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Lecture notes on Mean Curvature Flow

Jingyi Chen

University of British Columbia
Department of Mathematics
Vancouver BC, V6T 1Z2
Canada

jychen@math.ubc.ca

LECTURE NOTES ON MEAN CURVATURE FLOW

JINGYI CHEN

1. Fundamental equations for submanifolds

Let Σ be a n-dimensional manifold and M be an $(n+k)$-dimensional manifold with a Riemannian metric g. Let $F : \Sigma \to M$ be an smooth immersion. The induced metric via F on Σ is $h = F^*g$. Denote the Levi-Civita connections of g and h by $\overline{\nabla}$ and ∇, respectively. Let $T\Sigma, N\Sigma$ be the tangent bundle and the normal bundle of Σ in M respectively. For each $p \in \Sigma$ and $X, Y \in T\Sigma$, the <u>second fundamental form</u> of Σ in (M, g)

$$A : T\Sigma \times T\Sigma \to N\Sigma$$

is given by <u>the Gauss formula</u>:

$$\overline{\nabla}_{X_p} Y = \nabla_{X_p} Y + A(X_p, Y_p). \tag{1}$$

The following are some of the most important forlumas/equations for submanifolds.

<u>Gauss' Equations</u>:

$$\langle \overline{R}(X,Y)Z, W\rangle = \langle R(X,Y), Z, W\rangle + \langle A(X,Z), A(Y,W)\rangle - \langle A(Y,Z), A(X,W)\rangle \tag{2}$$

for all tangent vectors $X, Y, Z, W \in T_p\Sigma$.

<u>Weingarten equations</u>:

$$\overline{\nabla}_X \nu = -B_\nu(X) + D_X \nu \tag{3}$$

where D is the connection on $N\Sigma$ and $B : T\Sigma \times N\Sigma \to T\Sigma$ is the <u>sharp operator</u> defined by

$$g(B_\nu(X), Y) = g(A(X,Y), \nu)$$

<u>Codazzi's equation</u>: let $\nu_{n+1}, ..., \nu_{n+k}$ be a local orthoformal section of $N\Sigma$, then

$$\begin{aligned} \langle \overline{R}(X,Y)Z, \nu\rangle &= \big((\nabla_X A^i)(Y,Z) - (\nabla_Y A^i)(X,Z)\big)\langle \nu_i, \nu\rangle \\ &\quad + A^i(Y,Z)\langle D_X\nu_i, \nu\rangle - A^i(X,Z)\langle D_Y \nu_i, \nu\rangle \\ &= (\widetilde{\nabla}_X A)(Y,Z) - (\widetilde{\nabla}_Y A)(X,Z) \end{aligned} \tag{4}$$

where $\widetilde{\nabla}$ is the covariant differentiation on $\mathrm{Hom}(TM \times TM, NM)$ determined by ∇, D.

<u>Ricci equation</u>:

$$\big(\overline{R}(X,Y)\nu\big)^\perp = A(B_\nu(X), Y) - A(B_\nu(Y), X) + D_X(D_Y\nu) - D_Y(D_X\nu) - D_{[X,Y]}\nu \tag{5}$$

The <u>mean curvature vector</u> of Σ in M is the trace of second fundamental form.

The second fundamental form for the immersion $F : \Sigma \to M$ is given in local coordinate by

$$A = h^\alpha_{ij} \frac{\partial F}{\partial x^i} \otimes \frac{\partial F}{\partial x^j} \otimes \nu_\alpha \tag{6}$$

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with $\alpha = n+1, ..., n+k$, where x^i's are coordinates on Σ and ν_α's form a basis for $N\Sigma$. The coefficients are given by

$$h_{ij}^\alpha = \frac{\partial^2 F^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial F^\alpha}{\partial x^k} + \overline{\Gamma}_{\beta\gamma}^\alpha(F) \frac{\partial F^\beta}{\partial x^i} \frac{\partial F^\gamma}{\partial x^j}. \tag{7}$$

In particular, when M is the euclidean space $\mathbb{R}^{n+k}$, the mean curvature vector of Σ is

$$H = \Delta F \tag{8}$$

where Δ is the Laplace-Beltrami operator of the induced metric. To see ΔF is indeed normal to Σ, we compute

$$\begin{aligned}
\langle \Delta F, \partial_l F \rangle &= \langle \sqrt{g}^{-1} \partial_i (g^{ij} \sqrt{g} \partial_j F), \partial_l F \rangle \\
&= \langle g^{ij} \partial_{ij}^2 F, \partial_l F \rangle + \frac{1}{2} \langle g^{ij} g^{st} \partial_i g_{st} \partial_j F, \partial_l F \rangle - \langle g^{is} \partial_i g_{st} g^{tj} \partial_j F, \partial_l F \rangle \\
&= \langle g^{ij} \partial_{ij}^2 F, \partial_l F \rangle + \frac{1}{2} g^{st} \partial_l g_{st} - g^{is} \partial_i g_{sl} \\
&= \frac{1}{2} g^{ij} \partial_l \langle \partial_i F, \partial_j F \rangle - g^{ij} \langle \partial_j F, \partial_{il}^2 F \rangle \\
&= 0.
\end{aligned}$$

Therefore, $\Delta F \in N\Sigma$. Here we have used the useful formula

$$\partial_l \det C = \det C \cdot c^{ij} \partial_l c_{ij}$$

for any invertible matrix $C = (c_{ij})$.

A submanifold Σ is called a minimal submanifold in M if $H \equiv 0$. In particular, if Σ is a minimal submanifold in a euclidean space, then its coordinates are harmonic functions on Σ by (8), hence there does not exists any compact minimal submanifold in euclidean space, except points.

