimate

$$\|v_\varepsilon\|_{\mathcal{C}^{2,\alpha}(M')} \leq \, c\, \|v_\varepsilon\|_{L^p(M_{2\varepsilon})}\,.$$

But

$$\|v_\varepsilon\|_{L^p(M_{2\varepsilon})} \leq \|v\|_{L^p(M)}\,.$$

Therefore, the sequence of functions $(v_\varepsilon)_\varepsilon$ is bounded in $\mathcal{C}^{2,\alpha}(M')$. Using the compactness of Proposition 2.3.2, we obtain the existence of a convergent subsequence to v almost everywhere and hence $v \in \mathcal{C}^{2,\alpha}(M')$. A simple bootstrap argument then implies that v is smooth in the interior of M'.

3.5 The weak maximum principle

When the function v is a classical solution (namely $v \in \mathcal{C}^2$) of

$$L\,v \leq 0$$

in M and when the function d in the definition of L is negative then

$$\inf_M v \geq \inf_{\partial M}\, \min(0, v)$$

Indeed, let us assume that v achieves a local minimum at a point $p \in \mathring{M}$, then, at this point

$$\sum_{i,j} a_{ij} \partial_{x_i} \partial_{x_j} v \geq 0$$

(simply use (3.1)) and

$$\sum_{j} b_j \, \partial_{x_j} v = 0,$$

since the gradient of v certainly vanishes at the point p) and hence

$$v \leq 0$$

(since $d\,v \leq L\,v \leq 0$ and $d < 0$) at a point $p \in \mathring{M}$, where v. The inequality then follows at once.

This result will often be used under slightly stronger hypothesis, namely we assume that $L\,v < 0$ and $v > 0$ on ∂M, then the previous argument implies that $v > 0$ in M.

More generally, we have the :

Proposition 3.5.1. *Assume that $d \leq 0$. Let $v \in W^{1,2}(M) \cap \mathcal{C}^0(M)$ be a solution of $L\,v \leq 0$ in M. Then*

$$\inf_M v \geq \inf_{\partial M} \min(0, v)$$

3.6 The spectrum of self adjoint operators

Here we assume that (M, g) is a compact manifold without boundary and L is an elliptic second order operator. We will say that λ is an eigenvalue of $-L$ if there exists a function φ such that

$$-L\,\varphi = \lambda\,\varphi\,.$$

The operator L is said to be self-adjoint if

$$\int_M Lu\,v\,\mathrm{dvol}_g = \int_M v\,Lu\,\mathrm{dvol}_g$$

for all $u, v \in \mathcal{C}^\infty(M)$. In this case, the following result generalizes the corresponding result for the Laplace Beltrami operator.

Theorem 3.6.1. *Assume that L is a self-adjoint elliptic operator. Then the eigenvalues of $-L$ form an increasing sequence of numbers*

$$\lambda_0 < \lambda_1 < \lambda_2 < \ldots$$

tending to ∞. The dimension of the eigenspace E_j associated to λ_j is finite. Moreover, the union of L^2-orthonormal basis of the E_j form a Hilbert basis of $L^2(M)$.

3.7 Bibliography

Most of the above results can be found in the book of D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order* Springer-Verlag, Berlin, (1983).

Chapter 4

A simple model problem

We agree that B_R (resp. $\bar{B}_R$) denotes the open (closed) ball of radius $R > 0$ in $\mathbb{R}^n$ and $B_R^* := B_R \setminus \{0\}$ (resp. $\bar{B}_R^*$) denotes the corresponding punctured ball.

If Ω is a bounded domain of $\mathbb{R}^n$ containing 0, the closure of Ω is denoted by $\bar{\Omega}$ and $\Omega^* := \Omega \setminus \{0\}$ (resp. $\bar{\Omega}^*$) denotes the corresponding punctured domain.

Finally, if $R_0 > 0$ is fixed small enough so that $B_{R_0} \subset \Omega$ and for all $R < R_0$, we define

$$\Omega_R := \Omega \setminus \bar{B}_R.$$

Given $\nu \in \mathbb{R}$ and a function

$$f : \Omega^* \subset \mathbb{R}^n \longrightarrow \mathbb{R},$$

satisfying

$$\| |x|^{-\nu} f \|_{L^\infty(\Omega)} < +\infty \,,$$

we would like to study the solvability of the Dirichlet problem

$$\left\{ \begin{array}{rclll} |x|^2 \, \Delta u & = & f & \text{in} & \Omega^* \\ u & = & 0 & \text{on} & \partial\Omega \,. \end{array} \right. \tag{4.1}$$

Observe that we are not looking for the solvability of (4.1) on all of Ω but only away from the origin. A solution of this equation is understood in the sense of distributions, namely u is a solution of (4.1) if $u \in L^1_{loc}(\bar{\Omega}^*)$ and if

$$\int_\Omega u \, \Delta v \, dx = \int_\Omega f \, v \, |x|^{-2} \, dx$$

for all $v \in \mathcal{C}_0^\infty(\Omega^*)$, the space of smooth functions with compact support in Ω^*.

We claim that :

Proposition 4.0.1. *Assume that $n \geq 3$ and $\nu \in (2-n, 0)$. Then there exists a constant $c > 0$ (depending on ν and n) and for all $f \in L^\infty_{loc}(\Omega^*)$ there exists $u \in L^\infty_{loc}(\bar{\Omega}^*)$ solution of (4.1) which satisfies*

$$\||x|^{-\nu}\, u\|_{L^\infty(\Omega)} \leq c\, \||x|^{-\nu}\, f\|_{L^\infty(\Omega)} \,.$$

Proof : The proof of this result turns out to be a simple consequence of the maximum principle. First, we recall the expression of the Euclidean Laplacian in polar coordinates (1.4)

$$\Delta = \partial_r^2 + \frac{n-1}{r}\,\partial_r + \frac{1}{r^2}\,\Delta_{g_{S^{n-1}}} \,.$$

Using this expression we get at once

$$|x|^2\, \Delta |x|^\nu = -\nu\,(2-n-\nu)\,|x|^\nu \,,$$

away from the origin. Now, observe that the constant

$$c_{n,\nu} := \nu\,(2-n-\nu) > 0,$$

precisely when $\nu \in (2-n, 0)$ (this is where we use the fact that $n \geq 3$!).

The existence of a solution of (4.1) can then be obtained arguing as follows : Given $R \in (0, R_0/2)$, we first solve the problem

$$\left\{ \begin{array}{rclcl} |x|^2\, \Delta u_R & = & f & \text{in} & \Omega_R \\ u_R & = & 0 & \text{on} & \partial\Omega_R \,. \end{array} \right. \tag{4.2}$$

Since $f \in L^\infty(\Omega_R)$, the existence of a solution $u_R \in W^{2,p}(\Omega_R)$ for any $p \in (1, \infty)$ follows from Proposition 3.1.2 which we apply to Ω_R and to the function f which satisfies

$$f \in L^\infty(\Omega_R) \subset L^p(\Omega_R),$$

for all $p \in (1, \infty)$. Hence we find

$$u_R \in W^{2,p}(\Omega_R) \cap W^{1,p}_0(\Omega_R),$$

a (strong) solution of (4.2).

One can make use of the Sobolev Embedding Theorem in Proposition 2.2.1 to show that $u_R \in \mathcal{C}^{1,\alpha}(\bar{\Omega}_R)$ for all $\alpha \in (0,1)$. Now, observe that the function

$$w(x) = \frac{1}{c_{n,\nu}}\, \||x|^{-\nu}\, f\|_{L^\infty(\Omega)}\, |x|^\nu - u_R(x)$$

is positive on $\partial\Omega_R$. Moreover

$$\Delta w \leq 0$$

in Ω_R. Therefore one can apply the maximum principle Proposition 3.5.1 which yields the pointwize bound

$$u_R(x) \leq \frac{1}{c_{n,\nu}} \, \||x|^{-\nu} f\|_{L^\infty(\Omega)} \, |x|^\nu$$

for all $x \in \bar{\Omega}_R$. Observe that this bound is independent of the value of R. Applying the same reasoning to $-u_R$ we obtain

$$|u_R(x)| \leq \frac{1}{c_{n,\nu}} \, \||x|^{-\nu} f\|_{L^\infty(\Omega)} \, |x|^\nu \, . \tag{4.3}$$

Now, we would like to pass to the limit, as R tends to 0. To this aim, we use the *a priori* estimates for solutions of (4.2) which are provided by Proposition 3.2.2 with $M = \Omega_R$, $T_1 = \partial\Omega$ and $M' = \Omega_{2R}$ together with the *a priori* bound (4.3). We conclude that, for all $R \in (0, 1/2)$ there exists a constant $c = c(n, \nu, R) > 0$ such that

$$\|u_{R'}\|_{W^{2,p}(\Omega_{2R})} \leq c \, \||x|^{-\nu} f\|_{L^\infty(\Omega)},$$

for all $R' \in (0, R)$.

It is now enough to apply the Sobolev Embedding Theorem together with a standard diagonal argument to show that there exists a sequence $(R_i)_i$ tending to 0 such that the sequence of functions u_{R_i} converges to some continuous function u on compacts of $\bar{\Omega}^*$. Obviously u will be a solution of (4.1) and, passing to the limit in (4.3), will satisfy

$$c_{n,\nu} \, \||x|^{-\nu} u\|_{L^\infty(\Omega)} \leq \, \||x|^{-\nu} f\|_{L^\infty(\Omega)} \tag{4.4}$$

We have thus obtained the existence of a solution of (4.1) satisfying (4.4), when $\nu \in (2-n, 0)$. This completes the proof of the Proposition. $\square$

Exercice : Prove the uniqueness of the solution obtained in the previous proof.

Exercice : Assume that $n \geq 3$, $\nu \in (2-n, 0)$ and Ω is a smooth bounded domain in $\mathbb{R}^n$. Show that there exists a constant $c > 0$ and for all $f \in L^\infty_{loc}(\mathbb{R}^n \setminus \bar{\Omega})$ there exists $u \in L^\infty_{loc}(\mathbb{R}^n \setminus \bar{\Omega})$ a solution of

$$\left\{ \begin{array}{rclcl} |x|^2 \, \Delta u & = & f & \text{in} & \mathbb{R}^n \setminus \bar{\Omega} \\[2mm] u & = & 0 & \text{on} & \partial\Omega \, , \end{array} \right. \tag{4.5}$$

which satisfies

$$\||x|^{-\nu} u\|_{L^\infty(\mathbb{R}^n \setminus \bar{\Omega})} \leq c \, \||x|^{-\nu} f\|_{L^\infty(\mathbb{R}^n \setminus \bar{\Omega})} \, .$$

Hint : Consider the change of functions and variables

$$\tilde{u}(x) = |x|^{2-n} \, u(x/|x|^2) \qquad \text{and} \qquad \tilde{f}(x) = |x|^{2-n} \, f(x/|x|^2) \, ,$$

and compute Δv.

Exercice : Given points $x_1, \ldots, x_m \in \mathbb{R}^n$ and weight parameters $\mu, \nu_1, \ldots, \nu_m \in \mathbb{R}$, we define two positive smooth functions

$$\pi : \mathbb{R}^n \setminus \{x_1, \ldots, x_m\} \longrightarrow (0, \infty) \qquad \text{and} \qquad \gamma : \mathbb{R}^n \setminus \{x_1, \ldots, x_m\} \longrightarrow (0, \infty) ,$$

such that :

(i) For each $i = 1, \ldots, m$, $\pi(x) = |x - x_i|^{-\nu_i}$ and $\gamma(x) = |x - x_i|$ in a neighborhood of the point x_i.

(ii) $\pi(x) = |x|^{-\mu}$ and $\gamma(x) = |x|$ away from a compact subset of $\mathbb{R}^n$.

We assume that $n \geq 3$ and $\mu, \nu_1, \ldots, \nu_m \in (2 - n, 0)$. Show that, given a function

$$f : \mathbb{R}^n \setminus \{x_1, \ldots, x_n\} \longrightarrow \mathbb{R} ,$$

satisfying

$$\|\pi\, f\|_{L^\infty(\mathbb{R}^n)} < \infty ,$$

it is possible to find a solution of the equation

$$\gamma^2\, \Delta u = f ,$$

which satisfies

$$\|\pi\, u\|_{L^\infty(\mathbb{R}^n)} \leq c\, \|\pi\, f\|_{L^\infty(\mathbb{R}^n)} ,$$

for some constant $c > 0$ independent of f.

We end this chapter by the following simple remark. If the functions u and f solve

$$|x|^2\, \Delta u = f ,$$

in B_1^* and if we define for $(t, z) \in (0, \infty) \times S^{n-1}$ the functions

$$\hat{u}(t, z) := e^{\frac{n-2}{2} t}\, u(e^{-t}\, z) , \qquad \text{and} \qquad \hat{f}(t, z) := e^{\frac{n-2}{2} t}\, f(e^{-t}\, z) .$$

Then $\hat{u}$ and $\hat{f}$ satisfy

$$\left(\partial_t^2 + \Delta_{g_{S^{n-1}}} - \left(\tfrac{n-2}{2}\right)^2 \right) \hat{u} = \hat{f} ,$$

on $(0, \infty) \times S^{n-1}$. Recall that $\partial_t^2 + \Delta_{g_{S^{n-1}}}$ is the Laplace-Beltrami operator for the product metric $g = dt^2 + g_{S^{n-1}}$ on the cylinder $\mathbb{R} \times S^{n-1}$.

Chapter 5

Indicial roots

Assume that (Σ, h) is a compact $(n-1)$-dimensional Riemannian manifold. We denote by $C := \mathbb{R} \times \Sigma$ the cylinder with base Σ. In this chapter, we study the asymptotic behavior of the solutions of the homogeneous problem $L\,u = 0$ at $+\infty$ or $-\infty$, where L is a second order elliptic operator on C. It turns out that the asymptotic behavior of the solutions of the homogeneous problem $L\,u = 0$ is intimately related to the notion of *indicial roots* we now define.

Definition 5.0.1. *The indicial roots of the operator L at $+\infty$ (resp. $-\infty$) are the real numbers δ for which the following holds : There exists a non-zero function $v \in \mathcal{C}^2(C)$ and there exists $\delta' < \delta$ (resp. $\delta' > \delta$) such that*

$$\liminf \|v\|_{L^\infty(\{t\}\times\Sigma)} > 0$$

and

$$\lim e^{-\delta' t}\, L\,(e^{\delta t}\, v) = 0$$

as t tends to $+\infty$ (resp. $-\infty$).

Observe that, if $v \in \mathcal{C}^2(C)$ then, we always have

$$L\,(e^{\delta t}\, v) = \mathcal{O}(e^{\delta t})\,,$$

while, for δ to be an indicial root of L at $+\infty$, we ask that $e^{-\delta t}\, L\,(e^{\delta t}\, v)$ tends to 0 exponentially fast at $+\infty$. Finally, observe that, in order to define the indicial roots of L at $+\infty$ it is enough to consider functions v which are supported on a half cylinder $(t_0, +\infty) \times \Sigma$.

Without any further assumption on the structure of L, the determination of the indicial roots of a given operator L is a hopeless task. Henceforth, we now restrict our attention to some special class of operators, namely we assume that

$$L = \partial_t^2 + L_\Sigma + a\,, \tag{5.1}$$

where L_Σ is a self-adjoint second order elliptic operator and where the function a defined on C only depends on t and is t_0-periodic. For this class of operators, it turns out that the determination of the indicial roots is equivalent to the study of the asymptotic behavior of the solutions of the homogeneous problem $L\,u = 0$ at ∞.

Given the special structure of our operator, in order to understand the indicial roots of the operator L, we perform the eigenfunction decomposition of a function u as

$$u(t,\cdot) = \sum_{j\geq 0} u_j(t,\cdot)$$

where, for all $t \in (0,\infty)$, $u_j(t,\cdot) \in E_j$ the j-th eigenspace of $-L_\Sigma$ on Σ. The eigenfunction decomposition induces a splitting of L into

$$L\,u = \sum_{j\geq 0} \left(\partial_t^2 - \lambda_j + a\right)\, u_j$$

In this special case, the study of the asymptotic behavior at ∞ of a function u, solution of $L\,u = 0$, reduces to the study of the solutions of the second order ordinary differential equations

$$\left(\partial_t^2 - \lambda_j + a\right)\, u_j = 0\,,$$

where λ_j are the eigenvalues of $-L_\Sigma$ associated to the j-th eigenspace.

5.1 ODE analysis

When the function a is constant, the determination of the solutions of the homogeneous equation $\left(\partial_t^2 - (\lambda_j - a)\right)\, u_j = 0$ is completely elementary but, for applications, it is also interesting to consider the case where a is a (smooth) periodic function in which case the corresponding study is classical even though it does not seem to be common knowledge.

Therefore, we consider in this section some operator of the form

$$\partial_t^2 + b\,,$$

where b is a smooth t_0-periodic function. Concerning the existence and characterization of the solutions of the homogeneous problem

$$(\partial_t^2 + b)\, w = 0\,, \tag{5.2}$$

we recall the classical :

Proposition 5.1.1. *There exist $\delta + i\,\mu \in \mathbb{C}$, w^+ and w^-, two real valued independent solutions of (5.2), such that the following holds :*

(i) either $\delta > 0$ and the functions

$$t \longmapsto e^{\mp\delta\, t}\, w^{\pm}(t)$$

are $2\, t_0$-periodic.

(ii) or $\delta = 0$ and the functions

$$t \longmapsto e^{\mp\, i\, \mu\, t}\, (w^+ \pm i w^-)(t)$$

are $2\, t_0$-periodic.

(iii) or $\delta = 0$ and there exists $\nu \in \mathbb{R} \setminus \{0\}$ such that the functions

$$t \longmapsto w^+(t) \qquad \textit{and} \qquad t \longmapsto w^-(t) - \nu\, t\, w^+(t),$$

are $2\, t_0$-periodic.

Proof : Given initial data (a_0, a_1) we consider w the unique solution of

$$\partial_t^2 w + b\, w = 0 \tag{5.3}$$

with $w(0) = a_0$ and $\partial_t w(0) = a_1$. We then define the mapping

$$B(a_0, a_1) = (w(t_0), \partial_t w(t_0))\,,$$

where we recall that t_0 is the period of the function b. Clearly B is linear and $\mathrm{Tr} B \in \mathbb{R}$. We claim that

$$\det B = 1.$$

To see this, consider the solution w_0 associated to the initial data $w_0(0) = 1$ and $\partial_t w_0(0) = 0$ and the solution w_1 associated to the initial data $w_1(0) = 0$ and $\partial_t w_1(0) = 1$. The Wronskian of the two solutions

$$W := w_1\, \partial_t w_0 - w_0\, \partial_t w_1$$

does not depend on t and by construction $W(0) = 1$. Now, given the definition of B we check that

$$W(t_0) = (\det B)\, W(0)$$

and hence $\det B = 1$ as claimed. We now distinguish a few cases according to the spectrum of B.

Case 1. Assume that B can be diagonalized (in $\mathbb{C}^2$). Since the determinant of B is equal to 1 then the eigenvalues are given by λ and $1/\lambda$ where $\lambda \in \mathbb{C}$ and $|\lambda| \geq 1$. We write

$$\lambda = e^{(\delta + i\, \mu)\, t_0}$$

where $\delta \geq 0$ and $\mu\, t_0 \in \mathbb{R}$.

Observe that if $|\lambda| > 1$ then necessarily $\lambda \in \mathbb{R}$ since the trace of B is a real number therefore, $\delta > 0$ and $\mu\, t_0 \in \pi\mathbb{Z}$. The coordinates of the eigenvector of B associated to λ (resp. $1/\lambda$) are the Cauchy data of a function w^+, solution of (5.3), which satisfies

$$w^{\pm}(t + 2\,t_0) = e^{\pm 2\,(\delta + i\mu)\,t_0}\, w^{\pm}(t) = e^{\pm 2\,\delta\,t_0}\, w^{\pm}(t),$$

and hence blows up (resp. tends to 0) exponentially at infinity.