When Σ is a hypersurface in $\mathbb{R}^{n+1}$, locally it is the graph of a smooth function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. The unit upward normal vector ν is given by

$$\nu = \frac{(-Df, 1)}{\sqrt{1 + |Df|^2}}$$

and the mean curvature vector is

$$H = \operatorname{div} \left(\frac{Df}{\sqrt{1 + |Df|^2}} \right) \nu.$$

The first variation formula for volume reads:

$$\frac{d}{dt} V(\Sigma_t) \bigg|_{t=0} = \int \operatorname{div} X = - \int H \cdot X$$

where X is an arbitrary vector field with compact support generated by a 1-parameter family of diffeomorphisms φ_t of the ambient space:

$$X = \frac{d\varphi_t}{dt} \bigg|_{t=0}$$

and $\Sigma_t = \varphi_t(\Sigma), \varphi_0$ is the identity map.

2. Mean curvature flow and Geometric evolution equations

2.1. Mean curvature flow. If $\widetilde{F} : \Sigma \times [0, T) \to M$ is smooth and satisfies

$$\left(\frac{\partial \widetilde{F}}{\partial t}\right)^{\perp} = H(\widetilde{F})$$

we say F satisfies the mean curvature flow. One can show that there exists a t-dependent family of tangential diffeomorphisms ϕ such that $F = \widetilde{F} \circ \phi$ satisfies

$$\frac{\partial F}{\partial t} = H(F) \tag{9}$$

and we will always use this equation to refer mean curvature flows (MCF).

If Ω is a domain in $\mathbb{R}^{n+1}$ and $u : \Omega \times [0, T) \to \mathbb{R}$ is a smooth function which satisfies

$$\frac{\partial u}{\partial t} = \sqrt{1 + |Du|^2}\,\mathrm{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) \tag{10}$$

then the time dependent graph $\Sigma = \{(x, u(x, t)) | x \in \Omega, t \in [0, T)\}$ evolves by MCF. This can be seen as follows

$$\begin{aligned}
\left(\frac{\partial F}{\partial t}\right)^{\perp} &= \left(\frac{\partial F}{\partial t} \cdot \nu\right)\nu \\
&= \left((0, u_t) \cdot \frac{(-Du, 1)}{\sqrt{1 + |Du|^2}}\right)\nu \\
&= \frac{u_t}{\sqrt{1 + |Du|^2}}\nu \\
&= \mathrm{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right)\nu \\
&= H.
\end{aligned}$$

Expanding (10), we have

$$\frac{\partial u}{\partial t} = \left(\delta_{ij} - \frac{D_i u D_j u}{1 + |Du|^2}\right) D_i D_j u. \tag{11}$$

When $n = 1$, the graphical MCF (11) reduces to

$$\frac{\partial u}{\partial t} = \frac{u_{xx}}{1 + u_x^2} = \frac{\partial}{\partial x} \arctan u_x. \tag{12}$$

Example 2.1. A special solution to (12) is the so-called grim reaper given by

$$u(x, t) = -\log\cos x + t,\ \ x \in (-\pi/2, \pi/2).$$

This solution is not defined on entire domain $\mathbb{R}$ and it translates in the y-direction by the unit vector $(0, 1)$.

Example 2.2. Let Σ_t be the round sphere $\partial B_{r(t)}(0)$ in $\mathbb{R}^{n+1}$.

$$r(t) = \sqrt{R^2 - 2nt}$$

solves the O.D.E. on $(-\infty, R^2/2n)$

$$r' = -\frac{n}{r},\ r(0) = R.$$

The MCF, w.r.t. inward pointing normal vector, is reduced to the above O.D.E., it shrinks to a point at time $R^2/2n$. So

$$F(x,t) = F(x)\sqrt{1-\frac{2nt}{R^2}}$$

is a self-similar solution.

Example 2.3. Let Σ_t be the spherical cylinder $\partial B^{n+1-k}_{r(t)}(0)\times\mathbb{R}^k$ in $\mathbb{R}^{n+1}$. The MCF reduces to

$$r' = -\frac{n-k}{r},\ r(0) = R.$$

The solution is

$$r(t) = \sqrt{R^2 - 2(n-k)t}$$

for $t \in (-\infty, R^2/2(n-k))$. This is an ancient solution as it exists from $-\infty$.

MCF equation for $F : \Sigma \times [0,T) \to M$ can also be written as

$$\frac{\partial F}{\partial t} = g^{ij}\left(\frac{\partial^2 F^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial F^\alpha}{\partial x^k} + \overline{\Gamma}^\alpha_{\beta\gamma}\frac{\partial F^\beta}{\partial x^i}\frac{\partial F^\gamma}{\partial x^j}\right)\frac{\partial}{\partial y^\alpha} \tag{13}$$

where g is the time dependent induced metric and Γ^k_{ij} its connection and $\overline{\Gamma}^\alpha_{\beta\gamma}$ is the connection on M. This is a quasilinear parabolic system, and its short time existence and uniqueness is guaranteed when Σ is compact and smooth [17]. Note that for each fixed t, the right hand side of (13) is the tension field of the isometric immersion $F : (\Sigma, g) \to (M, h)$ and when the tension field vanishes F is a harmonic map.

2.2. **Basic evolutions equations for geometric quantities.** We consider the mean curvature flow from a closed n-dimensional manifold in a m-dimensional Riemannian manifold M with a Riemannian metric. Given an embedding $F_0 : \Sigma \to M$, we consider a one-parameter family of smooth maps $F_t = F(\cdot,t) : \Sigma \to M$ with corresponding images $\Sigma_t = F_t(\Sigma)$ are embedded submanifolds in M and F satisfies the mean curvature flow equation:

$$\begin{cases} \dfrac{d}{dt}F(x,t) = H(x,t) \\ F(x,0) = F_0(x). \end{cases} \tag{14}$$

Here $H(x,t)$ is the mean curvature vector of Σ_t at $F(x,t)$ in M. Denote by A the second fundamental form of Σ_t in M and the Riemannian metric on M by $\langle\cdot,\cdot\rangle$. In a normal coordinates around a point in Σ, the induced metric on Σ_t from $\langle\cdot,\cdot\rangle$ is given by

$$g_{ij} = \langle \partial_i F, \partial_j F\rangle$$

where ∂_i $(i = 1,\cdots,n)$ are the partial derivatives with respect to the local coordinates. In the sequel, we denote by Δ and ∇ the Laplace operator and covariant derivative for the induced metric on Σ_t respectively. We choose a local field of orthonormal frames $e_1, ..., e_n$, $v_1, ..., v_{m-n}$ of M along Σ_t such that $e_1, ..., e_n$ are tangent vectors of Σ_t and $v_1, ..., v_{m-n}$ are in the normal bundle over Σ_t. We can write:

$$\begin{aligned} A &= A^\alpha v_\alpha \\ H &= -H^\alpha v_\alpha. \end{aligned}$$

Let $A^\alpha = (h^\alpha_{ij})$, where (h^α_{ij}) is a matrix. By the Weingarten equation, we have

$$h^\alpha_{ij} = \langle \overline{\nabla}_i v_\alpha, e_j\rangle = \langle \overline{\nabla}_j v_\alpha, e_i\rangle = h^\alpha_{ji}$$

where $\overline{\nabla}$ is the Levi-Civita connection on M. The trace and the norm of the second fundamental form of Σ_t in M are:

$$H^\alpha = g^{ij} h^\alpha_{ij} = h^\alpha_{ii}$$

$$|A|^2 = \sum_\alpha |A^\alpha|^2 = g^{ij} g^{kl} h^\alpha_{ik} h^\alpha_{jl} = h^\alpha_{ik} h^\alpha_{ik}.$$

We first derive the evolution equation of the induced metric.

Lemma 2.4. *Along a smooth MCF, we have*

$$\frac{\partial g_{ij}}{\partial t} = -2\langle h_{ij}, H\rangle \tag{15}$$

Proof. Write $\partial_i = \partial_i F$ and $g_{ij} = \langle \partial_i, \partial_j\rangle$. Then

$$\begin{aligned} \frac{\partial g_{ij}}{\partial t} &= \langle \overline{\nabla}_H \partial_i, \partial_j\rangle + \langle \partial_i, \overline{\nabla}_H \partial_j\rangle \\ &= \langle \overline{\nabla}_{\partial_i} H, \partial_j\rangle + \langle \partial_i, \overline{\nabla}_{\partial_j} H\rangle \\ &= -2\langle H, h_{ij}\rangle. \end{aligned}$$

Here we have used $[\partial_t F, \partial_i] = 0$. □

Consequently, volume decreases along mean curvature flow:

Lemma 2.5. *Let Σ be a compact submanifold and let $F : \Sigma \times [0, T) \to M$ satisfy MCF. Then*

$$\frac{d}{dt}\int_\Sigma \sqrt{g}\, dx = -\int_\Sigma |H|^2 \sqrt{g}\, dx.$$

Lemma 2.6. *Along MCF, we have*

$$\frac{\partial}{\partial t} h^\alpha_{ij} = H^\alpha_{,ji} - H^\beta R_{\beta ij\alpha} - H^\beta h^\beta_{jl} h^a_{il} - h^\beta_{ij}\langle v_\beta, \overline{\nabla}_H v_\alpha\rangle.$$

Proof. Set $e_i = \partial_i F$. We have

$$h^\alpha_{ij} = -\langle \overline{\nabla}_{e_i} e_j, v_\alpha\rangle.$$

Then we have

$$\begin{aligned} \frac{\partial}{\partial t} h^\alpha_{ij} &= \langle \overline{\nabla}_H \overline{\nabla}_{e_i} e_j, v_\alpha\rangle + \langle \overline{\nabla}_{e_i} e_j, \overline{\nabla}_H v_\alpha\rangle \\ &= \langle \overline{\nabla}_{e_i} \overline{\nabla}_H e_j, v_\alpha\rangle - \langle R(H, e_i)e_j, v_\alpha\rangle + \langle \overline{\nabla}_{e_i} e_j, \overline{\nabla}_H v_\alpha\rangle \\ &= \langle \overline{\nabla}_{e_i} \overline{\nabla}_{e_j} H, v_\alpha\rangle - \langle R(H, e_i)e_j, v_\alpha\rangle + \langle \overline{\nabla}_{e_i} e_j, \overline{\nabla}_H v_\alpha\rangle \\ &= \langle \overline{\nabla}_{e_i} (\overline{\nabla}^T_{e_j} H + \overline{\nabla}^N_{e_j} H), v_\alpha\rangle - \langle R(H, e_i)e_j, v_\alpha\rangle + \langle \overline{\nabla}_{e_i} e_j, \overline{\nabla}_H v_\alpha\rangle \\ &= \langle \overline{\nabla}^N_{e_i} \overline{\nabla}^N_{e_j} H, v_\alpha\rangle - H^\beta R_{\beta ij\alpha} - \langle \overline{\nabla}^T_{e_j} H, \overline{\nabla}_{e_i} v_\alpha\rangle + \langle \overline{\nabla}_{e_i} e_j, \overline{\nabla}_H v_\alpha\rangle \\ &= H^\alpha_{,ji} - H^\beta R_{\beta ij\alpha} - \langle \overline{\nabla}^T_{e_j} H, \overline{\nabla}_{e_i} v_\alpha\rangle + \langle \overline{\nabla}_{e_i} e_j, \overline{\nabla}_H v_\alpha\rangle \\ &= H^\alpha_{,ji} - H^\beta R_{\beta ij\alpha} - H^\beta h^\beta_{jl} h^a_{il} + h^\beta_{ij}\langle v_\beta, \overline{\nabla}_H v_\alpha\rangle \end{aligned}$$

where we have used $\overline{\nabla}_{e_i} e_j = h^\beta_{ij} v_\beta$ at the center of the normal coordinates of Σ_t. □