If $|\lambda| = 1$, then $\delta = 0$ and the real part and imaginary part of the coordinates of the eigenvectors of B are the Cauchy data of $w^{\pm}$, two real valued independent solutions of (5.3), which satisfy

$$(w^+ + i\, w^-)(t + 2t_0) = e^{2i\mu t_0}\,(w^+ + i\, w^-)(t)\,,$$

and hence these two independent solutions are bounded.

Case 2. To complete this discussion, we consider the case where B cannot be diagonalized. In this case the eigenvalue λ necessarily satisfies $\lambda^2 = 1$ and we can write

$$\lambda = e^{i\,\mu\,t_0}\,,$$

where $\mu\, t_0 \in \pi\,\mathbb{Z}$. The coordinates of the eigenvector e_1 of B associated to the eigenvalue λ are the Cauchy data of w^+ a $2\,t_0$-periodic solution of (5.3). Since the operator B is not diagonalized, there exists a vector e_2 (independent of e_1) such that

$$B\, e_2 = \lambda\,(\, e_2 + \nu\, t_0\, e_1)\,,$$

for some $\nu \in \mathbb{R}$. In other words e_1, e_2 is a Jordan basis associated to B. We denote by w^- the solution of (5.3) whose Cauchy data are the coordinates of e_2. By definition we have

$$e_1 = (w^+(0), \partial_t w^+(0)) \qquad \text{and} \qquad e_2 = (w^-(0), \partial_t w^-(0))$$

Now, on the one hand

$$B\, e_2 = (w^-(t_0), \partial_t w^-(t_0))$$

and on the other hand

$$B\, e_2 = \lambda\,\big((w^-(0), \partial_t w^-(0)) + \nu\, t_0\,(w^+(0), \partial_t w^+(0))\big)$$

Therefore, we have the identity

$$(w^-(t_0), \partial_t w^-(t_0)) = \lambda\,\big((w^-(0), \partial_t w^-(0)) + \nu\, t_0\,(w^+(0), \partial_t w^+(0))\big)$$

which implies that

$$w^-(t + t_0) = \lambda\,(w^-(t) + \nu\, t_0\, w^+(t))$$

for all $t \in \mathbb{R}$ (simply use the uniqueness of the solutions of (5.3) with given initial data). This shows that the function v defined by

$$v(t) = w^-(t) - \nu\, t\, w^+(t)$$

satisfies

$$v(t + 2t_0) = \lambda^2\, v(t) = v(t).$$

In other words, the function v is $2\,t_0$-periodic and hence we can write

$$w^-(t) = v(t) + \nu\, t\, w^+(t)$$

where both v and w^+ are $2\,t_0$-periodic. In this last case, we will say that w^- is "linearly growing". This completes the proof of the result. □

Recall that the knowledge of the solutions w^+ and w^- allows one to solve the equation

$$(\partial_t^2 + b)\, w = f\,, \tag{5.4}$$

using the "variation of the constant formula" namely, all the solutions of (5.4) are given by

$$w = \frac{1}{W}\left(w^+ \int^t w^-\, f\, ds - w^- \int^t w^+\, f\, ds\right) + a^+\, w^+ + a^-\, w^-\,, \tag{5.5}$$

where W denotes the Wronskian of the two solutions $w^\pm$ and $a^\pm$ are free parameters.

Building on the above analysis, we show the following result.

Proposition 5.1.2. *Assume that $v \in \mathcal{C}^2(\mathbb{R})$ satisfies*

$$\lim_{t \to +\infty} e^{-\tilde{\delta}' t}\, (\partial_t^2 + b)\, (e^{\tilde{\delta} t}\, v) = 0\,,$$

for some $\tilde{\delta}' < \tilde{\delta}$. Then, $\tilde{\delta} \in \{-\delta, \delta\}$ (where δ is given in Proposition 5.1.1) and there exists w a non trivial solution of $(\partial_t^2 + b)\, w = 0$ such that

$$\lim_{t \to +\infty} e^{-\tilde{\delta} t}\, (e^{\tilde{\delta} t}\, v - w) = 0\,.$$

Proof : Without loss of generality, we can always assume that $\tilde{\delta}' \notin \{-\delta, \delta\}$. We set $f := (\partial_t^2 + b)\, (e^{\tilde{\delta} t}\, v)$ and use (5.5) to write

$$v = e^{-\tilde{\delta} t}\, \left(u + a^+\, w^+ + a^-\, w^-\right)\,,$$

where

$$u(t) := \frac{1}{W}\left(w^+ \int_*^t w^-\, f\, ds - w^- \int_{**}^t w^+\, f\, ds\right)\,.$$

The bounds of integrations $*$ and $**$ are chosen according to the location of $\tilde{\delta}'$ with respect to $\pm\delta$. We choose $* = ** = +\infty$ if $\tilde{\delta}' < -\delta$, $* = +\infty$ and $** = -\infty$ if $\tilde{\delta}' \in (-\delta, \delta)$ and $* = ** = -\infty$ if $\tilde{\delta}' > \delta$.

Direct estimates (using the fact that $\tilde{\delta}' < \tilde{\delta}$) show that

$$\lim_{t\to+\infty} \|e^{-\tilde{\delta}t}\, u\|_{\mathcal{C}^2([t,t+1])} = 0.$$

This completes the proof of the result. □

5.2 Indicial roots of L

As a byproduct of the analysis in the previous section, we have the :

Proposition 5.2.1. *Assume that a is a t_0-periodic function. Then the indicial roots of the operator*

$$\partial_t^2 + L_\Sigma + a$$

at $+\infty$ (or $-\infty$) are equal to $\pm\delta_j$ where δ_j is the parameter δ, associated to the potential $b = a - \lambda_j$, which appears in the statement of Proposition 5.1.1.

We give a few examples of application of this result.

Example : In the case where the function a is constant, the indicial roots of the operator

$$\partial_t^2 + L_\Sigma + a\,,$$

are given by

$$\delta_j = \Re\,\sqrt{\lambda_j - a}\,,$$

where λ_j are the eigenvalues of $-L_\Sigma$.

Example : Assume that the metric on the cylinder C is conformal to the product metric g_{cyl}, namely

$$g = \varphi^2\,(dt^2 + h)\,,$$

for some smooth t_0-periodic function φ, only depending on t. We consider the operator

$$\Delta_g + a = \tfrac{1}{\varphi^n}\,\partial_t\left(\varphi^{n-2}\,\partial_t\right) + \tfrac{1}{\varphi^2}\,\Delta_h + a\,,$$

where Δ_g is the Laplace-Beltrami operator associated to the metric g and where a is a smooth t_0-periodic function. Observe that the conjugate operator

$$\varphi^{\frac{n+2}{2}}\,(\Delta_g + a)\,(\varphi^{\frac{2-n}{2}}\,w) = \left(\partial_t^2 + \Delta_h + \varphi^2\,a - \tfrac{(n-2)(n-4)}{4}\,(\log\varphi)^2\right)\,w\,,$$

has precisely the form studied above. Obviously, the indicial roots of $\Delta_g + a$ are equal to the indicial roots of the conjugate operator $\varphi^{\frac{n+2}{2}}\,(\Delta_g + a)\,\varphi^{\frac{2-n}{2}}$.

Example : We can also find the indicial roots of the operator

$$L := \partial_t^2 + a\,\partial_t + b\,L_\Sigma + c\,,$$

where a, b and c are t_0-periodic functions only depending on t. We assume that the function b is positive. Indeed, we define the functions φ and ψ by

$$\partial_t \varphi = \sqrt{b} \qquad \text{and} \qquad \partial_t \left(\log(\psi^2\, \partial_t \varphi) \right) = -a,$$

with $\varphi(0) = 0$ and $\psi(0) = 1$. The indicial roots of L can be derived from the knowledge of the indicial roots of the conjugate operator

$$\tilde{L} := \frac{1}{\psi}\, L\,(\psi\, \cdot)\,,$$

once the change of variables $s = \varphi(t)$ has been performed. We find explicitly

$$\tilde{L} = \partial_s^2 + L_\Sigma + \left(\frac{\partial_t^2 \psi + a\, \partial_t \psi + c\, \psi}{b\, \psi} \right) \circ \varphi^{-1}\,.$$

The relation between the indicial roots δ_j of L and $\tilde{\delta}_j$ of $\tilde{L}$ is then given by

$$\delta_j = \frac{1}{t_0} \int_0^{t_0} (b\, \tilde{\delta}_j - \tfrac{1}{2}\, a)\, dt\,.$$

Chapter 6

Analysis in weighted spaces on a cylinder

Assume that (Σ, h) is a compact Riemannian manifold. We denote by $C = \mathbb{R} \times \Sigma$ the cylinder with base Σ which is endowed with the product metric

$$g_{cyl} = dt^2 + h \, .$$

Given $\delta \in \mathbb{R}$, we define the space

$$L^2_\delta(C) := e^{\delta\, t} \, L^2(C) \, ,$$

endowed with the norm

$$\|u\|_{L^2_\delta(C)} := \|e^{-\delta t} \, u\|_{L^2(C)} \, .$$

More generally, for $k \in \mathbb{N}$, we define $W^{k,2}_\delta(C) := e^{\delta t} \, W^{k,2}(C)$ endowed with the norm

$$\|u\|_{W^{k,2}_\delta(C)} := \|e^{-\delta t} \, u\|_{W^{k,2}(C)} \, .$$

It is easy to check that $(W^{k,2}_\delta(C), \| \cdot \|_{W^{k,2}_\delta(C)})$ is a Banach space.

Given a bounded smooth function a defined on C, we are interested in the mapping properties of the (unbounded) operator A_δ defined by

$$\begin{array}{rccc} A_\delta : & L^2_\delta(C) & \longrightarrow & L^2_\delta(C) \\ & u & \longmapsto & (\partial_t^2 + L_\Sigma + a) \, u \, , \end{array}$$

where $L_\Sigma : \mathcal{C}^\infty(\Sigma) \longrightarrow \mathcal{C}^\infty(\Sigma)$ is a second order elliptic operator. The domain of this operator is defined to be the set of functions $u \in L^2_\delta(C)$ such that $f := A_\delta u$

(in the sense of distributions) belongs to $L^2_\delta(C)$. This means that

$$\int_C u\,(\partial_t^2 + L_\Sigma + a)\,v\,dt\,\mathrm{dvol}_h = \int_C f\,v\,dt\,\mathrm{dvol}_h\,,$$

for all $\mathcal{C}^\infty$ functions v with compact support in C (we shall return to this point later on).

6.1 Consequences of standard elliptic estimates

We start with some properties of $\partial_t^2 + L_\Sigma + a$ which are inherited from the corresponding more classical properties for elliptic problems.

Proposition 6.1.1. *Assume that $\delta \in \mathbb{R}$ is fixed. There exists a constant $c > 0$ such that for all $u, f \in L^2_\delta(C)$ satisfying $(\partial_t^2 + L_\Sigma + a)\,u = f$ in the sense of distributions, we have*

$$\|u\|_{W^{2,2}_\delta(C)} \leq c\,(\|f\|_{L^2_\delta(C)} + \|u\|_{L^2_\delta(C)})\,.$$

Proof : First of all, applying Proposition 3.4.1 and Proposition 3.1.2, we see that $u \in W^{2,2}_{loc}(C)$. Indeed, we have

$$(\partial_t^2 + L_\Sigma + a)\,u = f$$

and since the right hand side belongs to $L^2_{loc}(C)$, for all $t_2 > t_1$, we can use the result of Proposition 3.1.2 to get a function $\bar{u} \in W^{2,2}([t_1, t_2] \times \Sigma)$ which solves

$$(\partial_t^2 + L_\Sigma + a)\,\bar{u} = f\,,$$

in $[t_1, t_2] \times \Sigma$ (and for example belongs to $W^{1,2}_0([t_1, t_2] \times \Sigma)$). Therefore, $u - \bar{u}$ solves

$$(\partial_t^2 + L_\Sigma + a)\,(u - \bar{u}) = 0\,,$$

and the result of Proposition 3.4.1 can be applied to conclude that $u - \bar{u} \in \mathcal{C}^\infty((t_1, t_2) \times \Sigma)$. In particular $u = \bar{u} + (u - \bar{u}) \in W^{2,2}((t_1, t_2) \times \Sigma)$.

Now, given $t_* \in \mathbb{R}$ we apply the result of Proposition 3.2.1 with $\Omega = [t_* - 2, t_* + 2] \times \Sigma$ and $\Omega' = [t_* - 1, t_* + 1] \times \Sigma$ and we conclude that

$$\|u\|^2_{W^{2,2}([t_*-1,t_*+1]\times\Sigma)} \leq c\left(\|u\|^2_{L^2([t_*-2,t_*+2]\times\Sigma)} + \|f\|^2_{L^2([t_*-2,t_*+2]\times\Sigma)}\right).$$

It remains to multiply this inequality by $e^{-2\delta t_*}$, and sum the result over $t_* \in \mathbb{Z}$, to obtain

$$\|u\|^2_{W^{2,2}_\delta(C)} \leq c\left(\|u\|^2_{L^2_\delta(C)} + \|f\|^2_{L^2_\delta(C)}\right).$$

This completes the proof of the result. $\square$

6.2 The role of the indicial roots

We now assume that the function a only depends on t and is periodic. We further assume that L_Σ is a self-adjoint second order elliptic operator. The next result explains the importance of the indicial roots δ_j in the study of the operator $\partial_t^2 + L_\Sigma + a$ when defined between weighted L^2-spaces.

Proposition 6.2.1. *Assume that $\delta \neq \pm\delta_j$ for $j \in \mathbb{N}$. Then there exists a constant $c > 0$ such that, for all $u, f \in L^2_\delta(C)$ satisfying*

$$(\partial_t^2 + L_\Sigma + a)\, u = f\,,$$

in the sense of distributions, we have

$$\|u\|_{L^2_\delta(C)} \leq c\, \|f\|_{L^2_\delta(C)}\,.$$

Proof : To prove the result let us perform the eigenfunction decomposition of both u and f. We write

$$u(t,z) = \sum_{j\geq 0} u_j(t,z) \qquad \text{and} \qquad f(t,z) = \sum_{j\geq 0} f_j(t,z)$$

where, for each $j \geq 0$, the functions $u_j(t,\,\cdot\,)$ and $f_j(t,\,\cdot\,)$ belong to E_j for a.e. t. In particular

$$L_\Sigma\, u_j = -\lambda_j\, u_j \qquad \text{and} \qquad L_\Sigma\, f_j = -\lambda_j\, f_j,$$

wherever this makes sense.

Observe that

$$\int_C |u|^2\, e^{-2\delta t}\, dt\, \mathrm{dvol}_h = \sum_{j\geq 0} \int_C |u_j|^2\, e^{-2\delta t}\, dt\, \mathrm{dvol}_h = \sum_{j\geq 0} \int_0^\infty \|u_j\|^2_{L^2(\Sigma)}\, e^{-2\delta t}\, dt\,,$$

and

$$\int_C |f|^2\, e^{-2\delta t}\, dt\, \mathrm{dvol}_h = \sum_{j\geq 0} \int_C |f_j|^2\, e^{-2\delta t}\, dt\, \mathrm{dvol}_h = \sum_{j\geq 0} \int_0^\infty \|f_j\|^2_{L^2(\Sigma)}\, e^{-2\delta t}\, dt\,,$$

where $\|\cdot\|_{L^2(\Sigma)}$ is the $L^2(\Sigma)$-norm. In addition, the functions u_j and f_j satisfy

$$(\partial_t^2 - \lambda_j + a)\, u_j = f_j\,, \tag{6.1}$$

in the sense of distributions in $\mathbb{R}$. Indeed, making use of

$$\int_C u\, (\partial_t^2 + L_\Sigma + a)\, v\, dt\, \mathrm{dvol}_h = \int_C f\, v\, dt\, \mathrm{dvol}_h\,,$$

with test functions of the form $v(t,z) = \varphi(t)\,\phi(z)$ where $\phi \in E_j$ and φ is a smooth function with compact support in $(0,\infty)$, we find that u_j is a E_j-valued function solution of (6.1). Moreover

$$\int_{\mathbb{R}} \|u_j\|^2_{L^2(\Sigma)}\, e^{-2\delta t}\, dt < \infty \qquad \text{and} \qquad \int_{\mathbb{R}} \|f_j\|^2_{L^2(\Sigma)}\, e^{-2\delta t}\, dt < \infty\,.$$

Observe that $u_j \in W^{2,2}_{loc}(\mathbb{R})$ and hence we find by Sobolev embedding that $u_j \in \mathcal{C}^{1,1/2}_{loc}(\mathbb{R})$. Also, arguing as in the proof of Proposition 6.1.1, we conclude that

$$\int_{\mathbb{R}} \|\partial_t u_j\|^2_{L^2(\Sigma)}\, e^{-2\delta t}\, dt < \infty \qquad \text{and} \qquad \int_{\mathbb{R}} \|\partial_t^2 u_j\|^2_{L^2(\Sigma)}\, e^{-2\delta t}\, dt < \infty\,. \tag{6.2}$$

Let j_0 denote the least index in $\mathbb{N}$ such that

$$\delta^2 + a < \lambda_{j_0}\,, \tag{6.3}$$

for all $t \in \mathbb{R}$. The proof of Proposition 6.2.2 is now decomposed into two parts.

Part 1 : We multiply the equation (6.1) by $e^{-2\,\delta\,t}\,u_j$ and integrate over C. We obtain

$$-\int_C u_j\,\partial_t^2 u_j\, e^{-2\,\delta\,t}\, dt\, \mathrm{dvol}_h + \int_C (\lambda_j - a)\, u_j^2\, e^{-2\delta t}\, dt\, \mathrm{dvol}_h = -\int_C u_j\, f_j\, e^{-2\delta\,t}\, dt\, \mathrm{dvol}_h\,.$$

We integrate the first integral twice by parts to get

$$\begin{aligned}\int_C |\partial_t\, u_j|^2\, e^{-2\delta t}\, dt\, \mathrm{dvol}_h + \int_C (\lambda_j - a - 2\,\delta^2)\, u_j^2\, e^{-2\delta t}\, dt\, \mathrm{dvol}_h \\ = -\int_C u_j\, f_j\, e^{-2\delta t}\, dt\ \mathrm{dvol}_h\,.\end{aligned} \tag{6.4}$$

Formally, this computation follows from the fact that

$$u_j\,\partial_t^2 u_j\ e^{-2\delta t} \quad = \quad (-|\partial_t\, u_j|^2 + 2\,\delta^2\, u_j^2)\, e^{-2\delta t} + \partial_t\left((u_j\,\partial_t u_j + \delta\, u_j^2)\, e^{-2\delta t}\right)\,.$$

However, some care is needed to justify the integration by parts at ∞ and this can be achieved by making use of (6.2).

We shall now make use of the following Hardy type inequality

Lemma 6.2.1. *The following inequality holds*

$$\delta^2 \int_{\mathbb{R}} v^2\, e^{-2\delta t}\, dt \leq \int_{\mathbb{R}} |\partial_t v|^2\, e^{-2\delta t}\, dt\,,$$

provided the integral on the left hand side is finite.

Using this Lemma together with (6.4) we conclude that

$$(\lambda_j - a - \delta^2) \int_C u_j^2 \, e^{-2\delta t} \, dt \, \mathrm{dvol}_h \leq \int_C |f_j| \, |u_j| \, e^{-2\delta \, t} \, dt \, \mathrm{dvol}_h \, .$$

where the constant $c > 0$ depends on δ but does not depend on j.

Let us now assume that $j \geq j_0$. Since $\lambda_j - a - \delta^2 > 0$, we conclude, using Cauchy-Schwarz inequality, that

$$(\lambda_j - a - \delta^2)^2 \int_C u_j^2 \, e^{-2\delta t} \, dt \, \mathrm{dvol}_h \leq \int_C f_j^2 \, e^{-2\delta \, t} \, dt \, \mathrm{dvol}_h \, . \tag{6.5}$$

We now sum these inequalities over $j \geq j_0$ to conclude that

$$(\lambda_{j_0} - a - \delta^2)^2 \int_C \tilde{u}^2 \, e^{-2\delta t} \, dt \, \mathrm{dvol}_h \leq \int_C \tilde{f}^2 \, e^{-2\delta \, t} \, dt \, \mathrm{dvol}_h \, . \tag{6.6}$$

where we have defined

$$\tilde{u} = \sum_{j \geq j_0} u_j \qquad \text{and} \qquad \tilde{f} = \sum_{j \geq j_0} f_j \, .$$

Observe that the constant $c > 0$ only depends on δ.