Lemma 2.7. *Along MCF, we have*

$$\begin{aligned} H^\alpha_{,ji} &= \Delta h^\alpha_{ij} + h^\beta_{il}h^\beta_{lm}h^\alpha_{mj} - H^\beta h^\beta_{im}h^\alpha_{mj} + h^\beta_{ij}h^\beta_{lm}h^\alpha_{ml} - h^\beta_{im}h^\beta_{lj}h^\alpha_{ml} \\ &\quad -\nabla_l R_{\alpha jil} - \nabla_i R_{\alpha ljl} + R_{illm}h^\alpha_{mj} + R_{iljm}h^\alpha_{ml} - R_{\alpha\beta il}h^\beta_{lj}. \end{aligned}$$

Proof. Let K_{ijkl} be the curvature tensor on Σ_t and $K_{\alpha\beta ij}$ be the curvature tensor for $D^\perp$. The Codazzi equations, the Gauss equation, and the Ricci equation read:

$$\nabla_k h^\alpha_{mn} - \nabla_m h^\alpha_{kn} = -R_{\alpha mnk}$$

and

$$K_{ijkl} = (h^\beta_{ik}h^\beta_{jl} - h^\beta_{il}h^\beta_{jk}) + R_{ijkl}$$

and

$$K_{\alpha\beta il} = (h^\alpha_{ik}h^\beta_{kl} - h^\alpha_{lk}h^\beta_{ki}) + R_{\alpha\beta il}$$

where R is the curvature tensor of M. We have

$$\begin{aligned} \nabla_i\nabla_j H^\alpha &= \nabla_i\nabla_j h^\alpha_{ll} \\ &= \nabla_i(\nabla_l h^\alpha_{jl} + R_{\alpha ljl}) \\ &= \nabla_i\nabla_l h^\alpha_{lj} + \nabla_i R_{\alpha ljl} \\ &= \nabla_l\nabla_i h^\alpha_{lj} + (h^\beta_{il}h^\beta_{lm} - h^\beta_{im}h^\beta_{ll})h^\alpha_{mj} + R_{illm}h^\alpha_{mj} \\ &\quad +(h^\beta_{ij}h^\beta_{lm} - h^\beta_{im}h^\beta_{lj})h^\alpha_{ml} + R_{iljm}h^\alpha_{ml} \\ &\quad +R_{\alpha\beta il}h^\beta_{lj} + (h^\alpha_{ik}h^\beta_{kl} - h^\alpha_{lk}h^\beta_{ki})h^\beta_{lj} + \nabla_i R_{\alpha ljl} \\ &= \nabla_l(\nabla_l h^\alpha_{ij} + R_{\alpha jil}) + h^\beta_{il}h^\beta_{lm}h^\alpha_{mj} - H^\beta h^\beta_{im}h^\alpha_{mj} \\ &\quad +h^\beta_{ij}h^\beta_{lm}h^\alpha_{ml} - h^\beta_{im}h^\beta_{lj}h^\alpha_{ml} + (h^\alpha_{ik}h^\beta_{kl} - h^\alpha_{lk}h^\beta_{ki})h^\beta_{lj} \\ &\quad +R_{illm}h^\alpha_{mj} + R_{iljm}h^\alpha_{ml} + R_{\alpha\beta il}h^\beta_{lj} + \nabla_i R_{\alpha ljl} \\ &= \Delta h^\alpha_{ij} + h^\beta_{il}h^\beta_{lm}h^\alpha_{mj} - H^\beta h^\beta_{im}h^\alpha_{mj} \\ &\quad +h^\beta_{ij}h^\beta_{lm}h^\alpha_{ml} - h^\beta_{im}h^\beta_{lj}h^\alpha_{ml} + (h^\alpha_{ik}h^\beta_{kl} - h^\alpha_{lk}h^\beta_{ki})h^\beta_{lj} \\ &\quad +\nabla_l R_{\alpha jil} + \nabla_i R_{\alpha ljl} + R_{illm}h^\alpha_{mj} + R_{iljm}h^\alpha_{ml} + R_{\alpha\beta il}h^\beta_{lj}. \end{aligned}$$

This proves the lemma. □

Then Lemma 2.6 and Lemma 2.7 immediately imply

Lemma 2.8. *For the mean curvature flow, the second fundamental form satisfies*

$$\begin{aligned} (\frac{\partial}{\partial t} - \Delta)h^\alpha_{ij} &= h^\alpha_{ik}h^\beta_{kl}h^\beta_{lj} - h^\alpha_{lk}h^\beta_{ki}h^\beta_{lj} + h^\beta_{il}h^\beta_{lm}h^\alpha_{mj} - H^\beta(h^\beta_{im}h^\alpha_{mj} + h^\beta_{jl}h^\alpha_{il}) \\ &\quad +h^\beta_{ij}h^\beta_{lm}h^\alpha_{ml} - h^\beta_{im}h^\beta_{jl}h^\alpha_{ml} + \nabla_l R_{\alpha jil} + \nabla_i R_{\alpha ljl} \\ &\quad +R_{illm}h^\alpha_{mj} + R_{iljm}h^\alpha_{ml} + R_{\alpha\beta il}h^\beta_{lj} - h^\beta_{ij}\langle v_\beta, \overline{\nabla}_H v_\alpha\rangle. \end{aligned}$$

Now we prove the main result in this section.