Proof of Lemma 6.2.1: We provide a short proof of the Hardy type inequality we have used. Assume that $\delta \neq 0$ and also that

$$\int_{\mathbb{R}} |\partial_t v|^2 \, e^{-2\delta t} \, dt < \infty \, ,$$

since otherwise there is nothing to prove. We start with the identity

$$-2\,\delta \int_{\mathbb{R}} v^2 \, e^{-2\delta t} \, dt = \int_{\mathbb{R}} v^2 \, \partial_t (e^{-2\delta t}) \, dt = -2 \int_{\mathbb{R}} v \, \partial_t v \, e^{-2\delta t} \, dt \, ,$$

where the last equality follows from an integration by parts. Use Cauchy-Schwarz inequality to conclude that

$$\delta^2 \int_{\mathbb{R}} v^2 \, e^{-2\delta t} \, dt \leq \int_{\mathbb{R}} |\partial_t v|^2 \, e^{-2\delta t} \, dt \, .$$

Observe that, in order to justify the integration by parts, it is enough to assume that $\displaystyle\int_{\mathbb{R}} v^2 \, e^{-2\delta t} \, dt$ converges. This completes the proof of Lemma 6.2.1. $\square$

Part 2 : It remains to estimate u_j, for $j = 0, \ldots, j_0 - 1$. Here we simply use the fact that we have an explicit expression for u_j in terms of f_j. We use the analysis of the previous chapter and define $w_j^\pm$ to be the two independent solutions of the homogeneous problem

$$(\partial_t^2 - \lambda_j + a) \, w_j^\pm = 0 \, ,$$

which have been defined in Proposition 5.1.1. Let W_j denote the Wronskian of these two solutions. We define the function $\tilde{u}_j$ by

$$\tilde{u}_j(t,\,\cdot) = \frac{1}{W_j}\left(w_j^+(t)\int_*^t w_j^-(s)\, f_j(s,\cdot\,)\, ds - w_j^-(t)\int_{**}^t w_j^+(s)\, f_j(s,\cdot\,)\, ds\right) \tag{6.7}$$

where the bounds of integration $*$ and $**$ have to be chosen according to the position of δ with respect to $\pm\delta_j$. In fact, it is easy to see that :

(1) We choose $* = ** = +\infty$ when $\delta < -\delta_j$.

(2) We choose $* = ** = -\infty$ when $\delta > \delta_j$.

(3) We choose $* = +\infty$ and $** = -\infty$ when $-\delta_j < \delta < \delta_j$.

It is easy to check that

$$(\partial_t^2 + L_\Sigma + a)\,\tilde{u}_j = f_j\,,$$

in C. We claim that

$$\int_C \tilde{u}_j^2\, e^{-2\delta t}\, dt\, \mathrm{dvol}_h \le c \int_C f_j^2\, e^{-2\delta t}\, dt\, \mathrm{dvol}_h\,, \tag{6.8}$$

for some constant $c > 0$ depending on j and δ.

Proof of the claim : Let us describe the basic strategy in order to obtain the relevant *a priori* estimate. We first assume that we are either is case (i) or (ii) described in Proposition 5.1.1. Namely, either $\delta_j > 0$, or $\delta_j = 0$ and we assume that $w_j^\pm$ are both bounded. Consider a quantity of the form

$$v_j(t) = \frac{1}{W_j}\, w_j^\pm(t)\int_\bullet^t w_j^\mp(s)\, f_j(s)\, ds$$

where $\bullet$ is either $*$ or $**$ and where the function f_j satisfies

$$\int_{\mathbb{R}} f_j^2(t)\, e^{-2\delta t}\, dt < \infty$$

We need to prove that

$$\int_{\mathbb{R}} v_j^2(t)\, e^{-2\delta t}\, dt \le c\int_{\mathbb{R}} f_j^2(t)\, e^{-2\delta t}\, dt$$

Clearly it is enough to get the corresponding estimate for the function

$$\tilde{v}(t) = e^{\tilde{\delta} t}\int_\bullet^t e^{-\tilde{\delta} s}\, \tilde{f}(s)\, ds$$

where $\tilde{\delta}$ is either δ_j or $-\delta_j$ and $\tilde{f} = |f_j|$. Here $\bullet = -\infty$ if $\delta > \tilde{\delta}$ and $\bullet = +\infty$ if $\delta < \tilde{\delta}$.

Pointwise bound Using Cauchy-Schwarz inequality, we already get the pointwise bound

$$\tilde{v}^2(t) \leq \frac{1}{2\,|\delta - \tilde{\delta}|}\, e^{2\,\delta\, t} \int_{\mathbb{R}} \tilde{f}^2(s)\, e^{-2\delta s}\, ds\,. \tag{6.9}$$

This is where, it is important that $\delta \neq \tilde{\delta}$.

Global estimate Next, using an integration by parts, we get

$$\int_{t_1}^{t_2} \tilde{v}^2(t)\, e^{-2\delta t}\, dt \quad = \quad \frac{1}{2\,(\tilde{\delta} - \delta)} \left[e^{-2\delta s}\, \tilde{v}^2(s) \right]_{t_1}^{t_2} - \frac{1}{\tilde{\delta} - \delta} \int_{t_1}^{t_2} \tilde{f}(s)\, \tilde{v}(s)\, e^{-2\delta s}\, ds$$

for all $0 \leq t_1 < t_2$. Here again, it is important that $\delta \neq \tilde{\delta}$.

The pointwise bound (6.9) yields

$$\int_{t_1}^{t_2} \tilde{v}^2(t)\, e^{-2\delta t}\, dt \quad \leq \quad \frac{1}{4\,(\tilde{\delta} - \delta)^2} \int_{\mathbb{R}} \tilde{f}^2(t)\, e^{-2\delta t}\, dt - \frac{1}{\tilde{\delta} - \delta} \int_{t_1}^{t_2} \tilde{f}(s)\, \tilde{v}(s)\, e^{-2\delta s}\, ds \tag{6.10}$$

Letting t_1 tend to $-\infty$ and t_2 tend to $+\infty$, and using Cauchy-Schwarz inequality, we conclude that

$$\begin{aligned} \int_{\mathbb{R}} \tilde{v}^2(t)\, e^{-2\delta t}\, dt \quad &\leq \quad \frac{1}{4\,(\tilde{\delta} - \delta)^2} \int_{\mathbb{R}} \tilde{f}^2(s)\, e^{-2\delta s}\, ds \\ &+ \quad \frac{1}{|\tilde{\delta} - \delta|} \left(\int_{\mathbb{R}} \tilde{f}^2(s)\, e^{-2\delta s}\, ds \right)^{1/2} \left(\int_{\mathbb{R}} \tilde{v}^2(s)\, e^{-2\delta s}\, ds \right)^{1/2} \end{aligned}$$

from which (using Young's inequality) it is straightforward to conclude that

$$\int_{\mathbb{R}} \tilde{v}^2(t)\, e^{-2\delta t}\, dt \leq \frac{c}{(\tilde{\delta} - \delta)^2} \int_{\mathbb{R}} \tilde{f}^2(s)\, e^{-2\delta s}\, ds\,.$$

This completes the proof of (6.8) in these two cases.

Finally, let us describe the changes which are needed to handle case (iii) described in Proposition 5.1.1. This time we decompose

$$w_j^-(t) = \tilde{w}_j(t) + \nu\, t\, w_j^+(t)\,,$$

where w_j^+ and $\tilde{w}_j$ are periodic. It is easy to check (using some integration by parts formula) that (6.7) can also be written as

$$\begin{aligned} \tilde{u}_j(t, \cdot) \quad &= \quad \frac{1}{W_j} \left(w_j^+(t) \int_*^t \tilde{w}_j(s)\, f_j(s, \cdot)\, ds - \tilde{w}_j(t) \int_*^t w_j^+(s)\, f_j(s, \cdot)\, ds \right) \\ &- \quad \frac{1}{W_j}\, w_j^+(t) \int_*^t \int_*^s w_j^+(\zeta)\, f_j(\zeta, \cdot)\, d\zeta\, ds\,, \end{aligned}$$

where $* = -\infty$ if $\delta > 0$ and $* = +\infty$ if $\delta < 0$. The firt term can obviously be estimated as we have already done. While, w_j^+ being bounded, in order to estimate the second term, it is enough to obtain the relevant estimate for quantities of the form

$$\tilde{v} = \int_*^t \int_*^s \tilde{f}(\zeta)\, d\zeta\, ds$$

where the function $\tilde{f}$ satisfies

$$\int_0^\infty \tilde{f}^2(t)\, e^{-2\delta t}\, dt < \infty\,.$$

Clearly, using the previous strategy, we get for the function

$$\hat{v} = \int_*^t \hat{f}(s)\, ds$$

the weighted estimate

$$\int_{\mathbb{R}} |\hat{v}|^2(t)\, e^{-2\delta t}\, dt \leq \frac{c}{\delta^2} \int_{\mathbb{R}} \hat{f}^2(t)\, e^{-2\delta t}\, dt\,.$$

Using this estimate twice, we conclude that

$$\int_{\mathbb{R}} |\tilde{v}|^2(t)\, e^{-2\delta t}\, dt \leq \frac{c}{\delta^4} \int_{\mathbb{R}} \tilde{f}^2(t)\, e^{-2\delta t}\, dt\,.$$

The proof of (6.8) then follows at once. □

Granted the estimate (6.8), it remains to evaluate the difference between the functions u_j and $\tilde{u}_j$. Since

$$(\partial_t^2 + L_\Sigma + a)\,(u_j - \tilde{u}_j) = 0\,,$$

we find that

$$u_j - \tilde{u}_j = w_j^+\, \phi + w_j^-\, \psi\,,$$

where $\phi, \psi \in E_j$. Remembering that $u_j - \tilde{u}_j \in L^2_\delta(C)$ we find that $\phi = \psi = 0$. This completes the proof of the result. □

As a byproduct, we also get the localized version of Proposition 6.2.1. For all $t > 0$ we define

$$C_t := [t, \infty) \times \Sigma\,,$$

the half cylinder , which is also assumed to be endowed with the metric g_{cyl}. We have the :

Proposition 6.2.2. *Assume that $\delta \neq \pm\delta_j$ for $j \in \mathbb{N}$. Then there exists a constant $c > 0$ such that, for all $u, f \in L^2_\delta(C_0)$ satisfying*

$$(\partial_t^2 + L_\Sigma + a)\, u = f\,,$$

in the sense of distributions, we have

$$\|u\|_{L^2_\delta(C_0)} \leq c\,(\|f\|_{L^2_\delta(C_0)} + \|u\|_{L^2([0,1]\times\Sigma)})\,.$$

Proof : Consider χ a cutoff function which is identically equal to 1 for $t > 3/4$ and identically equal to 0 for $t < 1/4$. Define

$$\tilde{f} = (\partial_t^2 + L_\Sigma + a)\,(\chi\, u)$$

Clearly

$$\|\tilde{f}\|_{L^2_\delta(C)} \leq \|\chi\, f\|_{L^2_\delta(C)} + c\,\|u\|_{W^{2,2}([1/4,3/4]\times\Sigma)}$$

But, it follows from the proof of Proposition 3.2.1 that

$$\|u\|_{W^{2,2}([1/4,3/4]\times\Sigma)} \leq c\,(\|f\|_{L^2([0,1]\times\Sigma)} + \|u\|_{L^2([0,1])\times\Sigma)})\,.$$

Finally, it follows from Proposition 6.2.1 that

$$\|\chi\, u\|_{L^2_\delta(C)} \leq c\,\left(\|f\|_{L^2_\delta(C_0)} + \|u\|_{L^2([0,1])\times\Sigma)}\right),$$

The result then follows at once from these estimates. □

Remark : It is easy to check that, in the main estimate in the statement of Proposition 6.2.2, one can replace $\|u\|_{L^2([0,1]\times\Sigma)}$ by $\|u\|_{L^1([0,1]\times\Sigma)}$.

6.3 Generalization

The results of this chapter hold in greater generality. Indeed, the product metric on the cylinder C can be replaced by *any* metric on C which is t_0 periodic. Indeed, all the norms in the function spaces which have been defined using the metric g_{cyl} are equivalent to the corresponding norms defined (on the same spaces) with the metric g.

More important, the operator $\partial_t^2 + L_\Sigma + a$ can be replaced by any operator of the form

$$L := d\,\left(\partial_t^2 + a\,\partial_t + b\,L_\Sigma + c\right)\,,$$

where a, b, c and d are t_0-periodic functions only depending on t, provided we assume that the functions b and d are positive. We still assume that L_Σ is a self-adjoint second order elliptic operator.

Indeed, as already mentioned at the end of the previous chapter, we can define the functions φ and ψ by

$$\partial_t \varphi = \sqrt{b} \qquad \text{and} \qquad \partial_t \left(\log(\psi^2\, \partial_t \varphi)\right) = -a,$$

with $\varphi(0) = 0$ and $\psi(0) = 1$ and define the conjugate operator

$$\tilde{L} := \frac{1}{d\,\psi}\, L\,(\psi\,\cdot)\,,$$

which, once the change of variables $s = \varphi(t)$ has been performed, can be expressed as

$$\tilde{L} = \partial_s^2 + L_\Sigma + \left(\frac{\partial_t^2 \psi + a\,\partial_t \psi + c\,\psi}{b\,\psi}\right) \circ \varphi^{-1}\,.$$

At the end of the previous Chapter, we have already mentioned the relation between the indicial roots of L and $\tilde{L}$. Now, it is clear that if

$$L\,u = f$$

then

$$\tilde{L}\tilde{u} = \tilde{f}$$

where $u = \psi\,\tilde{u}$ and $f = d\,\psi\,\tilde{f}$. Therefore, all the estimates we have obtained for $\tilde{L}\,\tilde{u} = \tilde{f}$ translate into the corresponding estimates for the equation $L\,u = f$.

Chapter 7

Fourier analysis on a cylinder

The main result of the previous chapter was certainly Proposition 6.2.1 which states that

$$\|u\|_{L^2_\delta(C)} \leq c\,\|(\partial_t^2 + L_\Sigma + a)\,u\|_{L^2_\delta(C)}.$$

provided δ is not an indicial root of the operator $\partial_t^2 + L_\Sigma + a$. The proof of this result relied on the crucial fact that L_Σ and a did not depend on t and hence we could use separation of variables together with eigenfunction decomposition on the cross section to analyze this operator. In the present chapter we recover this result when the potential a is constant. This time the main tool is Fourier analysis along the axis of the cylinder.

This analysis complements the previous analysis. It also has the advantage to provide an explicit expression of the constant c which appears in the above estimate. This explicit expression will allow us to analyze the bounded kernel of operators of the form $\partial_t^2 + L_\Sigma + a$ when the potential a is *any* bounded function.

7.1 Fourier analysis

Recall that $C = \mathbb{R} \times \Sigma$ is the cylinder endowed with the product metric $g_{cyl} = dt^2 + h$. We now assume that the potential function a is constant and, without loss of generality, it can be assumed to be equal to 0, since this simply amounts to change $L_\Sigma + a$ into L_Σ. So, let us assume that $a \equiv 0$. As usual, we denote the eigenvalues of $-L_\Sigma$ by $\lambda_0 < \lambda_1 < \lambda_2 \ldots$ and the associated eigenspaces by $E_0, E_1, E_2, \ldots$.

As promised, we recover the result of Proposition 6.2.1 and we also provide an explicit formula for the constant c which appeared in the statement of Proposition 6.2.1.

Proposition 7.1.1. *Assume that $\delta \in \mathbb{R}$ is chosen so that $\delta^2 \neq \lambda_j$ for all $j \geq 0$. Then, for all $u, f \in L^2_\delta(C)$ satisfying*

$$(\partial_t^2 + L_\Sigma)\, u = f\,,$$

in the sense of distributions, we have

$$\|u\|_{L^2_\delta(C)} \leq \sup_{\xi \in \mathbb{R}} \sup_{j \in \mathbb{N}} |(\delta + i\xi)^2 - \lambda_j|^{-1}\, \|f\|_{L^2_\delta(C)}.$$

Proof : First observe that, if $u, f \in L^2_\delta(C)$ satisfy

$$(\partial_t^2 + L_\Sigma)\, u = f,$$

then $U = e^{-\delta t}\, u$ and $F = e^{-\delta t}\, f$ satisfy

$$e^{-\delta t}\, \left(\partial_t^2 + L_\Sigma\right) (e^{\delta t}\, U) = F$$

and $U, F \in L^2(C)$. Therefore, what we really need to prove is the estimate

$$\|U\|_{L^2(C)} \leq \sup_{\xi \in \mathbb{R}} \sup_{j \in \mathbb{N}} |(\delta + i\xi)^2 - \lambda_j|^{-1}\, \|F\|_{L^2(C)}$$

provided $B_\delta\, U = F$ where the operator B_δ is defined by

$$B_\delta := e^{-\delta t}\, \left(\partial_t^2 + L_\Sigma\right) (e^{\delta t}\, \cdot) = \partial_t^2 + 2\,\delta\,\partial_t + \delta^2 + L_\Sigma\,,$$

We perform the Fourier transform of both u and f in the t variable. Hence, we write

$$\hat{U}(\xi, \cdot) = \frac{1}{\sqrt{2\,\pi}} \int_{\mathbb{R}} U(t, \cdot)\, e^{i\xi t}\, dt$$

and

$$\hat{F}(\xi, \cdot) = \frac{1}{\sqrt{2\,\pi}} \int_{\mathbb{R}} F(t, \cdot)\, e^{i\xi t}\, dt\,.$$

It is easy to check that $\hat{U}(\xi, \cdot)$ and $\hat{F}(\xi, \cdot)$ are solutions (in the sense of distributions) of

$$\left(L_\Sigma + (\delta + i\,\xi)^2\right)\, \hat{U} = \hat{F}\,.$$

We define the operator

$$\hat{B}_{\delta + i\,\xi} := L_\Sigma + (\delta + i\,\xi)^2$$

acting on complex valued functions and, to keep notations short, we set

$$z := \delta + i\,\xi \in \mathbb{C}$$

Remember that the eigenvalues of $-L_\Sigma$ are denoted by λ_j, with $\lambda_0 < \lambda_1 < \lambda_2 \ldots$ and let us denote the corresponding eigenspaces by $E_0, E_1, E_2 \ldots$. If

$$z^2 \neq \lambda_j\,, \tag{7.1}$$

for all $j \in \mathbb{N}$, (namely if $z^2 \in \mathbb{C}$ is not an eigenvalue of $-L_\Sigma$), then there exists a bounded operator

$$\hat{R}_z : L^2(\Sigma) \longmapsto L^2(\Sigma)\,,$$

which is a right inverse of for $\hat{B}_z$, namely $\hat{B}_z \circ \hat{R}_z = I$ in $L^2(\Sigma)$. Using the eigendata decomposition, we even have an explicit formula for $\hat{R}_z$

$$\hat{R}_z\,\hat{F} = \sum_{j\in\mathbb{N}} \frac{1}{z^2-\lambda_j}\,\hat{F}_j \qquad \text{if} \qquad \hat{F} = \sum_{j\in\mathbb{N}} \hat{F}_j\,,$$

with $\hat{F}_j \in E_j$. Plancherel formula then implies that

$$\|\hat{R}_z\,\hat{F}\|^2_{L^2(\Sigma)} = \sum_{j\in\mathbb{N}} |z^2-\lambda_j|^{-2}\,\|\hat{F}_j\|^2_{L^2(\Sigma)}$$

and hence

$$\|\hat{R}_z\,\hat{F}\|_{L^2(\Sigma)} \le \sup_{j\in\mathbb{N}} |z^2-\lambda_j|^{-1}\,\|\hat{F}\|_{L^2(\Sigma)}\,. \tag{7.2}$$

Hence we get the explicit bound

$$\|\hat{R}_z\,\hat{f}\|_{L^2(\Sigma)} \le c_\delta\,\|\hat{R}_z\,\hat{f}\|_{L^2(\Sigma)}. \tag{7.3}$$

where

$$c_\delta := \sup_{\xi\in\mathbb{R}}\sup_{j\in\mathbb{N}} |(\delta+i\xi)^2-\lambda_j|^{-1}$$

By Fourier inverse formula, we have

$$U(t,y) = \frac{1}{\sqrt{2\,\pi}} \int_{\mathbb{R}} e^{-i\xi t}\,\hat{U}(\xi,y)\,d\xi$$

and

$$F(t,y) = \frac{1}{\sqrt{2\,\pi}} \int_{\mathbb{R}} e^{-i\xi t}\,\hat{F}(\xi,y)\,d\xi$$

Therefore, if $\delta^2 \neq \lambda_j$, for all $j \ge 0$, we can write

$$\hat{U} = \hat{R}_z\,\hat{F}\,,$$

and we have the (formal) expression of u in terms of $\hat{f}$, which is given by

$$U(t,y) = \frac{1}{\sqrt{2\,\pi}} \int_{\mathbb{R}} e^{-i\xi t}\,\hat{R}_z\,\hat{F}(\xi,y)\,d\xi$$

In any case, (7.3) implies that

$$\|\hat{U}\|_{L^2(\Sigma)} \le c_\delta\,\|\hat{F}\|_{L^2(\Sigma)}$$

and, using the fact that the Fourier transform is an isometry in $L^2(\mathbb{R})$, we can estimate

$$\begin{aligned}\|U\|^2_{L^2(C)} &= \|\hat{U}\|^2_{L^2(C)} = \int_{\mathbb{R}} \int_{\Sigma} |\hat{U}|^2 \, d\xi \, \mathrm{dvol}_h \\ &\leq c_\delta \int_{\mathbb{R}} \int_{\Sigma} |\hat{F}|^2 \, d\xi \, \mathrm{dvol}_h \\ &\leq c_\delta \, \|\hat{F}\|^2_{L^2(C)} = c_\delta \, \|F\|^2_{L^2(C)}\end{aligned}$$

This completes the proof of the Proposition. □

Observe that, if $\delta^2 + \lambda_j \geq 0$ then

$$|(\delta + i\xi)^2 - \lambda_j|^2 \geq |\delta^2 - \lambda_j|^2$$

for all $\xi \in \mathbb{R}$ while, in the case where $\delta^2 + \lambda_j \leq 0$ then

$$|(\delta + i\xi)^2 - \lambda_j|^2 \geq -4\, \lambda_j \, \delta^2$$

for all $\xi \in \mathbb{R}$. Hence, we have the general formula

$$\sup_{\xi \in \mathbb{R}} \sup_{j \in \mathbb{N}} |(\delta + i\xi)^2 - \lambda_j|^{-1} = \max \left(\sup_{j \geq j_\delta} |\delta^2 - \lambda_j|^{-1}, \max_{j \leq j_\delta} (-4\, \lambda_j \, \delta^2)^{-1/2} \right)$$

where $j_\delta := \min\{j \in \mathbb{N} \quad : \quad \lambda_j + \delta^2 \geq 0\}$. In the case where $\delta^2 + \lambda_0 \geq 0$ then $j_\delta = 0$ and the above formula simplifies into

$$\sup_{\xi \in \mathbb{R}} \sup_{j \in \mathbb{N}} |(\delta + i\xi)^2 - \lambda_j|^{-1} = \sup_{j \geq 0} |\delta^2 - \lambda_j|^{-1}. \tag{7.4}$$

7.2 The bounded kernel of Schödinger type operators

Building on the previous result, we analyze some properties of the space of functions which are bounded and which belong to the kernel of

$$\partial_t^2 + L_\Sigma + a,$$

when the potential a is only assumed to be in $L^\infty(C)$. Obviously, when a is constant then this space if finite dimensional. The analysis of Chapter 5 shows that this space is still finite dimensional when a is assumed to be periodic and only to depend on t.

Now, we do not assume that a is constant nor that it is a periodic function of t and we nevertheless show that the bounded kernel of $\partial_t^2 + L_\Sigma + a$ is finite dimensional provided there are large enough gaps between some consecutive eigenvalues of Λ_Σ. To be more precise, we assume that :

There exists $d > 0$ and $j_0 \in \mathbb{N}$ such that:

$$\lambda_{j_0+1} - \lambda_{j_0} > (1+d)\,\|V\|_{L^\infty} \qquad \text{and} \quad \lambda_{j_0} + \lambda_0 \geq 0\,. \tag{7.5}$$

Under this assumption, we define

$$\delta := \sqrt{\frac{\lambda_{j_0+1} + \lambda_{j_0}}{2}}$$

Using the results of the previous section, we prove the :

Proposition 7.2.1. *Assume that (7.5) holds. Then there exists $c > 0$ (depending on (Σ, h), L_Σ, $\|a\|_{L^\infty}$, λ_0 and d), such that*

$$\|e^{-\delta\,|t|}\,w\|_{L^2(C)} \leq c\,\|w\|_{L^2((-1,1)\times\Sigma)} \tag{7.6}$$

for any function $w \in L^\infty(C)$ such that $(\partial_t^2 + L_\Sigma + a)\,w = 0$.

Proof : To keep notations simple, we agree that the constants $c > 0$ may increase from line to line but are constants which only depend on (Σ, h), L_Σ, $\|a\|_{L^\infty}$, , λ_0 and d and the choice of δ (which itself depend on the previous data). Assume that

$$(\partial_t^2 + L_\Sigma + a)\,w = 0$$

and that $w \in L^\infty(C)$. Let χ be a cutoff function only depending on t, equal to 0 for $t < -1/2$ and equal to 1 for $t > 1/2$.

Observe that the second part of (7.5) implies that $\delta^2 + \lambda_0 \geq 0$ and hence we can use (7.4). Then, the first part of (7.5) together with the choice of δ, implies that

$$\|a\|_{L^\infty} \sup_{\xi\in\mathbb{R}} \sup_{j\in\mathbb{N}} |(\delta + i\xi)^2 - \lambda_j|^{-1} < \frac{1}{1+d}. \tag{7.7}$$

We set

$$f := (\partial_t^2 + L_\Sigma)\,(\chi\,w) = -a\,\chi\,w\, + [L_\Sigma, \chi]\,w$$

where $[L_\Sigma, \chi]\,w := L_\Sigma(\chi\,w) - w\,L_\Sigma\,w$. Observe that $[L_\Sigma, \chi]\,w$ is supported in $(-1/2, 1/2)\times\Sigma$ and depends on w and the first partial derivatives of w. Therefore, we have the estimate

$$\|e^{-\delta t}\,[L_\Sigma, \chi]\,w\|_{L^2(C)} \leq c\,\|w\|_{W^{1,2}((-1/2,1/2)\times\Sigma)}$$

We now apply the result of Proposition 7.1.1 to the functions $u := \chi\,w$ and f, the operator $\partial_t^2 + L_\Sigma$ and the parameter δ, to conclude that

$$\begin{aligned} \|e^{-\delta t}\,\chi\,w\|_{L^2(C)} \;&\leq\; \sup_{\xi\in\mathbb{R}} \sup_{j\in\mathbb{N}} |(\delta + i\xi)^2 - \lambda_j|^{-1} \|a\|_{L^\infty}\,\|e^{-\delta t}\,\chi\,w\|_{L^2(C)} \\ &+\; c\,\|w\|_{W^{1,2}((-1/2,1/2)\times\Sigma)} \end{aligned}$$

Here we have implicitly used the fact that $\partial_t \chi$ and $\partial_t^2 \chi$ is supported in $(-1/2, 1/2) \times \Sigma$. Thanks to (7.7) we conclude that

$$\|e^{-\delta t}\, \chi\, w\|_{L^2(C)} \leq \frac{1}{1+d}\, \|e^{-\delta t}\, \chi\, w\|_{L^2(C)} + c\, \|w\|_{W^{1,2}((-1/2,1/2)\times\Sigma)}$$

and hence,

$$\|e^{-\delta t}\, \chi\, w\|_{L^2(C)} \leq c\, \|w\|_{W^{1,2}((-1/2,1/2)\times\Sigma)} \tag{7.8}$$

We apply the result of Proposition 1.1 to the functions $u := (1-\chi)\, w$ and this time we change δ into $-\delta$. Arguing as above, we conclude that

$$\|e^{\delta t}\, (1-\chi)\, w\|_{L^2(C)} \leq c\, \|w\|_{W^{1,2}((-1/2,1/2)\times\Sigma)} \tag{7.9}$$

Collecting (7.8) and (7.9), we get

$$\|e^{-\delta\, |t|}\, w\|_{L^2(C)} \leq c\, \|w\|_{W^{1,2}((-1/2,1/2)\times\Sigma)} \tag{7.10}$$

To complete the proof, it suffices to observe that classical elliptic estimates applied to the solution w of

$$(\partial_t^2 + L_\Sigma)\, w = -a\, w$$

imply that

$$\|w\|_{W^{1,2}((-1/2,1/2)\times\Sigma)} \leq c\, \|w\|_{L^2((-1,1)\times\Sigma)} \tag{7.11}$$

The proof of the estimate is now complete. □

Using the above estimate, we get

Proposition 7.2.2. *Assume that (7.5) holds. Then the dimension of the space of functions which are bounded and belong to the kernel of $\partial_t^2 + L_\Sigma + a$ is bounded by a constant only depending on (Σ, h), L_Σ, $\|a\|_{L^\infty}$, λ_0 and d.*

Proof : To keep notations simple, we agree that the constants $c > 0$ may increase from line to line but are constants which only depend on the data. Assume that w is bounded solution of $(\partial_t^2 + L_\Sigma + a)\, w = 0$. The estimate of the previous proposition implies that

$$\|w\|_{L^2([-2,2]\times\Sigma)} \leq c\, \|w\|_{L^2([-1,1]\times\Sigma)}$$

But elliptic regularity implies that

$$\|w\|_{L^\infty([-1,1]\times\Sigma)} \leq c\, \|w\|_{L^2([-2,2]\times\Sigma)}$$

So we conclude that

$$\|w\|_{L^\infty([-1,1]\times\Sigma)} \leq c\, \|w\|_{L^2([-1,1]\times\Sigma)} \tag{7.12}$$

for any bounded solution of $(\partial_t^2 + L_\Sigma + a)\, w = 0$.

Now, let us denote by $\phi_1, \ldots, \phi_m$ an orthonormal basis (in $L^2([-1,1]\times\Sigma)$) of the bounded kernel of $\partial_t^2 + L_\Sigma + a$. We consider the Bergman kernel associated to the orthogonal projection in $L^2([-1,1]\times\Sigma)$ onto the space spanned by the bounded kernel of $\partial_t^2 + L_\Sigma + a$. We have explicitly

$$K(x,y) = \sum_j \phi_j(x)\phi_j(y)$$

Observe that K is independent of the choice of the orthonormal basis. Also

$$m = \int_{[-1,1]\times\Sigma} K(x,x)\, dvol$$

is the dimension of the bounded kernel of $\partial_t^2 + L_\Sigma + a$. Obviously, there exists $x_0 \in [-1,1]\times\Sigma$ such that

$$K(x_0,x_0)\mathrm{Vol}([-1,1]\times\Sigma) \geq m$$

Consider the evaluation form

$$\mathcal{E}_{x_0}(\phi) = \phi(x_0)$$

and choose the orthonormal basis $\phi_1, \ldots, \phi_m$ such that $\mathcal{E}_{x_0}(\phi_j) = 0$ for $j = 2, \ldots, m$. Then

$$K(x_0,x_0)\,\mathrm{Vol}([-1,1]\times\Sigma) = \phi_1(x_0)^2\,\mathrm{Vol}([-1,1]\times\Sigma) \geq m$$

But (9.4) implies that

$$\|\phi_1\|_{L^\infty([-1,1]\times\Sigma)} \leq c\,\|\phi_1\|_{L^2([-1,1]\times\Sigma)} = c$$

Therefore $m \leq c$ This completes the proof. □

Many generalizations are possible (under suitable assumption on the gaps in the spectrum of the operator L_Σ. For example one can prove a similar result for solutions in the kernel of which are bounded by $(\cosh t)^\delta$.

7.3 Bibliography

The proof of Proposition 7.1.1 is essentially the one developed in the paper by R.B. Lockhart and R.C McOwen, *Elliptic differential operators on noncompact manifolds*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 1, no. 3, (1985) 409-447. It has the advantage to be simple to implement and also to extend immediately to higher order elliptic operators whose coefficients do not depend on the t variable.

Proposition 7.2.2 is inspired by the recent work of T. Colding, C. de Lellis and W. Minicozzi *Three circles theorems for Schrödinger operators on cylindrical ends* *and geometric applications*, mathDG/0701302 where similar results are proven for Schrödinger operators defined on manifolds with cylindrical ends.

The proof of Proposition 7.2.2 seems to be classical. Similar arguments can be found in the paper by H. Donelly *Eigenfunctions of the Laplacian on compact Riemannian manifolds*, Asian J. Math. Vol. 10, No. 1, pp. 115126, (2006). In this paper, the starting point is Hörmander's estimate

$$\|\phi\|_{L^\infty} \leq c\,\lambda^{\frac{n-1}{4}}\,\|\phi\|_{L^2},$$

for eigenfunctions ϕ of $-\Delta_g$ on a compact manifold (M,g). Here the constant $c>0$ does not depend on λ. This estimate plays the role of the result of Proposition 7.2.1). Then following the proof of Proposition 7.2.2, one shows that the multiplicity of λ is bounded by a constant (independent of λ) times $\lambda^{\frac{n?1}{2}}$.

Chapter 8

Analysis on manifolds with cylindrical ends

We will say that a complete, noncompact n-dimensional manifold (M, g) is a manifold with cylindrical ends if it can be decomposed into the union of a compact piece $K \subset\subset M$ and finitely many ends $\mathcal{E}^{(1)}, \ldots, \mathcal{E}^{(m)}$, each of which is diffeomorphic to

$$C_0^{(i)} := [0, \infty) \times \Sigma^{(i)} ,$$

where $(\Sigma^{(i)}, h^{(i)})$ is a $(n-1)$-dimensional compact Riemannian manifold. Some information about the metric g will be required. We would like to transplant the previous analysis to the setting.

8.1 The function spaces

In this chapter we simply need to assume that the geometry of g is controlled uniformly along each end $\mathcal{E}^{(i)}$. A simple way to express the assumption needed (nevertheless covering almost all geometrically interesting situations) is to require that the metric g restricted to $\mathcal{E}^{(i)}$ is asymptotic to some metric $g_{per}^{(i)}$ defined on $\mathbb{R} \times \Sigma^{(i)}$ which is $t_0^{(i)}$-periodic in the t variable.

Henceforth, we assume that there exists $\theta > 0$ and $c > 0$ such that the coefficients of g satisfy

$$\|e^{\theta\, t}\,(g - g_{per}^{(i)})_{ab}\|_{\mathcal{C}^2(C_0^{(i)}, g_{per}^{(i)})} \leq c\,, \tag{8.1}$$

on each $\mathcal{E}^{(i)}$. This assumption ensures that the spaces $W^{2,2}(\mathcal{E}^{(i)}, g)$, $W^{2,2}(C_0^{(i)}, g_{per}^{(i)})$ and $W^{2,2}(C_0^{(i)}, dt^2 + h^{(i)})$ are the same.

Given a m-tuple

$$\bar{\delta} = (\delta^{(1)}, \ldots, \delta^{(m)}) \in \mathbb{R}^m\,.$$

Let

$$\Gamma_{\bar\delta} : M \longrightarrow (0, \infty)\,,$$

be a smooth positive function which coincides with $e^{\delta^{(i)} t}$ on the end $\mathcal{E}^{(i)}$ of M.

We define

$$L^2_{\bar\delta}(M) = \Gamma_{\bar\delta}\, L^2(M, g)\,.$$

The norm in this weighted space is naturally defined as

$$\|u\|_{L^2_{\bar\delta}(M)} := \|\Gamma_{\bar\delta}^{-1}\, u\|_{L^2(M,g)}\,.$$

More generally, for $k \in \mathbb{N}$, we define $W^{k,2}_{\bar\delta}(M) := \Gamma_{\bar\delta}\, W^{k,2}(M)$ endowed with the norm

$$\|u\|_{W^{k,2}_{\bar\delta}(M)} := \|\Gamma_{\bar\delta}^{-1}\, u\|_{W^{k,2}(M,g)}\,.$$

It is easy to check that $(W^{k,2}_{\bar\delta}(M), \|\cdot\|_{W^{k,2}_{\bar\delta}(M)})$ is a Banach space.

Example : We consider the simple case where $M := \mathbb{R} \times \Sigma$ is endowed with the product metric

$$g = ds^2 + h\,,$$

where h is a metric on Σ and s is the coordinate on $\mathbb{R}$. This manifold has 2 cylindrical ends $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$ and we can choose $t = -s$ for $s < 0$ to describe the end $\mathcal{E}^{(1)}$, and $t = s$, for $s > 0$ to describe the other end $\mathcal{E}^{(2)}$. With these choices

$$L^2_\delta(C) := e^{\delta s}\, L^2(\mathbb{R} \times \Sigma) = L^2_{(-\delta,\delta)}(\mathbb{R} \times \Sigma)\,.$$

Where $L^2_\delta(C)$ is the space already defined in Chapter 6. While

$$(\cosh s)^\delta\, L^2(\mathbb{R} \times \Sigma) = L^2_{(\delta,\delta)}(\mathbb{R} \times \Sigma)\,.$$

Warning : one should not confuse $L^2_\delta(C)$ with $\delta \in \mathbb{R}$ and $L^2_{\bar\delta}(C)$ with $\bar\delta = (\delta_1, \delta_2) \in \mathbb{R}^2$.

Remark : Observe that there is no uniqueness in the choice of the coordinate t describing a given end $\mathcal{E}$ of M since one can for example change t into $t = \lambda\, \tilde t$ and which would have the effect to change the definition of the weight and hence of the weighted spaces. We shall assume from now on that on each end a particular choice of coordinate t has been done and one will easily check that the forthcoming results will be independent of such a choice.

For all $t > 0$, it will be convenient to define $\mathcal{E}^{(i)}(t) := [t, \infty) \times \Sigma^{(i)}$ and

$$M_t := M \setminus \mathring{\mathcal{E}}^{(1)}(t) \cup \ldots \cup \mathring{\mathcal{E}}^{(m)}(t)\,.$$

8.2 Elliptic operators

On (M, g) we would like to study some elliptic operator $\mathcal{L}$ which has the property to be asymptotic to the model operator

$$L^{(i)} := d^{(i)} \left(\partial_t^2 + a^{(i)} \, \partial_t + b^{(i)} \, L_{\Sigma^{(i)}} + c^{(i)} \right),$$

on the end $\mathcal{E}^{(i)}$. Here the functions $a^{(i)}, b^{(i)}, c^{(i)}$ and $d^{(i)}$ are smooth, $t_0^{(i)}$-periodic functions on each $\mathcal{E}^{(i)}$ and $b^{(i)}$ and $d^{(i)}$ are positive. This means that there exists $\theta > 0$ and $c > 0$ such that on each $\mathcal{E}^{(i)}$.

$$\|e^{\theta \, t} \, (\mathcal{L} - L^{(i)}) w\|_{L^2(\mathcal{E}^{(i)})} \leq c \, \|w\|_{W^{2,2}(\mathcal{E}^{(i)})} \, , \tag{8.2}$$

for any function $w \in W^{2,2}(\mathcal{E}^{(i)})$.