Proposition 2.9. *We have*

$$\begin{aligned}(\frac{\partial}{\partial t}-\Delta)|A|^2 &= -2|\tilde{\nabla}A|^2+2h^\alpha_{ij}(\overline{\nabla}_l R_{\alpha jil}+\overline{\nabla}_i R_{\alpha ljl})+8R_{\alpha\beta lm}h^\alpha_{il}h^\alpha_{im}\\ &\quad +2h^\alpha_{ij}h^\beta_{ij}R_{\alpha l\beta l}+4h^\alpha_{ij}R_{illm}h^\alpha_{mj}+4h^\alpha_{ij}R_{iljm}h^\alpha_{ml}\\ &\quad +2\sum_{\alpha,\beta,i,j}(h^\alpha_{il}h^\beta_{jl}-h^\alpha_{jl}h^\beta_{il})^2+2\sum_{i,j,l,m}(h^\alpha_{ij}h^\alpha_{lm})^2.\end{aligned}$$

where $\tilde{\nabla}$ *is the covariant differentiation on* $\mathrm{Hom}(T\Sigma_t\times T\Sigma_t, Nor\Sigma_t)$ *determined by the covariant differentiation on* $T\Sigma_t$ *and* D *on the normal bundle,* D *is the normal connection for the embedding* $\Sigma_t\subset M$*, and* $\overline{\nabla}$ *is the connection on* M*,* R *is the curvature operator of* M*.*

Proof. Using the previous lemmas, we have

$$\begin{aligned}\frac{\partial}{\partial t}|A|^2 &= \frac{\partial}{\partial t}(g^{ik}g^{jl}h^\alpha_{ij}h^\alpha_{kl})\\ &= 4H^\alpha h^\alpha_{ik}h^\beta_{ij}h^\beta_{kj}+2h^\alpha_{ij}\frac{\partial}{\partial t}h^\alpha_{ij}\\ &= 2h^\alpha_{ij}(\Delta h^\alpha_{ij}+h^\alpha_{ik}h^\beta_{kl}h^\beta_{lj}-h^\alpha_{lk}h^\beta_{ki}h^\beta_{lj}+h^\beta_{il}h^\beta_{lm}h^\alpha_{mj}\\ &\quad +h^\beta_{ij}h^\beta_{lm}h^\alpha_{ml}-h^\beta_{im}h^\beta_{jl}h^\alpha_{ml}+\nabla_l R_{\alpha jil}+\nabla_i R_{\alpha ljl}\\ &\quad +R_{illm}h^\alpha_{mj}+R_{iljm}h^\alpha_{ml}+R_{\alpha\beta il}h^\beta_{lj}+H^\beta R_{\alpha j\beta i})\\ &= 2h^\alpha_{ij}\Big(\Delta h^\alpha_{ij}+2h^\beta_{il}h^\beta_{lm}h^\alpha_{mj}+h^\beta_{ij}h^\beta_{lm}h^\alpha_{ml}-2h^\beta_{im}h^\beta_{lj}h^\alpha_{ml}+H^\beta R_{\alpha j\beta i}\\ &\quad +\nabla_l R_{\alpha jil}+\nabla_i R_{\alpha ljl}+R_{illm}h^\alpha_{mj}+R_{iljm}h^\alpha_{ml}+R_{\alpha\beta il}h^\beta_{lj}\Big).\end{aligned} \tag{16}$$

Covariant differentiation of the curvature tensor leads to the following formula:

$$\nabla_q R_{\alpha mnp}=\overline{\nabla}_q R_{\alpha mnp}+R_{\alpha m\beta p}h^\beta_{nq}+R_{\alpha mn\beta}h^\beta_{pq}+R_{\alpha\beta np}h^\beta_{mq}-R_{smnp}h^\alpha_{sq},$$

and in turn we have

$$\nabla_l R_{\alpha jil}=\overline{\nabla}_l R_{\alpha jil}+R_{\alpha\beta il}h^\beta_{jl}+R_{\alpha j\beta l}h^\beta_{il}+R_{\alpha ji\beta}h^\beta_{ll}-R_{mjil}h^\alpha_{ml},$$

and

$$\nabla_i R_{\alpha ljl}=\overline{\nabla}_i R_{\alpha ljl}+R_{\alpha\beta jl}h^\beta_{li}+R_{\alpha l\beta l}h^\beta_{ij}+R_{\alpha lj\beta}h^\beta_{il}-R_{mljl}h^\alpha_{mi}.$$

The first Bianchi identity implies that

$$R_{\alpha j\beta l}+R_{\alpha l\beta j}=R_{\alpha\beta jl}.$$

It is clear that

$$\Delta|A|^2=2h^\alpha_{ij}\Delta h^\alpha_{ij}+2\nabla_l h^\alpha_{ij}\nabla_l h^\alpha_{ij}.$$

Substituting the last four identities into (16), we get the identity in the proposition. □

2.3. **Higher derivatives of the second fundamental form.** Let Rm, K be the Riemann curvature tensors of M, Σ respectively. For simplicity, we will use $E*F$, for tensors E, F, to denote any linear combination of tensors formed by contraction on $E_{i\ldots j}F_{k,\ldots l}$ using the metric.