Given a m-tuple $\bar{\delta} = (\delta^{(1)}, \ldots, \delta^{(m)}) \in \mathbb{R}^m$, we define the unbounded operator

$$\begin{array}{rccc} A_{\bar{\delta}} : & L^2_{\bar{\delta}}(M) & \longrightarrow & L^2_{\bar{\delta}}(M) \\ & u & \longmapsto & \mathcal{L}\, u \end{array}$$

8.3 The domain of the operator $A_{\bar{\delta}}$

The result we have obtained in Proposition 6.1.1 immediately translates into :

Proposition 8.3.1. *Assume $\bar{\delta} \in \mathbb{R}^m$ is fixed. There exists a constant $c > 0$ (depending on $\bar{\delta}$) such that for all $u \in L^2_{\bar{\delta}}(M)$ satisfying $\mathcal{L}\, u \in L^2_{\bar{\delta}}(M)$, we have*

$$\|u\|_{W^{2,2}_{\bar{\delta}}(M)} \leq c \, (\|\mathcal{L}\, u\|_{L^2_{\bar{\delta}}(M)} + \|u\|_{L^2_{\bar{\delta}}(M)}) \, .$$

Proof : The proof of the result goes as follows : Let χ be a cutoff function identically equal to 0 in $M_{1/2}$ and identically equal to 1 on each $\mathcal{E}^{(i)}(1)$.

Applying Proposition 6.1.1 to the operator $L^{(i)}$ and the function $\chi\, u$ defined on $[0, +\infty) \times \Sigma^{(i)}$, we get

$$\|\chi\, u\|_{W^{2,2}_{\delta^{(i)}}(\mathcal{E}^{(i)})} \leq c \left(\|L^{(i)} \, (\chi\, u)\|_{L^2_{\delta^{(i)}}(\mathcal{E}^{(i)})} + \|\chi\, u\|_{L^2_{\delta^{(i)}}(\mathcal{E}^{(i)})} \right) \, .$$

Using (8.2), we evaluate

$$\|(\mathcal{L} - L^{(i)}) \, (\chi\, u)\|_{L^2_{\delta^{(i)}}(\mathcal{E}^{(i)})} \leq c \, \|\chi\, u\|_{W^{2,2}_{\delta^{(i)} - \theta}(\mathcal{E}^{(i)})} \, .$$

Collecting these two inequalities, we conclude easily that

$$\|u\|_{W^{2,2}_{\bar{\delta}}(M)} \leq c \left(\|\mathcal{L}\, u\|_{L^2_{\bar{\delta}}(M)} + \|u\|_{L^2_{\bar{\delta}}(M)} + \|u\|_{W^{2,2}_{\bar{\delta} - \bar{\theta}}(M)} \right) \tag{8.3}$$

where $\bar\theta := (\theta, \dots, \theta)$. We fix $t > 0$ and decompose the last term on the right hand side of this inequality

$$\begin{aligned} \|u\|_{W^{2,2}_{\bar\delta-\bar\theta}(M)} &= \|u\|_{W^{2,2}_{\bar\delta-\bar\theta}(M_t)} + \|u\|_{W^{2,2}_{\bar\delta-\bar\theta}(M\setminus M_t)} \\ &\leq c\, e^{-\theta t}\, \|u\|_{W^{2,2}_{\bar\delta}(M)} + c_t\, \|u\|_{W^{2,2}(M_t)}\,, \end{aligned}$$

where $c_t > 0$ depends on $t > 0$ while $c > 0$ does not. Inserting this into (8.3), we conclude that

$$\begin{aligned} \|u\|_{W^{2,2}_{\bar\delta}(M)} &\leq c\, \left(\|\mathcal{L}\, u\|_{L^2_{\bar\delta}(M)} + \|u\|_{L^2_{\bar\delta}(M)} + e^{-\theta t}\, \|u\|_{W^{2,2}_{\bar\delta}(M)} \right) \\ &+ c_t\, \|u\|_{W^{2,2}(M_t)}\,, \end{aligned}$$

where $c_t > 0$ depends on $t > 0$ while $c > 0$ does not.

Now, we use the elliptic estimates provided by Proposition 3.2.2 with $\Omega = M_{t+1}$ and $\Omega' = M_t$ to show that

$$\|u\|_{W^{2,2}(M_t)} \leq c'_t\, \left(\|\mathcal{L}\, u\|_{L^2(M_{t+1})} + \|u\|_{L^2(M_{t+1})} \right)\,,$$

for some constants $c'_t > 0$ depending on $t > 0$.

Collecting these estimate, we get

$$\|u\|_{W^{2,2}_{\bar\delta}(M)} \leq c''_t\, \left(\|\mathcal{L}\, u\|_{L^2_{\bar\delta}(M)} + \|u\|_{L^2_{\bar\delta}(M)} \right) + c\, e^{-\theta t}\, \|u\|_{W^{2,2}_{\bar\delta}(M)}.$$

for some constant $c''_t > 0$ depending on $t > 0$ and some constant $c > 0$ independent of $t > 0$.

Finally, we choose $t > 0$ so that $c\, e^{-\theta t} \leq 1/2$ to conclude that

$$\|u\|_{W^{2,2}_{\bar\delta}(M)} \leq \bar c_t\, \left(\|\mathcal{L}\, u\|_{L^2_{\bar\delta}(M)} + \|u\|_{L^2_{\bar\delta}(M)} \right)\,.$$

This completes the proof of the result. □

The domain of the operator $A_{\bar\delta}$ is defined to be

$$\mathrm{Dom}\, A_{\bar\delta} = \{u \in L^2_{\bar\delta}(M)\, :\, \mathcal{L}\, u \in L^2_{\bar\delta}(M)\}$$

It follows from the previous Proposition that

$$\mathrm{Dom}\, A_{\bar\delta} = W^{2,2}_{\bar\delta}(M),$$

And it is a simple exercise to check that

$$\begin{aligned} \mathrm{Dom}\, A_{\bar\delta} = & \{u \in L^2_{\bar\delta}(M)\, :\, \exists (u_m)_m \in \mathcal{C}^\infty_0(M) \\ & \text{such that} \quad (u_m)_m \longrightarrow u \quad \text{and} \quad (\mathcal{L}\, u_m)_m \longrightarrow u \text{ in } L^2_{\bar\delta}(M)\} \end{aligned}$$

As a consequence, we also obtain the following Lemma whose proof is left to the reader.

Lemma 8.3.1. *The operator $A_{\bar\delta}$ has dense domain and closed graph.*

8.4 An *a priori* estimate

In order to further extend the results we have obtained so far to manifolds with cylindrical ends, we need some definition :

Definition 8.4.1. *The set of indicial roots of the operator $\mathcal{L}$, denoted by $\mathrm{Ind}\, L$, is the collection of $(\delta^{(1)}, \ldots, \delta^{(m)}) \in \mathbb{R}^m$ where $\delta^{(i)}$ is an indicial root of the restriction of $\mathcal{L}$ on $\mathcal{E}^{(i)}$.*

Observe that $\mathrm{Ind}\, \mathcal{L}$ is also the collection of $(\delta^{(1)}, \ldots, \delta^{(m)}) \in \mathbb{R}^m$ where $\delta^{(i)}$ is an indicial root of $L^{(i)}$ on $(0, \infty) \times \Sigma^{(i)}$.

The following result is a consequence of Proposition 6.2.2.

Proposition 8.4.1. *Assume that $\bar{\delta} \notin \mathrm{Ind}\mathcal{L}$. Then, there exists a constant $c > 0$ and a compact K in M such that, for all $u \in L^2_{\bar{\delta}}(M)$ satisfying $\mathcal{L}\, u \in L^2_{\bar{\delta}}(M)$, we have*

$$\|u\|_{L^2_{\bar{\delta}}(M)} \leq c \left(\|\mathcal{L}\, u\|_{L^2_{\bar{\delta}}(M)} + \|u\|_{L^2(K)} \right)$$

This result states that we can control the weighted L^2-norm of u in terms of the weighted L^2-norm of $\mathcal{L}\, u$ and some information about the function u away from the ends.

Proof : We keep the notations of the proof of Proposition 8.3.1. This time, we apply Proposition 6.2.2 (instead of Proposition 6.1.1) to the operator $L^{(i)}$ and the function $\chi\, u$ defined on $[0, +\infty) \times \Sigma^{(i)}$, we get

$$\|\chi\, u\|_{W^{2,2}_{\delta^{(i)}}(\mathcal{E}^{(i)})} \leq c\, \|L^{(i)}\, (\chi\, u)\|_{L^2_{\delta^{(i)}}(\mathcal{E}^{(i)})} \, ,$$

since by assumption, $\delta^{(i)}$ is not an indicial root of $L^{(i)}$. Arguing as in the previous proof, we get

$$\|u\|_{W^{2,2}_{\bar{\delta}}(M)} \leq c \left(\|\mathcal{L}\, u\|_{L^2_{\bar{\delta}}(M)} + \|u\|_{W^{2,2}_{\bar{\delta}-\bar{\theta}}(M)} \right) \, ,$$

instead of (8.3). The rest of the proof is left to the reader since it is identical to the proof of Proposition 8.3.1. $\square$

Chapter 9

Fredholm properties

We keep the notations and assumptions of the previous Chapter. In this chapter, we make use of the global estimates derived in the previous Chapter to show that the kernel of the operator $A_{\bar{\delta}}$ is always finite dimensional and also to show that this operator has closed range if $\bar{\delta}$ does not belong to the set of indicial roots of $\mathcal{L}$. The proofs of these results are standard and follow closely the classical proofs of Fredholm properties which can be found in many textbooks but is is interesting to see where the result of Proposition 8.4.1 enters into the proof. We keep the notations of the previous chapter.

9.1 The kernel of the operator $A_{\bar{\delta}}$

Recall that we have defined $\operatorname{Ind}\mathcal{L}$, the set of indicial roots of the operator L, as the collections of $(\delta^{(1)}, \ldots, \delta^{(m)}) \in \mathbb{R}^m$ where $\delta^{(i)}$ is any indicial root of $L^{(i)}$ on $(0, \infty) \times \Sigma^{(i)}$. We keep the notations of the previous chapter and prove that the kernel of $A_{\bar{\delta}}$ is always finite dimensional.

Theorem 9.1.1. *The kernel of $A_{\bar{\delta}}$ is finite dimensional.*

Proof : Increasing the values of the entries of $\bar{\delta} = (\delta^{(1)}, \ldots, \delta^{(m)})$ if this is necessary, we can assume that $\bar{\delta} \notin \operatorname{Ind}\mathcal{L}$. Indeed, if $u \in \operatorname{Ker} A_{\bar{\delta}}$ then $u \in \operatorname{Ker} A_{\bar{\delta}'}$ for any $\bar{\delta}' = (\bar{\delta}^{(1)\,\prime}, \ldots, \bar{\delta}^{(m)\,\prime})$ for which $\delta^{(i)\,\prime} \geq \delta^{(i)}$. Therefore, one can always reduce to the case where $\bar{\delta}' \notin \operatorname{Ind}\mathcal{L}$.

We give two proofs of this result :

First proof. We argue by contradiction and assume that the result is not true. Then, there would exist a sequence $(u_j)_j$ of linearly independent elements of $L^2_{\bar{\delta}}(M)$ which satisfy $A_{\bar{\delta}}\, u_j = 0$. Without loss of generality we can assume that this sequence is normalized so that

$$\int_M |u_j|^2\, \Gamma_{\bar{\delta}}^{-2}\, \mathrm{dvol}_g = 1\,, \tag{9.1}$$

and also that
$$\int_M u_j\, u_{j'}\, \Gamma_{\bar\delta}^{-2}\, \mathrm{dvol}_g = 0, \tag{9.2}$$
for all $j \neq j'$. Using the result of Proposition 8.4.1 we obtain
$$\|u_j - u_{j'}\|_{L^2_{\bar\delta}(M)} \leq c\, \|u_j - u_{j'}\|_{L^2(K)}\,, \tag{9.3}$$
where $c > 0$ does not depend on j nor on j'.

Using (9.1) together with the result of Proposition 8.3.1 we conclude that the sequence $(u_j)_j$ is bounded in $W^{1,2}(K)$. Now, we Rellich's compactness result allows us to extract some subsequence (which we will still denote by $(u_j)_j$) which converges in $L^2(K)$. In particular, the sequence $(u_j)_j$ is a Cauchy sequence in $L^2(K)$. In view of (9.3) we see that the sequence $(u_j)_j$ is also a Cauchy sequence in $L^2_{\bar\delta}(M)$. This space being a Banach space, we conclude that this sequence converges in $L^2_{\bar\delta}(M)$ to some function u.

Clearly, passing to the limit in (9.1) we see that
$$\int_M |u|^2\, \Gamma_{\bar\delta}^{-2}\, \mathrm{dvol}_g = 1\,.$$
While, passing to the limit $m' \longrightarrow \infty$ in (9.2), we get
$$\int_M u_j\, u\, \Gamma_{\bar\delta}^{-2}\, \mathrm{dvol}_g = 0\,,$$
and then passing to the limit as m tends to ∞, we conclude that
$$\int_M u^2\, \Gamma_{\bar\delta}^{-2}\, \mathrm{dvol}_g = 0\,.$$
This is clearly a contradiction and this completes the first proof of the Proposition.

Second proof. The second proof is more in the spirit of the proof of Proposition 7.2.2. Assume that $w \in L_{\bar\delta}(M)$ is bounded solution of $A_{\bar\delta}\, w = 0$. Proposition 8.4.1 implies that
$$\|w\|_{L^2_{\bar\delta}(M)} \leq c\, \|w\|_{L^2(K)}$$
But elliptic regularity (applied on some bounded open set containing K) implies that
$$\|w\|_{L^\infty(K)} \leq c\, \|w\|_{L^2_{\bar\delta}(M)}$$
So we conclude that
$$\|w\|_{L^\infty(K)} \leq c\, \|w\|_{L^2(K)} \tag{9.4}$$
for any solution of $A_{\bar\delta}\, w = 0$ which belongs to $L^2_{\bar\delta}(M)$.

Now the proof is identical to the one we gave in the proof of Proposition 7.2.2. We denote by $\phi_1, \ldots, \phi_m$ an orthonormal basis (in $L^2(K)$) of the kernel of $A_{\bar\delta}$ and we consider the Bergman kernel associated to the orthogonal projection in $L^2(K)$ onto the space spanned by the kernel of $A_{\bar{\delta}}$. We have explicitly

$$K(x,y) = \sum_{j=1}^{m} \phi_j(x)\phi_j(y)$$

Integration over K implies that

$$m = \int_K K(x,x) \,\mathrm{dvol}_g$$

which is the dimension of the kernel of $A_{\bar{\delta}}$. Now, there exists $x_0 \in K$ such that $K(x_0,x_0)\mathrm{Vol}(K) \geq m$ and we consider the evaluation form

$$\mathcal{E}_{x_0}(\phi) = \phi(x_0).$$

We choose the orthonormal basis $\phi_1, \dots, \phi_m$ such that $\mathcal{E}_{x_0}(\phi_j) = 0$ for $j = 2, \dots, m$. With these choices,

$$K(x_0,x_0)\,\mathrm{Vol}(K) = \phi_1(x_0)^2\,\mathrm{Vol}(K) \geq m$$

But (9.4) implies that

$$\|\phi_1\|_{L^\infty(K)} \leq c\,\|\phi_1\|_{L^2(K)} = c$$

Therefore $m \leq c$ This completes the second proof of the Proposition. □

9.2 The range of the operator $A_{\bar{\delta}}$

We pursue our quest of the mapping properties of the operators $A_{\bar{\delta}}$ by studying the range of this operator. Thanks to the results of the previous sections, we are in a position to prove the :

Theorem 9.2.1. *Assume that $\bar{\delta} \notin Ind\mathcal{L}$. Then the range of $A_{\bar{\delta}}$ is closed.*

Proof : Let $u_j, f_j \in L^2_{\bar{\delta}}(M)$ be sequences such that $f_j = \mathcal{L}\,u_j$ converges to f in $L^2_{\bar{\delta}}(M)$. Since we already know that $\mathrm{Ker}\,A_{\bar{\delta}}$ is finite dimensional, it is closed and we can project each u_j onto

$$\left\{ u \in L^2_{\bar{\delta}}(M) \quad : \quad \int_M u\,v\,\Gamma_{\bar{\delta}}^{-2}\,\mathrm{dvol}_g = 0 \quad \forall v \in \mathrm{Ker}\,A_{\bar{\delta}} \right\},$$

the orthogonal complement of $\mathrm{Ker}\,A_{\bar{\delta}}$ in $L^2_{\bar{\delta}}(M)$ with respect to the scalar product associated to the weighted norm. Therefore, without loss of generality, we can assume that u_j is $L^2_{\bar{\delta}}$-orthogonal to $\mathrm{Ker}\,A_{\bar{\delta}}$.

Since f_j converges in $L^2_{\bar\delta}(M)$, there exists $c > 0$ such that

$$\|f_j\|_{L^2_{\bar\delta}(M)} \leq c. \tag{9.5}$$

We claim that the sequence $(u_j)_j$ is bounded in $L^2_{\bar\delta}(M)$. To prove this claim, we argue by contradiction and assume that (at least for a subsequence still denoted $(u_j)_j$)

$$\lim_{j\to+\infty} \|u_j\|_{L^2_{\bar\delta}(M)} = \infty.$$

We set

$$\hat u_j := \frac{u_j}{\|u_j\|_{L^2_{\bar\delta}(M)}} \qquad \text{and} \qquad \hat f_j := \frac{f_j}{\|u_j\|_{L^2_{\bar\delta}(M)}}\,,$$

so that $\mathcal{L}\,\hat u_j = \hat f_j$. Applying the result of Proposition 8.3.1, we conclude that the sequence $(\hat u_j)_j$ is bounded in $W^{1,2}(K)$ and hence, using Rellich's Theorem, we conclude that a subsequence (still denoted $(\hat u_j)_j$) converges in $L^2(K)$. Now the result of Proposition 8.4.1 yields

$$\|\hat u_j - \hat u_{j'}\|_{L^2_{\bar\delta}(M)} \leq c\left(\|\hat f_j - \hat f_{j'}\|_{L^2_{\bar\delta}(M)} + \|\hat u_j - \hat u_{j'}\|_{L^2(K)}\right). \tag{9.6}$$

On the right hand side, the sequence $(\hat f_j)_j$ tends to 0 in $L^2_{\bar\delta}(M)$ and the sequence $(\hat u_j)_j$ converges in $L^2(K)$. Therefore, we conclude that $(\hat u_j)_j$ is a Cauchy sequence in $L^2_{\bar\delta}(M)$ and hence converges to $\hat u \in L^2_{\bar\delta}(M)$.

To reach a contradiction, we first pass to the limit in the identity $\mathcal{L}\,\hat u_j = \hat f_j$ to get that the function $\hat u$ is a solution of $\mathcal{L}\,\hat u = 0$ and hence $\hat u \in \operatorname{Ker} A_{\bar\delta}$. But by construction $\|\hat u\|_{L^2_{\bar\delta}(M)} = 1$ and also

$$\int_M \hat u_j\,\hat u\,\Gamma_{\bar\delta}^{-2}\,\mathrm{dvol}_g = 0\,,$$

(since $\hat u \in \operatorname{Ker} A_{\bar\delta}$) and, passing to the limit in this last identity we find that $\|\hat u\|_{L^2_{\bar\delta}(M)} = 0$. A contradiction.

Now that the claim is proven, we use the result of Proposition 8.3.1 together with Rellich's Theorem to extract, from the sequence $(u_j)_j$ some subsequence which converges to u in $L^2(K)$. Once more, Proposition 8.4.1 implies that

$$\|u_j - u_{j'}\|_{L^2_{\bar\delta}(M)} \leq c\left(\|f_j - f_{j'}\|_{L^2_{\bar\delta}(M)} + \|u_j - u_{j'}\|_{L^2(K)}\right). \tag{9.7}$$

This time, on the right hand side, the sequence $(f_j)_j$ converges in $L^2_{\bar\delta}(M)$ and the sequence $(u_j)_j$ converges in $L^2(K)$. Therefore, we conclude that $(u_j)_j$ is a Cauchy sequence in $L^2_{\bar\delta}(M)$ and hence converges to $u \in L^2_{\bar\delta}(M)$. Passing to the limit in the identity $\mathcal{L}\,u_j = f_j$ we conclude that $\mathcal{L}\,u = f$ and hence f belongs to the range of $A_{\bar\delta}$. This completes the proof of the result.