$$(\partial_t-\Delta)A=A*A*A+\nabla Rm+A*Rm \tag{17}$$

At the center p of a normal coordinate system of Σ_t, $\partial g(p)=0$. Taking the t-differentiation of the Christoffel symbols and using the evolution equation of g, we have at p

$$\begin{aligned}\partial_t \nabla A &= \nabla \partial_t A + \partial_t (g^{-1} * \partial g) * A \\ &= \nabla \partial_t A + g^{-1} * \nabla (A * H) * A \\ &= \nabla \partial_t A + \nabla A^2 * A \\ &= \nabla \partial_t A + \nabla A * A^2\end{aligned}$$

Changing order of covariant derivatives and using the Gauss equations, we have

$$\begin{aligned}\Delta \nabla A &= \nabla \Delta A + \nabla K * A + K * \nabla A \\ &= \nabla \Delta A + \nabla (Rm + A^2) + (Rm + A^2) * \nabla A\end{aligned}$$

Therefore, we obtain the evolution equation for ∇A:

$$\begin{aligned}(\partial_t - \Delta)\nabla A &= \nabla (A^3 + \nabla Rm + Rm * A) + \nabla A * A^2 \\ &\quad + Rm * \nabla A + \nabla Rm + A * \nabla A \\ &= \nabla A * (A^2 + A + Rm) + A * \nabla Rm + \nabla^2 Rm + \nabla Rm\end{aligned}$$

Next, we consider the second derivative of A:

$$\partial_t \nabla^2 A = \nabla(\partial_t \nabla A) + \nabla A^2 * \nabla A$$

and

$$\Delta \nabla^2 A = \nabla \Delta \nabla A + \nabla(\nabla A) * (Rm + A^2) + \nabla A * \nabla (Rm + A^2)$$

Hence, we have

$$\begin{aligned}(\partial_t - \Delta)\nabla^2 A &= \nabla(\nabla A * [A^2 + A + Rm] + A * \nabla Rm + \nabla^2 Rm + \nabla Rm) \\ &\quad + \nabla^2 A * (Rm + A^2) + \nabla A * \nabla (Rm + A^2) \\ &= \nabla^2 A * f_1(\nabla A, A, \nabla Rm, Rm) + f_2(\nabla A, A, \nabla Rm, Rm)\end{aligned} \tag{18}$$

Inductively, for general k, we have

$$\begin{aligned}(\partial_t - \Delta)\nabla^k A &= \nabla^k A * f_1(\nabla^{k-1} A, \nabla^{k-2} A, ..., A, \nabla^{k-1} Rm, ..., Rm) \\ &\quad + f_2(\nabla^{k-1} A, \nabla^{k-2} A, ..., A, \nabla^{k-1} Rm, ..., Rm)\end{aligned} \tag{19}$$

It then follows

$$\begin{aligned}(\partial_t - \Delta)|\nabla^k A|^2 &= \langle \nabla^k A, \nabla^k A * f_1(\nabla^{k-1} A, \nabla^{k-2} A, ..., A, \nabla^{k-1} Rm, ..., Rm)\rangle \\ &\quad + \langle \nabla^k A, f_2(\nabla^{k-1} A, \nabla^{k-2} A, ..., A, \nabla^{k-1} Rm, ..., Rm)\rangle \\ &\quad - 2|\nabla^{k+1} A|^2 \\ &\leq h_1(\nabla^{k-1} A, ..., A, \nabla^{k-1} Rm, ..., Rm)|\nabla^k A|^2 \\ &\quad + h_2(\nabla^{k-1} A, ..., A, \nabla^{k-1} Rm, ..., Rm)\end{aligned} \tag{20}$$

where we have used the inequality $2ab \leq a^2 + b^2$.

Therefore, if $|A|, ..., |\nabla^{k-1} A|$ are bounded on $[0,T), T < \infty$, and if the ambient space has bounded geometry, then

$$(\partial_t - \Delta)|\nabla^k A|^2 \leq C_1 |\nabla^k A|^2 + C_2.$$

By the maximum principle (assume Σ is compact),

$$\frac{d}{dt}\left(e^{-C_1 t} \max_{\Sigma_t} |\nabla^k A|^2\right) \leq C_2$$

This implies

$$\max_{\Sigma_t} |\nabla^k A|^2 \leq e^{C_1 t} \left(C_2 t + \max_{\Sigma_0} |\nabla^k A|^2 \right) \leq C(k, \Sigma_0, T) < \infty.$$

Hence, if $|A| \leq C$ on $[0,T), T < \infty$, then all derivatives of A are also bounded on $[0,T)$. The MCF can be extended to $[0, T+\epsilon)$ for some $\epsilon > 0$.

3. Monotonicity Formula

Assume the ambient space is euclidean. Using the standard heat kernel, we introduce

$$\rho(X,t) = (4\pi(t_0-t))^{-n/2}\exp\left(-\frac{|X-X_0|^2}{4(t_0-t)}\right) \tag{21}$$

for $t<t_0$. The following monotonicity formula, due to Huisken, is very useful.

Proposition 3.1. *Along MCF, we have*

$$\frac{\partial}{\partial t}\int_{M_t}\rho(F,t)d\mu_t = -\int_{M_t}\rho(F,t)\left|H+\frac{(F-X_0)^\perp}{2(t_0-t)}\right|^2 d\mu_t. \tag{22}$$

Proof. It is clear that

$$\begin{aligned}\frac{\partial}{\partial t}\int_{M_t}\rho(F,t)d\mu_t &= \int_{M_t}\frac{\partial}{\partial t}\rho(F,t)d\mu_t - \int_{M_t}\rho(F,t)|H|^2d\mu_t\\ &= \int_{M_t}(\frac{\partial}{\partial t}+\triangle)\rho(F,t)d\mu_t - \int_{M_t}\rho(F,t)|H|^2d\mu_t.\end{aligned}$$

Straightforward computation leads to

$$\frac{\partial}{\partial t}\rho(X,t) = \left(\frac{n}{2(t_0-t)} - \frac{1}{2(t_0-t)}\langle H, X-X_0\rangle - \frac{|X-X_0|^2}{4(t_0-t)^2}\right)\rho(X,t)$$

and

$$\nabla\exp\left(-\frac{|X-X_0|^2}{4(t_0-t)}\right) = -\exp\left(-\frac{|X-X_0|^2}{4(t_0-t)}\right)\frac{\langle X-X_0,\nabla X\rangle}{2(t_0-t)}$$

and

$$\triangle\exp\left(-\frac{|X-X_0|^2}{4(t_0-t)}\right) = \exp\left(-\frac{|X-X_0|^2}{4(t_0-t)}\right)\left(\frac{|\langle X-X_0,\nabla X\rangle|^2}{4(t_0-t)^2} - \frac{\langle X-X_0,\triangle X\rangle}{2(t_0-t)} - \frac{|\nabla X|^2}{2(t_0-t)}\right).$$

Note that in the induced metric on M_t

$$|\nabla X|^2 = n \text{ and } \triangle F = H,$$

so we have

$$\left(\frac{\partial}{\partial t}+\triangle\right)\rho(F,t) = -\left(\frac{\langle F-X_0,H\rangle}{(t_0-t)} + \frac{|(F-X_0)^\perp|^2}{4(t_0-t)^2}\right)\rho(F,t) \tag{23}$$

Note

$$|H|^2 + \frac{\langle F-X_0,H\rangle}{(t_0-t)} + \frac{|(F-X_0)^\perp|^2}{4(t_0-t)^2} = \left|H+\frac{(F-X_0)^\perp}{2(t_0-t)}\right|^2$$

Then the proposition follows. □

3.1. **Finite time singularity.** In general, MCF develops singularities, i.e., A becomes unbounded. In particular, MCF of any compact submanifold in the euclidean space must develop singularity in finite time. More generally, we have

Theorem 3.2. *Let $M^m = N^n\times\mathbb{R}^p$, where N is compact. Let Σ_t evolve along MCF where Σ_t is compact and $\dim\Sigma_t > n$. Then the MCF becomes singular in finite time.*

Proof. Define a function $f : N \times \mathbb{R}^p \to \mathbb{R}$ by
$$f(x, y) = \frac{1}{2}|y|^2, \;\; x \in N, y \in \mathbb{R}^p.$$
Let $F : \Sigma \to M$ be the MCF. Then, at the center of a normal coordinate on M, we have
$$\frac{\partial}{\partial t}(f \circ F) = \nabla^M f \cdot H = \nabla^M f \cdot \Delta_{\Sigma_t} F$$
and
$$\begin{aligned} \Delta(f \circ F) &= g^{ij}\left(\partial^2_{ij}(f \circ F) - \Gamma^k_{ij}\partial_k(f \circ F)\right) \\ &= g^{ij}\left(f_\alpha F^\alpha_{ij} + f_{\alpha\beta}F^\beta_j F^\alpha_i - \Gamma^k_{ij} f_\alpha F^\alpha_k\right) \\ &= \nabla f \cdot \Delta_{\Sigma_t} F + f_{\alpha\beta}F^\alpha_i F^\beta_j g^{ij}. \end{aligned}$$
Hence, we have
$$\frac{\partial f}{\partial t} = \Delta_{\Sigma_t} f - \sum_i \left(\nabla^M \nabla^M f\right)(e_i, e_i)$$
where e_i's is an orthonormal basis of $T\Sigma_t$. The Hessian of f has two eigenspaces: n-dimensional space E_0 of the eigenvalue 0 and the p-dimensional E_1 of the eigenvalue 1. The intersection of the tangent space to Σ_t and E_1 has dimension $\geq \dim \Sigma_t - n > 0$. It then follows
$$\frac{\partial}{\partial t} f \circ F \leq \Delta_{\Sigma_t} f \circ F - (\dim \Sigma_t - n)$$
From the maximum principle
$$\sup_{\Sigma_t} f \leq \sup_{\Sigma_0} f - t(\dim \Sigma_0 - n)$$
Therefore the MCF becomes singular at the latest at $T \leq \sup_{\Sigma_0} f/(\dim \Sigma_0 - n)$. □

Remark 3.3. One can use the monotonicity formula to show MCF develops finite time singularity when the ambient space is not a product described above but satisfies certain curvature condition.

3.2. **Type I singularity and self-similar solution.**

Definition 3.4. *We say $T > 0$ is a blow-up time if*
$$\limsup_{t \to T} \sup_{M_t} |A|^2 = \infty.$$

Proposition 3.5. *Let $U(t) = \max_{M_t} |A|^2$. If the mean curvature flow blows up at $T > 0$, the function $U(t)$ satisfies*
$$U(t) \geq \frac{1}{2(T - t)}.$$

Proof. By the parabolic maximum principle, we have
$$\frac{d}{dt}U(t) \;\; \leq \;\; 2(U(t))^2$$
So
$$\frac{d}{dt}U^{-1}(t) \geq -2.$$
We have
$$U^{-1}(T) - U^{-1}(t) \geq -2(T - t).$$

Since $U(T) = 0$, we get the desired inequality. □

Definition 3.6. *We say T is a type I singularity, if*

$$\max_{M_t} |A|^2 \le \frac{C}{T-t}.$$

Otherwise, we say it is a type II singularity.