Chapter 10

Duality theory

We keep the notations and assumptions of Chapter 8.

The adjoint of $A_{\bar\delta}$

$$A^*_{\bar\delta} : \left(L^2_{\bar\delta}(M)\right)' \longrightarrow \left(L^2_{\bar\delta}(M)\right)'$$

is defined as follows : An element $T \in \left(L^2_{\bar\delta}(M)\right)'$ belongs to $\mathrm{Dom}\,(A^*_{\bar\delta})$, the domain of $A^*_{\bar\delta}$, if and only if there exists $S \in \left(L^2_{\bar\delta}(M)\right)'$ such that

$$T(A_{\bar\delta}\, v) = S(v)\,,$$

for all $v \in \mathrm{Dom}\,(A_{\bar\delta})$. In this case, we will write $A^*_{\bar\delta}(T) = S$.

Classical properties for unbounded operators show that :

Theorem 10.0.2. *Assume that $\bar\delta \notin Ind\mathcal{L}$. Then*

$$Ker\, A_{\bar\delta} = \left(Im\, A^*_{\bar\delta}\right)^{\perp}$$

and

$$Im\, A_{\bar\delta} = \left(Ker\, A^*_{\bar\delta}\right)^{\perp}$$

The first equality follows from a classical result for unbounded operators with dense domain and closed graph. The second equality follows from classical results for unbounded operators with dense domains, closed graph and closed range.

Recall that if $E \subset B$

$$E^{\perp} := \{T \in B' \quad : \quad T(f) = 0 \quad \forall f \in E\}$$

while if $F \subset B'$,

$$F^{\perp} := \{f \in B \quad : \quad T(f) = 0 \quad \forall T \in F\}$$

10.1 Identification of $A^*_{\bar\delta}$.

We now further assume that the operator $\mathcal{L}$ is formally self-adjoint. Which means that

$$\int_M u\,\mathcal{L}v\,\mathrm{dvol}_g = \int_M v\,\mathcal{L}u\,\mathrm{dvol}_g$$

for any $u, v \in \mathcal{C}^2(M)$ with compact support in M.

It will be convenient to identify the dual of $L^2_{\bar\delta}(M)$ with $L^2_{-\bar\delta}(M)$. This is done using the scalar product

$$\langle u, v\rangle := \int_M u\,v\,\mathrm{dvol}_g \tag{10.1}$$

Clearly, given $v \in L^2_{-\bar\delta}(M)$, we can define $T_v \in \left(L^2_{\bar\delta}(M)\right)'$ by

$$T_v(u) = \langle u, v\rangle\,.$$

Moreover, we have

$$\|T_v\|_{\left(L^2_{\bar\delta}(M)\right)'} = \|v\|_{L^2_{-\bar\delta}(M)}$$

Conversely, given $T \in \left(L^2_{\bar\delta}(M)\right)'$, we can use Riez representation Theorem to shows that there exists a unique $v \in L^2_{-\bar\delta}(M)$ such that $\langle u, v\rangle = T(u)$ for all $u \in L^2_{\bar\delta}(M)$. Indeed, $\tilde u \longmapsto T(\Gamma_{\bar\delta}\tilde u)$ is a continuous linear functional on $L^2(M)$. Hence there exists a unique $\tilde v \in L^2(M)$ such that

$$\langle \tilde u, \tilde v\rangle = T(\Gamma_{\bar\delta}\,\tilde u)$$

Therefore, we have

$$\langle u, \Gamma^{-1}_{\bar\delta}\,\tilde v\rangle = T(\Gamma_{\bar\delta}\,\tilde u)$$

for any $u \in L^2_{\bar\delta}(M)$. It is enough to take $v = \Gamma^{-1}_{\bar\delta}\,\tilde v$ which clearly belongs to $L^2_{-\bar\delta}(M)$. Uniqueness is easy.

Given identification of $\left(L^2_{\bar\delta}(M)\right)'$ with $L^2_{\bar\delta}(M)$ it is easy to check that we can identify $A^*_{\bar\delta}$ with $A_{-\bar\delta}$. Indeed, if we write $T = T_u$ and $A^*_{\bar\delta}(T) = T_f$, for $u, f \in L^2_{-\bar\delta}(M)$, then, by definition

$$T_u(A_{\bar\delta}v) = \langle u, A_{\bar\delta}v\rangle$$

and

$$A^*_{\bar\delta}(T)(v) = \langle f, v\rangle$$

for all $v \in \mathrm{Dom}\,(A_{\bar\delta})$. Hence, we have

$$\int_M u\,\mathcal{L}\,v\,\mathrm{dvol}_g = \int_M f\,v\,\mathrm{dvol}_g\,,$$

for all $v \in \mathrm{Dom}\,(A_{\bar\delta})$. This in particular implies that $\mathcal{L}\,u = f$ in the sense of distributions. Since $u, f \in L^2_{-\bar\delta}(M)$, we conclude that $u \in \mathrm{Dom}\,(A_{-\bar\delta})$ and $f = A_{-\bar\delta}\,u$.

Conversely, if $u \in \mathrm{Dom}\,(A_{-\bar\delta})$, we can write for all $v \in \mathrm{Dom}\,(A_{\bar\delta})$

$$\begin{aligned}\langle u, A_{\bar\delta} v\rangle &= \int_M u\,\mathcal{L}\,v\,\mathrm{dvol}_g \\ &= \int_M v\,\mathcal{L}\,u\,\mathrm{dvol}_g \\ &= \langle A_{-\bar\delta}\,u, v\rangle\end{aligned}$$

The integrations by parts can be justified since, according to the result of Proposition 8.3.1, we have $v \in W^{2,2}_{\bar\delta}(M)$ and $u \in W^{2,2}_{-\bar\delta}(M)$. Therefore $T_u \in \mathrm{Dom}\,(A^*_{\bar\delta})$ and $A^*_{\bar\delta}(T_u) = T_{A_{-\bar\delta}\,u}$.

10.2 The result

With these identifications in mind, the results of the previous Chapter imply the :

Theorem 10.2.1. *Assume that $\bar\delta \notin Ind\mathcal{L}$. Then $A_{\bar\delta}$ is Fredholm. Moreover*

$$Ker\,A_{\bar\delta} = \left(Im\,A_{-\bar\delta}\right)^{\perp}$$

and

$$Im\,A_{\bar\delta} = \left(Ker\,A_{-\bar\delta}\right)^{\perp}$$

In particular

$$dim\,(KerA_{\bar\delta}) = codim\,(ImA_{-\bar\delta}).$$

Observe that, thanks to our identifications, if $E \subset L^2_{\bar\delta}(M)$, then

$$E^{\perp} := \left\{ v \in L^2_{-\bar\delta}(M) \quad : \quad \int_M u\,v\,\mathrm{dvol}_g = 0 \quad \forall v \in E \right\}.$$

Very useful for us will be the following consequence of this result :

Corollary 10.2.1. *Assume that $\bar\delta \notin Ind\mathcal{L}$. Then $A_{\bar\delta}$ is injective if and only if $A_{-\bar\delta}$ is surjective.*

In application, one often need to prove that the linear operator $A_{-\bar\delta}$ is surjective. Thanks to this Corollary, this amounts to prove that the operator $A_{\bar\delta}$ is injective.

10.3 Bibliography

This chapter relies essentially on classical results for unbounded operators which can be found in the book of H. Brezis, *Analyse Fonctionnelle.Théorie et applications.* Masson, Paris, (1983).

Chapter 11

The deficiency space

The results of the previous chapter are already of interest since they already provide right inverses for some class of elliptic operators acting on manifolds with cylindrical ends. However, more can be said in particular concerning the dimension of the kernel and of the cokernel of these operators.

11.1 Existence of a local parametrix

We consider the cylinder $C := \mathbb{R} \times \Sigma$ equipped with the product metric $g_{cyl} = dt^2 + h$. We go back to the study of the operator

$$L := \partial_t^2 + L_\Sigma + a$$

where L_Σ is an elliptic second order operator on Σ and where a is a periodic function of t. As usual, we denote by $\pm\delta_j$ the indicial roots of the operator L. Recall that, for all $t \in \mathbb{R}$, we have defined the half cylinder

$$C_t := [t, \infty) \times \Sigma.$$

To begin with, let us prove the following :

Lemma 11.1.1. *Assume that $\delta \neq \pm\delta_j$, for all $j \in \mathbb{N}$. Then, there exists an operator*

$$G_\delta : L^2_\delta(C_0) \longrightarrow L^2_\delta(C_0)$$

and a constant $c > 0$ (depending on δ) such that for all $f \in L^2_\delta(C_0)$, the function $u := G_\delta(f)$ is **a** *solution of*

$$L\, u = f$$

in C_0 and

$$\|u\|_{W^{2,2}_\delta(C_0)} \leq c\, \|f\|_{L^2_\delta(C_0)}$$

At first glance this result looks rather strange since we are not imposing any boundary data. Nevertheless, some boundary data are hidden in the construction of the operator G_δ. Observe that we state the existence of G_δ and do not state any uniqueness of this operator !

Proof : The proof of the existence of G_δ relies on the eigenfunction decomposition of the function f. We decompose as usual

$$f = \sum_{j \geq 0} f_j$$

where $f(t, \cdot) \in E_j$ for all $j \in \mathbb{N}$ and a.e. $t > 0$. Recall that E_j is the j-th eigenspace of $-L_\Sigma$ associated to the eigenvalue λ_J. Define $j_0 \in \mathbb{N}$ to be the least index for which

$$\delta^2 + a < \lambda_{j_0} .$$

We set

$$\tilde{f} = \sum_{j \geq j_0} f_j .$$

Clearly $\tilde{f} \in L^2_\delta(C_0)$ and, for all $T > 0$ one can solve

$$\left\{ \begin{array}{rclll} (\partial_t^2 + L_\Sigma + a)\, \tilde{u}_T & = & \tilde{f} & \text{in} & (0,T) \times \Sigma \\ \tilde{u}_T & = & 0 & \text{on} & \partial(0,T) \times \Sigma \end{array} \right.$$

The existence of $\tilde{u}_T$ follows at once from the fact that, since we restrict our attention to functions whose eigenfunction decomposition does not involve any term for $j = 0, \ldots, j_0 - 1$, there is no solution of $(\partial_t^2 + L_\Sigma + a)\, \tilde{u} = 0$ which is defined on $(0,T) \times \Sigma$ and vanishes on the boundary of this set. Therefore, this operator is injective and, since it is self-adjoint it is surjective. It then follows from Proposition 3.2.2 that we have the estimate

$$\|\tilde{u}_T\|_{L^2([0,T]\times\Sigma)} \leq c_T \, \|\tilde{f}\|_{L^2([0,T]\times\Sigma)}$$

for some constant $c_T > 0$ which *a priori* depends on T. We claim that there exists a constant $c > 0$ such that

$$\|e^{-\delta t}\, \tilde{u}_T\|_{L^2([0,T]\times\Sigma)} \leq c \, \|e^{-\delta t}\, \tilde{f}\|_{L^2([0,T]\times\Sigma)} \tag{11.1}$$

for some constant which does not depend on $T > 0$. The proof of the claim follows closely the proof of the corresponding estimate in the proof of Proposition 6.2.2. We leave the details to the reader.

Using elliptic estimates, as in the proof of Proposition 6.1.1, we conclude that there exists a constant $c > 0$ such that

$$\|e^{-\delta t}\, \tilde{u}_T\|_{W^{1,2}([0,T]\times\Sigma)} \leq c \, \|e^{-\delta t}\, \tilde{f}\|_{L^2_\delta([0,T]\times\Sigma)}$$

In particular, given $T' > 1$, there exists $c_{T'} > 0$ such that

$$\|\tilde{u}_T\|_{W^{1,2}([0,T']\times\Sigma)} \leq c\, \|e^{-\delta t}\, \tilde{f}\|_{L^2([0,T+1]\times\Sigma)}$$

for all $T > T' + 1$. Then, using Rellich's Theorem together with a simple diagonal argument, we conclude that there exists a sequence T_i tending to $+\infty$ such that the sequence $(\tilde{u}_{T_i})_i$ converges in $L^2([0,T']\times\Sigma)$, for all $T' > 1$. Passing to the limit in the equation we obtain a solution $\tilde{u}$ of

$$\begin{cases} (\partial_t^2 + L_\Sigma + a)\, \tilde{u} &= \tilde{f} \quad \text{in} \quad C_0 \\ \tilde{u} &= 0 \quad \text{on} \quad \partial C_0\,. \end{cases} \tag{11.2}$$

Moreover, passing to the limit in (11.1), we have the estimate

$$\|\tilde{u}\|_{L^2_\delta(C_0)} \leq c\, \|\tilde{f}\|_{L^2_\delta(C_0)}$$

To complete this study observe that the solution of (11.2) which belongs to $L^2_\delta(C_0)$ is unique. To see this, we assume that there exist two solutions and taking the difference we obtain a function $\tilde{w} \in L^2_\delta(C_0)$ satisfying

$$\begin{cases} (\partial_t^2 + L_\Sigma + a)\, \tilde{w} &= 0 \quad \text{in} \quad C_0 \\ \tilde{w} &= 0 \quad \text{on} \quad \partial C_0 \end{cases}$$

Performing the eigenfunction decomposition of $\tilde{w}$ as

$$\tilde{w} = \sum_{j\geq j_0} \tilde{w}_j,$$

where $w_j(t,\cdot) \in E_j$, the j-th eigenspace of $-L_\Sigma$ associated to the eigenvalue λ_j. We find that

$$\tilde{w}_j = w_j^+\, \phi_j + w_j^-\, \psi_j,$$

where ϕ_j, ψ_j belong to E_j, and where $w_j^\pm$ are the two independent solutions of the equation

$$(\partial_t^2 - \lambda_j + a)\, w_j^\pm = 0$$

which have been defined in Proposition 5.1.1. Using the fact that $\tilde{w}_j \in L^2_\delta(C_0)$ we conclude that $\psi_j = 0$. Next, using the fact that $\tilde{w}_j = 0$ on ∂C_0, we get $\phi_j = 0$ and hence $\tilde{w} = 0$.

Therefore, we can already define the operator G_δ on the space of functions whose eigenfunction decomposition does not involve any term for $j = 0, \ldots, j_0$, by

$$G_\delta(\tilde{f}) := \tilde{u}.$$

It remains to understand the definition of G_δ acting on f_j, when $j \leq j_0 - 1$. In this case we define $G_\delta(\tilde{f}_j) := \tilde{u}_j$, where $\tilde{u}_j$ is the function defined in (6.7) and for which we have already proven that

$$\int_0^\infty |G_\delta(\tilde{f}_j)|^2 \, e^{-2\delta t} \, dt \leq c \int_0^\infty f_j^2 \, e^{-2\delta t} \, dt \tag{11.3}$$

for some constant $c > 0$ depending on j and δ. This completes the proof of the result. $\Box$

Following the arguments given in Section 6.3, we see that the result of Lemma 11.1.1 holds when the operator L is of the form

$$L = d \left(\partial_t^2 + a \, \partial_t + b \, L_\Sigma + c \right) \tag{11.4}$$

where as usual, the functions a, b, c and d are periodic functions and b and d are positive.

Now, we keep the notations and assumptions of Chapter 8 and we explain how a perturbation argument allows to extend the previous result when the model operator L is replaced by the operator $\mathcal{L}$ defined on the end $\mathcal{E}^{(i)}$.

Lemma 11.1.2. *Assume that $\delta \neq \pm\delta_j^{(i)}$, for $j \in \mathbb{N}$. There exists $t_i > 0$, an operator*

$$G_\delta^{(i)} : L_\delta^2([t_i, \infty) \times \Sigma^{(i)}) \longrightarrow L_\delta^2([t_i, \infty) \times \Sigma^{(i)})$$

and $c > 0$ (depending on δ and i) such that for all $f \in L_\delta^2([t_i, \infty) \times \Sigma^{(i)})$, the function $u := G_\delta^{(i)}(f)$ is **a** *solution of*

$$\mathcal{L} \, u = f$$

in $(t_i, \infty) \times \Sigma^{(i)}$ and

$$\|u\|_{W^{2,2}_{([t_i,\infty)\times\Sigma^{(i)})}} \leq c \, \|f\|_{L_\delta^2([t_i,\infty)\times\Sigma^{(i)})}$$

Proof : This result follows from a simple perturbation argument. First observe that the result of Lemma 11.1.1 holds when $[0, \infty) \times \Sigma^{(i)}$ is replaced by $[t, \infty) \times \Sigma^{(i)}$. The corresponding operator will be denoted by $G_{\delta,t}$ and the estimate holds with a constant which does not depend on $t \in \mathbb{R}$.

Using the fact that the operators $\mathcal{L}$ and $L^{(i)}$ are asymptotic to each other, we can write

$$\|(\mathcal{L} - L^{(i)}) \, u\|_{L_\delta^2([t,\infty)\times\Sigma^{(i)})} \leq c \, e^{-2t} \, \|u\|_{W_\delta^{2,2}([t,\infty)\times\Sigma^{(i)})} ,$$

provided $t > 0$. This implies that

$$\|f - \mathcal{L} \circ G_{\delta,t} \, f\|_{L_\delta^2([t,\infty)\times\Sigma^{(i)})} \leq c \, e^{-2t} \, \|f\|_{L_\delta^2([t,\infty)\times\Sigma^{(i)})} ,$$

for some constant $c > 0$ which does not depend on $t > 0$. This clearly implies that the operator $\mathcal{L} \circ G_{\delta,R}$ is invertible provided t is fixed large enough, say $t = t_i$. To obtain the result, it is enough to define

$$G_\delta^{(i)} := G_{\delta,t_i} \circ (\mathcal{L} \circ G_{\delta,t_i})^{-1} .$$

The relevant estimate then follows at once. □

We now make use of the analysis of Chapter 5 and denote the indicial rrots of $L^{(i)}$ by $\pm\delta_j^{(i)}$. As usual, the index j refers to $\lambda_j^{(i)}$, the j-th eigenvalue of $-L_{\Sigma^{(i)}}$ associated to the eigenspace $E_j^{(i)}$. Proposition 5.1.1, provides for each $j \geq 0$ and each $i = 1, \ldots, k$, a function

$$(t, z) \longmapsto w_j^{\pm,(i)}(t)\, \phi(z) ,$$

which is a solution of the homogeneous problem

$$L^{(i)}\, (w_j^{\pm,(i)}\, \phi) = 0 ,$$

in $\mathbb{R} \times \Sigma^{(i)}$ and is associated to the indicial root $\delta_j^{(i)}$. Building on the result of the previous application, we now prove that one can perturb these functions to get, on any end $\mathcal{E}^{(i)}$ a solution of the homogeneous problem associated with the operator $\mathcal{L}$ which is asymptotic to $L^{(i)}$. This is the content of the following :

Lemma 11.1.3. *For all $i = 1, \ldots, k$, $j \in \mathbb{N}$ and $\phi \in E_j^{(i)}$, there exists $t_i > 0$ and $W_{j,\phi}^{\pm\,(i)}$ which is defined in $\mathcal{E}^{(i)}(t_i)$ and which satisfies*

$$\mathcal{L}\, W_{j,\phi}^{\pm,(i)} = 0 ,$$

in $\mathcal{E}^{(i)}(t_i)$. In addition,

$$W_{j,\phi}^{\pm,(i)} - w_j^{\pm,(i)}\, \phi \in L^2_\delta(\mathcal{E}^{(i)}(t_i))$$

for all $\delta < \pm\delta_j^{(i)} - \theta$, depending on the index $\pm$ in $W_{j,\phi}^{\pm\,(i)}$. Finally the mapping

$$\phi \longrightarrow W_{j,\phi}^{\pm\,(i)}$$

is linear.

In this result, t_i is the parameter given in Lemma 11.1.2.

Proof : The proof of this Lemma uses the following computation which follows at once from the expansion of the metric and the potential at infinity

$$\mathcal{L}\, w_j^{\pm,(i)} \phi = (\mathcal{L} - L^{(i)})\, w_j^{\pm,(i)} \phi \in L^2_\delta([t_i, \infty) \times \Sigma^{(i)})$$

for all $\delta < \pm\delta_j^{(i)} - \theta$. The result then follows from Lemma 11.1.2. □

For each $i = 1, \ldots, k$, we define $\chi^{(i)}$ to be a cutoff function which is identically equal to 1 in $\mathcal{E}^{(i)}(t_i + 1)$ and identically equal to 0 in $M \setminus \cup_{i=1}^k \mathcal{E}^{(i)}(t_i)$.

It will be convenient to define a partial order $\prec$ in $\mathbb{R}^k$ by

$$\bar{\delta} \prec \bar{\delta}' \qquad \text{if and only if} \qquad \delta^{(i)} < \delta'^{\,(i)}$$

for $i = 1, \ldots, k$. Similar definitions can be given for $\preceq$, $\succ$ and $\succeq$.

The main result of this chapter is :

Proposition 11.1.1. *Given $\bar{\delta}' \prec \bar{\delta}$, $\bar{\delta}, \bar{\delta}' \notin Ind\mathcal{L}$. Assume that $u \in L^2_{\bar{\delta}}(M)$ and $f \in L^2_{\bar{\delta}'}(M)$ satisfy*

$$\mathcal{L}\, u = f\,,$$

in M. Then, there exists $v \in L^2_{\bar{\delta}'}(M)$ such that

$$u - v \in D_{\bar{\delta},\bar{\delta}'} := Span\left\{\chi^{(i)}\, W^{\pm(i)}_{j,\phi}, \quad : \quad \phi \in E_j^{(i)}, \qquad \delta'^{\,(i)} < \pm\delta_j^{(i)} < \delta^{(i)}\right\}.$$

In addition, we have

$$\|v\|_{L^2_{\bar{\delta}'}(M)} + \|u - v\|_{D_{\bar{\delta},\bar{\delta}'}} \le c\,(\|f\|_{L^2_{\bar{\delta}'}(M)} + \|u\|_{L^2_{\bar{\delta}}(M)})$$

for some constant $c > 0$.

Observe that $D_{\bar{\delta},\bar{\delta}'}$ i finite dimensional and hence one can choose any norm $|\cdot\|_{D_{\bar{\delta},\bar{\delta}'}}$ on this space. The proof of this result relies on the corresponding result for operators of the form (11.4) on a half cylinder, but as usual, using a change of variables and conjugation, we can reduce to the corresponding result for the model operator $\partial_t^2 + L_\Sigma + a$ on a half cylinder.

Lemma 11.1.4. *Given $\delta' < \delta$, $\delta, \delta' \neq \pm\delta_j$, for all $j \in \mathbb{N}$. Assume that $u \in L^2_\delta(C_0)$ and $f \in L^2_{\delta'}(C_0)$ satisfy*

$$(\partial_t^2 + L_\Sigma + a)\, u = f$$

in C_0. Then, there exists $v \in L^2_{\delta'}(C_0)$ such that

$$u - v \in D_{\delta,\delta'} := Span\left\{w_j^{\pm}\, \phi, \quad : \quad \phi \in E_j, \qquad \delta' < \pm\delta_j < \delta\right\}.$$

In addition, we have

$$\|v\|_{L^2_{\delta'}(C_0)} + \|u - v\|_{D_{\delta,\delta'}} \le c\,(\|f\|_{L^2_{\delta'}(C_0)} + \|u\|_{L^2_\delta(C_0)})\,,$$

for some constant $c > 0$.

Proof : To prove the Lemma, we use the result of Lemma 11.1.1 and set $\bar v = G_{\delta'} f \in L^2_{\delta'}(C_0)$. Therefore

$$(\partial_t^2 + L_\Sigma + a)\,(u - \bar v) = 0\,,$$

in C_0. We have

$$\|\bar v\|_{L^2_{\delta'}(C_0)} \leq c\, \|f\|_{L^2_{\delta'}(C_0)}\,,$$

for some constant $c > 0$. We set $w = u - \bar v$ which we decompose as usual

$$w = \sum_{j \geq 0} w_j\,,$$

where $w_j(t,\cdot) \in E_j$. We fix j_0 to be the least index for which

$$|\delta| < \delta_{j_0} \qquad \text{and} \qquad |\delta'| < \delta_{j_0}\,.$$

We define

$$\tilde w = \sum_{j \geq j_0} w_j\,.$$

We claim that $\tilde w \in L^2_{\delta'}(C_0)$ and also that

$$\|\tilde w\|_{L^2_{\delta'}(C_0)} \leq c\, \|w\|_{L^2((0,1)\times\Sigma)}\,,$$

for some constant $c > 0$. The proof of the claim follows the arguments in the proof of Proposition 6.2.2. We omit the details.

Next, observe that, for $j = 0, \ldots, j_0 - 1$ the function w_j is given by

$$w_j = w_j^+\,\phi_j + w_j^-\,\psi_j\,,$$

for some $\phi_j, \psi_j \in E_j$. Observe that $\phi_j = 0$ if $\delta_j < \delta$ and $\psi_j = 0$ if $-\delta_j < \delta$ since $w_j \in L^2_\delta(C_0)$. It is easy to see that

$$\|\phi_j\|_{L^2(\Sigma)} + \|\psi_j\|_{L^2(\Sigma)} \leq c\, \|w_j\|_{L^2((0,1)\times\Sigma)}\,,$$

for some constant $c > 0$ (depending on j).

We set

$$v = \bar v + w + \sum_{j=0,\ldots,j_0-1,\quad \delta_j > \delta'} w_j^+\,\phi_j + \sum_{j=0,\ldots,j_0-1,\quad -\delta_j > \delta'} w_j^-\,\psi_j\,,$$

so that

$$u - v = \sum_{j=0,\ldots,j_0-1,\quad \delta < \delta_j < \delta'} w_j^+\,\phi_j + \sum_{j=0,\ldots,j_0-1,\quad \delta < -\delta_j < \delta'} w_j^-\,\psi_j\,.$$

The estimate follows from collecting the above estimates. This completes the proof of Lemma 11.1.4. □

We proceed with the proof of Proposition 11.1.1. Choose $\bar\delta'' \in \mathbb{R}^k$ such that

$$\bar\delta \succ \bar\delta'' \succeq \bar\delta', \qquad \bar\delta'' \succ \bar\delta - \bar\theta$$

and $\bar\delta'' \notin \mathrm{Ind}\,\mathcal{L}$. Recall that $\bar\theta := (\theta, \dots, \theta) \in \mathbb{R}^k$. Using the result of Proposition 8.3.1 we have

$$\|u\|_{W^{2,2}_{\bar\delta}(M)} \leq c\,(\|f\|_{L^2_{\bar\delta'}(M)} + \|u\|_{L^2_{\bar\delta}(M)})$$

Using the fact that the operators $\mathcal{L}$ and $L^{(i)}$ are asymptotic to each other, we conclude that, on each end $\mathcal{E}^{(i)}$, we have

$$\mathcal{L}\,u = f - (\mathcal{L} - L^{(i)})u \in L^2_{\bar\delta''}((t_i, \infty) \times \Sigma^{(i)}).$$

We apply the previous result to obtain the decomposition

$$u = v + \sum_{\delta''^{\,(i)} < \pm\delta_j^{(i)} < \delta^{\,(i)}} w_j^{\pm,(i)}\,\phi$$

on $\mathcal{E}^{(i)}$, where $\phi \in E_j^{(i)}$ and $v \in L^2_{\delta''^{\,(i)}}([t_i, \infty) \times \Sigma^{(i)})$. Next use the result of Lemma 11.1.3 and replace all $w_j^{\pm,(i)}\,\phi$ by $\chi^{(i)}\,W^{\pm(i)}_{j,\phi}$ to get the decomposition

$$u = \left(v + \sum_{i=1}^{k} \sum_{\delta''^{\,(i)} < \pm\delta_j^{(i)} < \delta^{\,(i)}} \chi^{(i)}\,(w_j^{\pm,(i)}\,\phi - W^{\pm,(i)}_{j,\phi}) \right) + \sum_{i=1}^{k} \sum_{\delta''^{\,(i)} < \pm\delta_j^{(i)} < \delta^{\,(i)}} \chi^{(i)}\,W^{\pm(i)}_{j,\phi}$$

Observe that the function

$$\tilde u = v + \sum_{i=1}^{k} \sum_{\delta''^{\,(i)} < \pm\delta_j^{(i)} < \delta^{\,(i)}} \chi^{(i)}\,(w_j^{\pm,(i)}\,\phi - W^{\pm,(i)}_{j,\phi}) \in L^2_{\bar\delta''}(M)$$

and also that $\mathcal{L}\,\tilde u = \tilde f \in L^2_{\bar\delta''}(M)$. If $\bar\delta'' = \bar\delta'$ then the roof is complete. If not, we apply the same argument with u replaced by $\tilde u$, f replaced by $\tilde f$ and $\bar\delta$ replaced by $\bar\delta''$ and argue inductively until the gap between $\bar\delta$ and $\bar\delta'$ is covered. □

11.2 The kernel of $A_{\bar\delta}$ revisited.

We keep the notations and assumptions of Chapter 8. Thanks to the result of Proposition 11.1.1 we can state the :

Lemma 11.2.1. *Fix $\bar\delta' \prec \bar\delta$ such that $\bar\delta, \bar\delta' \notin Ind\mathcal{L}$. Assume that $u \in L^2_{\bar\delta}(M)$ satisfies*

$$\mathcal{L}\, u = 0$$

in M. Then $u \in L^2_{\bar\delta'}(M)$ provided no element $\bar\delta'' \in Ind\mathcal{L}$ satisfies $\bar\delta' \prec \bar\delta'' \prec \bar\delta$.

This Lemma is a direct consequence of the result of Proposition 11.1.1. It essentially states that the kernel of the operator $A_{\bar\delta}$ does not change as $\bar\delta$ remains in some interval which does not contain any element of $\mathrm{Ind}\,\mathcal{L}$.

11.3 The deficiency space

Again, we keep the notations and assumptions of Chapter 8. We now define

Definition 11.3.1. *Given $\bar\delta \preceq 0$, $\bar\delta \notin Ind\mathcal{L}$, the deficiency space $D_{\bar\delta}$ is defined by*

$$D_{\bar\delta} := Span \left\{ \chi^{(i)}\, W^{\pm(i)}_{j,\phi}, \qquad : \qquad \phi \in E^{(i)}_j, \qquad \delta^{(i)}_j < -\delta^{(i)} \right\}.$$

Observe that the dimension of $D_{\bar\delta}$ can be computed explicitly and is in fact it is given by the formula

$$\dim D_{\bar\delta} = 2 \sum_{i,j, \quad \delta^{(i)}_j < -\delta^{(i)}} \dim E^{(i)}_j .$$

As a first byproduct, we obtain the :

Proposition 11.3.1. *Given $\bar\delta \preceq 0$, $\bar\delta \notin Ind\mathcal{L}$. Assume that $A_{\bar\delta}$ is injective. Then the operator*

$$\begin{array}{rccc} \tilde A_{\bar\delta}: & L^2_{\bar\delta}(M) \oplus D_{\bar\delta} & \longrightarrow & L^2_{\bar\delta}(M) \\ & u & \longmapsto & \mathcal{L}\, u \end{array}$$

is surjective and

$$Ker\, A_{-\bar\delta} = Ker\, \tilde A_{\bar\delta}$$

As a consequence of the previous Proposition, we have the following important result which is a relative index formula.

Corollary 11.3.1. *Given $\bar\delta \in \mathbb{R}^k$, $\bar\delta \preceq 0$ and $\bar\delta \notin Ind\mathcal{L}$. Assume that $A_{\bar\delta}$ is injective. Then*

$$dim\, Ker\, A_{-\bar\delta} = codim\, Im\, A_{\bar\delta} = \tfrac{1}{2}\, dim\, D_{\bar\delta}$$

Proof : Under the assumptions of the Corollary, we have

$$\dim \mathrm{Ker}\, A_{-\bar\delta} = \dim \mathrm{Ker}\, \tilde A_{\bar\delta}$$

and

$$\dim D_{\bar{\delta}} = \dim \dim \operatorname{Ker} A_{-\bar{\delta}} + \operatorname{codim} \operatorname{Im} A_{\bar{\delta}}$$

But, by duality, we have $\dim \operatorname{Ker} A_{-\bar{\delta}} = \operatorname{codim} \operatorname{Im} A_{\bar{\delta}}$. The result then follows at once. $\square$

Exercise 11.3.1. *Extend the results of Corollary 11.3.1 to the case there $A_{\bar{\delta}}$ is not injective.*

Chapter 12

Analysis in weighted Hölder spaces

As far as linear analysis is concerned the results of the previous chapter are sufficient. However, we would like to apply them to nonlinear problems for which is will be more convenient to work in the framework of Hölder spaces. The purpose of this chapter is to explain how the analysis of the previous chapter can be extended to weighted Hölder spaces.

We begin with the definition of weighted Hölder spaces.

Definition 12.0.2. *Given $\ell \in \mathbb{N}$, $\alpha \in (0,1)$ and $\bar{\delta} \in \mathbb{R}^k$, we define $\mathcal{C}^{\ell,\alpha}_{\bar{\delta}}(M)$ to be the space of functions $u \in \mathcal{C}^{\ell,\alpha}_{loc}(M)$ for which the following norm*

$$\|u\|_{\mathcal{C}^{\ell,\alpha}_{\bar{\delta}}(M)} := \|u\|_{\mathcal{C}^{\ell,\alpha}(M_1)} + \sum_{i=1}^{k} \|u\|_{\mathcal{C}^{\ell,\alpha}_{\delta^{(i)}}(\mathcal{E}^{(i)})}$$

is finite.

For example, the function $\Gamma_{\bar{\delta}} \in \mathcal{C}^{\ell,\alpha}_{\bar{\delta}'}(\Sigma)$ if and only if $\bar{\delta} \preceq \bar{\delta}'$. It also follows directly from this definition that

$$\mathcal{C}^{\ell,\alpha}_{\bar{\delta}}(M) \subset L^2_{\bar{\delta}'}(M)$$

for all $\bar{\delta} \prec \bar{\delta}'$.

Lemma 12.0.1. *The space $(\mathcal{C}^{\ell,\alpha}_{\bar{\delta}}(M), \|\cdot\|_{\mathcal{C}^{\ell,\alpha}_{\bar{\delta}}(M)})$ is a Banach space.*

It is easy to check that the embedding

$$\mathcal{C}^{\ell,\alpha}_{\bar{\delta}}(M) \longrightarrow \mathcal{C}^{\ell',\alpha'}_{\bar{\delta}'}(M)$$

is compact provided $\ell' + \alpha' < \ell + \alpha$ and $\bar{\delta} \prec \bar{\delta}'$.

The last easy observation is that the operator

$$\begin{array}{rccc} \mathcal{A}_{\bar{\delta}}: & \mathcal{C}^{2,\alpha}_{\bar{\delta}}(M) & \longrightarrow & \mathcal{C}^{0,\alpha}_{\bar{\delta}}(M) \\ & u & \longrightarrow & \mathcal{L}\, u \end{array}$$

is well defined and bounded.

12.1 Preliminary results

The proof of the main result of this Chapter requires some preliminary Lemma. Basically, the structure of this section parallels the one of Chapter 6.

We start with some elementary application of classical Schauder elliptic estimates. The result, which is not surprising, states that for solutions of $\mathcal{L}\, u = f$, if we *a priori* know that u belongs to $L^2_{\bar{\delta}}(M)$ and f belongs to $\mathcal{C}^{0,\alpha}_{\bar{\delta}}(M)$, then u belongs to $\mathcal{C}^{2,\alpha}_{\bar{\delta}}(M)$.

Lemma 12.1.1. *Assume that $\bar{\delta} \in \mathbb{R}^k$ is fixed. There exists $c > 0$ such that for all $u, f \in L^2_{\bar{\delta}}(M)$ satisfying*

$$\mathcal{L}\, u = f$$

in M, if $f \in \mathcal{C}^{0,\alpha}_{\bar{\delta}}(M)$ then $u \in \mathcal{C}^{2,\alpha}_{\bar{\delta}}(M)$ and

$$\|u\|_{\mathcal{C}^{2,\alpha}_{\bar{\delta}}(M)} \leq c \left(\|f\|_{\mathcal{C}^{0,\alpha}_{\bar{\delta}}(M)} + \|u\|_{L^2_{\bar{\delta}}(M)} \right) .$$

Proof : The result follows at once from elliptic estimates of Proposition 3.3.1 which we either apply on some compact subset M_t of M to get

$$\|u\|_{\mathcal{C}^{2,\alpha}(M_{t-1})} \leq c_t \left(\|f\|_{\mathcal{C}^{0,\alpha}(M_t)} + \|u\|_{L^2(M_t)} \right) ,$$

for some constant $c_t > 0$ depending on t, or which we apply to any sub-annulus $[t-2, t+2] \times \Sigma^{(i)}$ of the end $\mathcal{E}^{(i)}$ to get

$$\|u\|_{\mathcal{C}^{2,\alpha}([t-1,t+1]\times\Sigma^{(i)})} \leq c \left(\|f\|_{\mathcal{C}^{0,\alpha}([t-2,t+2]\times\Sigma^{(i)})} + \|u\|_{L^2([t-2,t+2]\times\Sigma^{(i)})} \right) ,$$

for some constant $c > 0$ independent of t. The result of the Lemma follows first by taking the first inequality with $t = 3$ and next multiplying the second inequality by $e^{-\delta^{(i)} t}$ and taking the supremum over $t > 2$. □

The last result didn't make use of the notion of indicial roots. In contrast, the next result will. The result is more difficult to prove, it states that for solutions of $L\, u = f$ on a half cylinder C_0, where L is an operator of the form (11.4), if we *a priori* know that u belongs to $\mathcal{C}^{2,\alpha}_{\delta}(C_0)$ but also that f belongs to some much smaller space $\mathcal{C}^{0,\alpha}_{\delta'}(C_0)$ for some $\delta' < \delta$, then u belongs to $\mathcal{C}^{2,\alpha}_{\delta'}(C_0)$, provided there is no indicial root of L between δ' and δ.

As usual, up to some change of variable and some conjugation, it is enough to prove the result for the operator $\partial_t^2 + L_\Sigma + a$ where a is a periodic function.

Lemma 12.1.2. *Assume that $\delta' < \delta \in \mathbb{R}$ and further assume that the interval $[\delta', \delta]$ does not contain any $\pm\delta_j$ for all $j \in \mathbb{R}$. Let $u \in \mathcal{C}^{2,\alpha}_{\delta}(\bar{C}_0)$ and $f \in \mathcal{C}^{0,\alpha}_{\delta'}(\bar{C}_0)$ satisfy*

$$(\partial_t^2 + L_\Sigma + a)\, u = f\,,$$

in C_0. Then $u \in \mathcal{C}^{2,\alpha}_{\delta'}(C_0)$ and

$$\|u\|_{\mathcal{C}^{2,\alpha}_{\delta'}(C_0)} \leq c\left(\|f\|_{\mathcal{C}^{0,\alpha}_{\delta'}(C_0)} + \|u\|_{\mathcal{C}^{2,\alpha}_{\delta}(C_0)}\right)\,,$$

Proof : As usual, we perform the eigenfunction decomposition of both u and f in C_0

$$u = \sum_{j\in\mathbb{N}} u_j \qquad \text{and} \qquad f = \sum_{j\in\mathbb{N}} f_j$$

We define $j_0 \in \mathbb{N}$ to be the least index for which

$$\delta^2 + a < \lambda_{j_0}.$$

For $j = 0, \ldots,, j_0 - 1$ one can use the explicit formula we have provided in the proof of Proposition 6.2.2 to show directly that $u_j \in \mathcal{C}^{2,\alpha}_{\delta'}(C_0)$ and that

$$\|u_j\|_{\mathcal{C}^{2,\alpha}_{\delta'}(C_0)} \leq c\left(\|f_j\|_{\mathcal{C}^{0,\alpha}_{\delta'}(C_0)} + \|u_j\|_{\mathcal{C}^{2,\alpha}_{\delta}(C_0)}\right)$$

We denote

$$\tilde{u} = \sum_{j\geq j_0} u_j \qquad \text{and} \qquad \tilde{f} = \sum_{j\geq j_0} f_j$$

The strategy is now to construct $\tilde{v} \in \mathcal{C}^{2,\alpha}_{\delta'}(C_0)$ solution of

$$(\partial_t^2 + L_\Sigma + a)\, \tilde{v} = \tilde{f},$$

in C_0 with $\tilde{v} = \tilde{u}$ on ∂C_0 and also to prove that

$$\|\tilde{v}\|_{\mathcal{C}^{2,\alpha}_{\delta'}(C_0)} \leq c\left(\|\tilde{f}\|_{\mathcal{C}^{0,\alpha}_{\delta'}(C_0)} + \|\tilde{u}\|_{\mathcal{C}^{2,\alpha}_{\delta}(C_0)}\right) \tag{12.1}$$

Assuming we have already done so, the difference $\tilde{u} - \tilde{v}$ satisfies

$$(\partial_t^2 + L_\Sigma + a)\,(\tilde{u} - \tilde{v}) = 0,$$

in C_0 and equal to 0 on ∂C_0. The eigenfunction decomposition of the function $\tilde{u} - \tilde{v}$ shows that

$$\tilde{u} - \tilde{v} = \sum_{j \geq j_0} \left(w_j^+ \, \phi_j + w_j^- \, \psi_j \right)$$

where $\phi_j, \psi_j \in E_j$. But $\tilde{u} - \tilde{v} \in \mathcal{C}^{2,\alpha}_\delta(C_0)$ and hence ψ_j all have to be equal to 0. Using the fact that $\tilde{u} - \tilde{v} = 0$ on ∂C_0, we also get that $\phi_j = 0$. Hence $\tilde{u} = \tilde{v}$. This will complete the proof of the Lemma.

Therefore, the only missing part is the existence of $\tilde{v}$ and the *a priori* estimate (12.1). To simplify the argument, let us first reduce to the case where $\tilde{u} = 0$ on ∂C_0. To this aim, we choose a cutoff function χ which only depends on t, is identically equal to 0 on $(0,1) \times \Sigma$ and identically equal to 1 in $(2, \infty) \times \Sigma$. Then, we define

$$\tilde{w} = \chi \, \tilde{v} \qquad \text{and} \qquad \tilde{g} = \chi \, \tilde{f} - [L, \chi] \, \tilde{v}$$

so that the equation we have to solve now reads

$$(\partial_t^2 + L_\Sigma + a) \, \tilde{w} = \tilde{g}$$

in C_0 with $\tilde{w} = 0$ on ∂C_0. Obviously the existence of $\tilde{v}$ is equivalent to the existence of $\tilde{w}$ and (12.1) will follow at once from

$$\|\tilde{w}\|_{\mathcal{C}^{2,\alpha}_{\delta'}(\bar{C}_0)} \leq c \, \|\tilde{g}\|_{\mathcal{C}^{0,\alpha}_{\delta'}(\bar{C}_0)},$$

since

$$\|\tilde{g}\|_{\mathcal{C}^{0,\alpha}_{\delta'}(\bar{C}_0)} \leq c \left(\|\tilde{f}\|_{\mathcal{C}^{0,\alpha}_{\delta'}(\bar{C}_0)} + \|\tilde{u}\|_{\mathcal{C}^{2,\alpha}_{\delta}(\bar{C}_0)} \right)$$

The existence of $\tilde{w}$ follows from the arguments already developed to prove Proposition 4.0.1. However the derivation of the estimate is more involved an requires new technics since it is not possible to construct barrier solutions anymore. In any case, for all $\bar{t} > 1$, we solve

$$(\partial_t^2 + L_\Sigma + a) \, \tilde{w}_{\bar{t}} = \tilde{g}$$

in $[0, \bar{t}] \times \Sigma$ with $\tilde{w}_{\bar{t}} = 0$ on $\{0, \bar{t}\} \times \Sigma$.

We claim that there exists a constant $c > 0$ such that

$$\sup_{[0,\bar{t}] \times \Sigma} e^{-\delta t} \, |\tilde{w}_{\bar{t}}| \leq c \sup_{[0,\bar{t}] \times \Sigma} e^{-\delta' t} \, |\tilde{g}|$$

When $\bar{t}$ remains bounded, the claim is certainly true and follows from standard elliptic estimate (use Proposition 3.1.2 and Proposition 2.2.1). In order to prove the claim, we argue by contradiction and assume that, for a sequence $\bar{t}_i$ tending to ∞, for a sequence of functions $\tilde{g}_i \in \mathcal{C}^{0,\alpha}_{\delta'}([0, \bar{t}_i] \times \Sigma)$, we have

$$\sup_{[0,\bar{t}_i] \times \Sigma} e^{-\delta' t} \, |\tilde{g}_i| = 1 \, ,$$

while, for the corresponding sequence of solutions $\tilde{w}_i$

$$A_i := \sup_{[0,\bar{t}_i]\times\Sigma} e^{-\delta' t}\,|\tilde{w}_i|\,,$$

tends to ∞. One should keep in mind that the eigenfunction decomposition of both $\tilde{g}_i$ and $\tilde{w}_i$ have no component over E_j for $j < j_0$.

Observe that the function $\tilde{w}_i$ is continuous and we can choose a point $t_i \in [0,\bar{t}_i]\times\Sigma$ where A_i is achieved. We define the rescaled functions

$$\hat{w}_i := A_i^{-1}\, e^{-\delta' t_i}\, \tilde{w}_i(t_i + \cdot, \cdot) \qquad \text{and} \qquad \hat{g}_i := A_i^{-1}\, e^{-\delta' t_i}\, \tilde{g}_{R_i}(t_i + \cdot, \cdot)\,.$$

Obviously

$$(\partial_t^2 + L_\Sigma + a(t_i + \cdot, \cdot))\, \hat{w}_i = \hat{g}_i\,.$$

Using the result of Proposition 3.2.2, we get the estimate

$$\|\nabla \tilde{w}_i\|_{L^\infty([0,1/2]\times\Sigma)} \leq c\, \left(\|\tilde{w}_i\|_{L^\infty([0,1]\times\Sigma)} + \|\tilde{g}_i\|_{L^\infty([0,1]\times\Sigma)}\right)\,,$$

for some constant $c > 0$ And hence

$$\|\nabla \tilde{w}_i\|_{L^\infty([0,1/2]\times\Sigma)} \leq c\,(1 + A_i)\,.$$

This implies that

$$e^{-\delta' t}\,|\tilde{w}_i| \leq c\,t\,(1 + A_i)\,,$$

for all $t \in [0, 1/2]$. Therefore, if $\rho > 0$ is fixed so that

$$c\,\rho \leq 1/2\,,$$

we conclude that, for i large enough, $t_i > \rho$.

Working near $\{\bar{t}_i\}\times\Sigma$ and using similar arguments one can show that there exists $\bar{\rho} > 0$ such that $\bar{t}_i - t_i > \bar{\rho}$. Therefore we conclude that

$$\rho \leq t_i \leq \bar{t}_i - \bar{\rho}\,. \tag{12.2}$$

As in the proof of Proposition 4.0.1 we pass to the limit for a subsequence of i tending to ∞ to obtain $\hat{w}$ a solution of

$$(\partial_t^2 + L_\Sigma + a(t_* + \cdot, \cdot))\, \hat{w} = 0\,,$$

in one of the following domains

(i) $\mathbb{R}\times\Sigma$ (which occurs when $t_* = \lim -t_i = -\infty$ and $t^* = \lim \bar{t}_i - t_i = \infty$).

(ii) $[t_*, \infty)\times\Sigma$ (which occurs when $t_* = \lim -t_i > -\infty$ and $t^* = \lim \bar{t}_i - t_i = \infty$).

(iii) $(-\infty, t^*]\times\Sigma$ (which occurs when $t_* = \lim -t_i = -\infty$ and $t^* = \lim \bar{t}_i - t_i < \infty$).

Observe that, using (12.2), we always have $t_* < 0 < t^*$.

In addition,

$$\sup e^{-\delta' t} |\hat{w}| = 1 \tag{12.3}$$

where the supremum is taken over the domain of definition of $\hat{w}$ and finally $\hat{w} = 0$ on $\{t_*\} \times \Sigma$ and/or on $\{t^*\} \times \Sigma$ if either t_* or t^* is finite. As usual, we perform the eigenfunction decomposition of $\hat{w}$ as

$$\hat{w} = \sum_{j \geq j_0} \hat{w}_j$$

Observe that, since we have chosen

$$\delta^2 + a < \lambda_{j_0}$$

a simple application of the maximum principle implies that

$$|\delta| < \delta_j$$

for all $j \geq j_0$. Also

$$\hat{w}_j = w_j^+ \, \phi_j + w_j^- \, \psi_j$$

for some $\phi_j, \psi_j \in E_j$. Using once more the fact that $\delta^2 + a < \lambda_{j_0}$ we obtain from the maximum principle that $w_j^\pm$ do not vanish.

In order to rule out case (ii), it is enough to look at the behavior of the function $\hat{w}_j$ near ∞ which implies that $\phi_j = 0$ and next using the fact that $\hat{w}_j = 0$ at $t = t_*$, we conclude that $\psi_j = 0$. Case (iii) can be ruled out using similar arguments. Finally, case (i) is ruled out inspecting the behavior of $\hat{w}_j$ at both $\pm\infty$.

Hence $\hat{w} \equiv 0$ and since this clearly contradicts (12.3), this completes the proof of the claim.

Now that we have proven the claim, we use elliptic estimates and Ascoli's Theorem to pass to the limit as $\bar{t}$ tends to 0 in the sequence $\tilde{w}_{\bar{t}}$ and obtain a solution of

$$(\partial_t^2 + L_\Sigma + a) \, \tilde{w} = \tilde{g}$$

in C_0 with $\tilde{w} = 0$ on ∂C_0 and

$$\sup_{C_0} e^{-\delta' t} |\tilde{w}| \leq c \sup_{C_0} e^{-\delta' t} |\tilde{g}|$$

To obtain the relevant estimates for the derivative, we use again the result of Proposition 3.3.2. This completes the proof of Lemma 12.1.2. □

12.2 The regularity result

The preliminary results we have proven allows us to extend our results in the framework of weighted Hölder. The key Proposition is :

Proposition 12.2.1. *Assume that $\bar\delta, \bar\delta' \in \mathbb{R}^k$ are fixed with $\bar\delta' \prec \bar\delta$. Further assume that no element $\bar\delta'' \in Ind\mathcal{L}$ satisfies $\bar\delta' \preceq \bar\delta'' \preceq \bar\delta$. Then, there exists $c > 0$ such that for all $u, f \in L^2_{\bar\delta}(M)$ satisfying*

$$\mathcal{L}\, u = f$$

in M, if $f \in \mathcal{C}^{0,\alpha}_{\bar\delta'}(M)$ then $u \in \mathcal{C}^{2,\alpha}_{\bar\delta'}(M)$ and

$$\|u\|_{\mathcal{C}^{2,\alpha}_{\bar\delta'}(M)} \leq c \left(\|f\|_{\mathcal{C}^{0,\alpha}_{\bar\delta'}(M)} + \|u\|_{L^2_{\bar\delta}(M)} \right)$$

Proof : We start by applying the result of Lemma 12.1.1 which implies that $u \in \mathcal{C}^{2,\alpha}_{\bar\delta}(M)$ and hence, thanks to the assumptions on the expansion of the metric and the potential, we can write on any of the ends $\mathcal{E}^{(i)}$

$$L^{(i)}\, u \in \mathcal{C}^{0,\alpha}_{\bar\delta''}(\mathcal{E}^{(i)})$$

for any $\bar\delta'' \succeq \bar\delta'$, $\bar\delta'' \succ \bar\delta - \bar\theta$. Next we apply the result of Lemma 12.1.2 which guaranties that $u \in \mathcal{C}^{2,\alpha}_{\bar\delta}(\mathcal{E}_i)$. If $\bar\delta'' = \bar\delta'$ then the proof is complete. If not, we iterate the argument starting from $\bar\delta''$ and proceed in this way until we reach $\bar\delta'$. The proof of the estimate follows from the estimates given in Lemma 12.1.1 and Lemma 12.1.2. □

12.3 The kernel of $A_{\bar\delta}$ revisited once more

The first application of the result of Proposition 12.2.1 is concerned with the kernel of the operator $A_{\bar\delta}$.

Lemma 12.3.1. *Assume that $\bar\delta \in \mathbb{R}^k$ is fixed with $\bar\delta \notin Ind\mathcal{L}$. Further assume that $u \in L^2_{\bar\delta}(M)$ is a solution of*

$$\mathcal{L}\, u = 0\,,$$

in M. Then $u \in \mathcal{C}^{2,\alpha}_{\bar\delta}(M)$.

In other words, in order to check the injectivity of $A_{\bar\delta}$, it is enough to check the injectivity of $\mathcal{A}_{\bar\delta}$, which in practical situation is easier to perform.

Observe that, if $u \in L^2_{\bar\delta}(M)$ is in the kernel of $A_{\bar\delta}$ then u is also in the kernel of $A_{\bar\delta'}$ for all $\delta \prec \delta'$ since $L^2_{\bar\delta}(M) \subset L^2_{\bar\delta'}(M)$. However, it follows from Proposition 12.2.1 that the following is also true :

Lemma 12.3.2. *Assume that $\bar\delta \in \mathbb{R}^k$ is fixed with $\delta \notin Ind\mathcal{L}$. Further assume that $u \in L^2_{\bar\delta}(M)$ is in the kernel of A_δ. Then u is also in the kernel of $A_{\bar\delta'}$ for all $\bar\delta' \prec \bar\delta$ for which no element $\delta'' \in Ind\mathcal{L}$ satisfies $\bar\delta' \preceq \bar\delta'' \preceq \bar\delta$.*

12.4 Mapping properties of $\mathcal{L}$ in weighted Hölder spaces

The third application of the result of Proposition 12.2.1 is concerned with the extension of the result of Proposition 11.1.1 in the framework of weighted Hölder spaces and this will be useful when dealing with nonlinear differential operators. We have the :

Proposition 12.4.1. *Given $\bar\delta' \prec \bar\delta$, $\bar\delta, \bar\delta' \notin Ind\mathcal{L}$. Assume that $u \in L^2_{\bar\delta}(M)$ and $f \in \mathcal{C}^{0,\alpha}_{\bar\delta'}(M)$ satisfy*

$$\mathcal{L}\, u = f$$

in M. Then, there exists $v \in \mathcal{C}^{2,\alpha}_{\bar\delta'}(M)$ such that

$$u - v \in D_{\bar\delta,\bar\delta'} := Span\left\{\chi^{(i)}\, W^{\pm(i)}_{j,\phi}, \quad : \quad \phi \in E^{(i)}_j, \qquad \delta'^{\,(i)} < \pm\delta^{(i)}_j < \delta^{(i)}\right\}.$$

In addition, we have

$$\|v\|_{\mathcal{C}^{2,\alpha}_{\bar\delta'}(M)} + \|u - v\|_{D_{\bar\delta,\bar\delta'}} \leq c\,(\|f\|_{\mathcal{C}^{0,\alpha}_{\bar\delta'}(M)} + \|u\|_{L^2_{\bar\delta}(M)}),$$

for some constant $c > 0$.

Let us now explain how this result is used. Given $\bar\delta \in \mathbb{R}^k$, $\bar\delta \prec 0$ and $\delta \notin \mathrm{Ind}\,\mathcal{L}$. Assume that $A_{\bar\delta}$ is injective, then, according to the result of Corollary 10.2.1, the operator $A_{-\bar\delta}$ is surjective and hence there exists

$$G_{-\delta} : L^2_{-\bar\delta}(M) \longrightarrow L^2_{-\bar\delta}(M),$$

a right inverse for $A_{-\bar\delta}$ (i.e. $A_{-\bar\delta} \circ G_{-\delta} = I$). In particular, given

$$f \in \mathcal{C}^{0,\alpha}_{\bar\delta}(M) \subset L^2_{-\bar\delta}(M),$$

the function $u := G_{-\bar\delta}\, f \in L^2_{-\bar\delta}(M)$ solves

$$\mathcal{L}\, u = f$$

in M. and we have

$$\|u\|_{L^2_{\bar\delta}(M)} \leq c\, \|f\|_{L^2_{-\bar\delta}(M)} \leq c\, \|f\|_{\mathcal{C}^{0,\alpha}_{\bar\delta}(M)}.$$

Applying the result of Proposition 12.4.1, we see that there exists $v \in \mathcal{C}^{2,\alpha}_{\bar\delta}(M)$ such that

$$u - v \in D_{\bar\delta} := \mathrm{Span}\left\{\chi^{(i)}\, W^{\pm(i)}_{j,\phi}, \quad : \quad \phi \in E^{(i)}_j, \qquad \delta^{(i)}_j < -\delta^{(i)}\right\},$$

and in addition, we have

$$\|v\|_{\mathcal{C}^{2,\alpha}_{\bar\delta}(M)} + \|u - v\|_{D_\delta} \leq c\,(\|f\|_{\mathcal{C}^{0,\alpha}_{\bar\delta}(M)} + \|u\|_{L^2_{\bar\delta}(M)}) \leq c\, \|f\|_{\mathcal{C}^{0,\alpha}_{\bar\delta}(M)},$$

for some constant $c > 0$.

We summarize this in the following :

Proposition 12.4.2. *Given $\bar{\delta} \notin Ind\mathcal{L}$ with $\bar{\delta} \prec 0$, let us assume that $A_{\bar{\delta}}$ is injective, then the operator*

$$\begin{array}{rlcl} \tilde{\mathcal{A}}_{\bar{\delta}} : & \mathcal{C}^{\ell,\alpha}_{\bar{\delta}}(M) \oplus D_{\bar{\delta}} & \longrightarrow & \mathcal{C}^{\ell,\alpha}_{\bar{\delta}}(M) \\ & u & \longrightarrow & \mathcal{L}\, u \end{array}$$

is well defined, bounded and surjective. In addition $dim\, Ker(\tilde{\mathcal{A}}_{\bar{\delta}}) = \frac{1}{2}\, dim\, D_{\bar{\delta}}$.

In particular, under the assumptions of the Proposition, there exists an operator

$$\mathcal{G}_{\bar{\delta}} : \mathcal{C}^{0,\alpha}_{\bar{\delta}}(M) \longrightarrow \mathcal{C}^{2,\alpha}_{\bar{\delta}}(M) \oplus D_{\bar{\delta}} \, .$$

which is a right inverse for the operator $\mathcal{L}$. In fact there exists a $\frac{1}{2}\dim D_{\bar{\delta}}$ dimensional family of such right inverses.

Let us define

$$\mathrm{Isom}\, \mathcal{L} := \left\{ \bar{\delta} \quad ; \quad \forall \bar{\delta}' \in \mathrm{Ind}\, \mathcal{L}, \quad \delta' \succeq 0 \Rightarrow -\bar{\delta}' \prec \bar{\delta} \prec \bar{\delta}' \right\}.$$

In other words, when $\bar{\delta} \succ 0$ and $\bar{\delta} \in \mathrm{Isom}\, \mathcal{L}$, then there is no element of $\mathrm{Ind}\, \mathcal{L}$ between $-\bar{\delta}$ and $\bar{\delta}$. In particular $D_{\bar{\delta}}$ is empty. Therefore, the above statement simplifies into the :

Proposition 12.4.3. *Assume that for some $\bar{\delta} \in Isom\, \mathcal{L}$, the operator $A_{\bar{\delta}}$ is injective, then the operator $\mathcal{A}_{\bar{\delta}'}$ is an isomorphism for any $\bar{\delta}' \in Isom\, \mathcal{L}$.*