Assume Σ is compact. At a type I singularity T, assume $x_i \to x_0 \in M$ such that $A(x_i, t_i) = \max_{x \in \Sigma_t, t \le t_i} |A(x,t)| \to \infty$ and $t_i \to T < \infty$. We may identify a neighborhood of x_0 in M with a euclidean domain via the exponential map centered at x_0. For simplicity, assume $M = \mathbb{R}^n$ and $x_0 = 0$. We consider the new flow which we call a rescaled flow,

$$\tilde{F}(x,s) = (2(T-t))^{-1/2} F(x,t),$$

where $x \in M$, $t > 0$ and

$$s = -\frac{1}{2} \log \left(\frac{T-t}{T} \right).$$

It is clear that

$$\frac{d}{ds} \tilde{F}(x,s) = \tilde{F}(x,s) + \tilde{H}$$

where $\tilde{H}$ is the mean curvature of $\tilde{F}(x,s)$. We denote the rescaled surface by $\tilde{M}_s$.

Proposition 3.7. *Let T be a type I singularity. For any $s \in [0, \infty)$, we have*

$$\max_{\tilde{M}_s} |\tilde{A}|^2 \le C,$$

and for any integer m,

$$\max_{\tilde{M}_s} |\nabla^m \tilde{A}|^2 \le C.$$

Lemma 3.8. *If $F(x,t) \to 0$ as $t \to T$, then $\tilde{F}(x,s)$ remains bounded for all $s \in [0, \infty)$.*

Proof.

$$|F(x,t) - F(x,T)| \le \int_t^T |H(x,\tau)| d\tau \le C \int_t^T \frac{1}{\sqrt{T-\tau}} d\tau \le C\sqrt{T-\tau},$$

which implies that

$$|\tilde{F}(x,s)| \le C.$$

The proof is complete. □

3.3. **Self-similar solution.** For the rescaled surface $\tilde{M}_s$, the monotonicity reads as follows. Let

$$\tilde{\rho} = \exp \left(\frac{-|\tilde{F}|^2}{2} \right),$$

then

$$\frac{d}{ds} \int_{\tilde{M}_s} \tilde{\rho} d\tilde{\mu}_s = - \int_{\tilde{M}_s} \tilde{\rho} |\tilde{H} + \tilde{F}^\perp|^2 d\tilde{\mu}_s.$$

It follows that

$$\int_0^\infty \int_{\tilde{M}_s} \tilde{\rho} |\tilde{H} + \tilde{F}^\perp|^2 d\tilde{\mu}_s \le \int_{\tilde{M}_0} \tilde{\rho} < \infty.$$

A subsequence $\tilde{M}_{s_i}$ converges to a smooth limiting surface $\tilde{M}_\infty$ as $s_i \to \infty$, in light of Proposition 3.7 and Lemma 3.8. Therefore, at infinity, we have

$$\tilde{H}_\infty + \tilde{F}_\infty^\perp = 0. \tag{24}$$

Assume $\Sigma_t = F(\Sigma, t)$ is a <u>self-similar solution</u>, i.e., it takes the form

$$F(x,t) = \lambda(t) F_0(x),$$

where $F_0 : \Sigma \to \mathbb{R}^m$ is an immersion satisfying

$$H_0 = c\, F_0^\perp$$

for some constant c. We may assume that $\lambda(0) = 1$. That $F_\infty(t)$ is a self-similar solution implies

$$\lambda \lambda^{'} = c.$$

Hence for some conatnts c and d,

$$\lambda^2 = ct + d.$$

MCF of the round sphere is an example of self-similar solution.

3.4. **Translating solitons.** Suppose that M_t is a translating soliton which translates in the direction of the constant vector T. That means $F_t = F + tT$, i.e, $M_t = M + tT$.

Example 3.9. An example of translating soliton. In $\mathbb{R}^2$, the t-family of curves

$$M_t = \{(x, -\ln \cos x + t) \mid |x| < \pi/2\}$$

is a translating soliton which translates in the direction of the constant vector $(0,1)$: $M_t = M_0 + t(0,1)$. In fact $(\cos x, \sin x)$ is a unit tangent vector and $(-\sin x, \cos x)$ is a unit normal vector, and $H = \cos x$. M_t as the graph of $u = -\ln \cos x + t$ satisfies the MCF

$$u_t = \frac{u_{xx}}{1 + u_x^2}.$$

3.5. **Density Function.** Let M be a smooth $(n+k)$-dimensional manifold with a Riemannian metric g. Assume that either M is closed or $M = M_1 \times \mathbb{R}^p$ for some closed manifold M_1 with g being the product metric. Further, we assume (M, g) has nonnegative sectional curvature and parallel Ricci curvature. Note this implies M_1 has nonnegative sectional curvature and parallel Ricci curvature.

Let $K(x, x_0, t) > 0$ be the heat kernel, *i.e.* the fundamental solution to the heat operator, of (M, g) for $t > 0$ and $x, x_0 \in M$ with the normalization

$$\int_M K(x, x_0, t) d\mu(x) = 1.$$

Then $K(x, x_0, t_0 - t)$ satisfies the backward heat equation

$$\left(\frac{\partial}{\partial t} + \Delta_M\right) K(x, x_0, t_0 - t) = 0$$

for $0 < t < t_0$, where Δ_M is the Laplace operator of (M, g). Define

$$\Phi_{(x_0,t_0)}(x,t) = (4\pi)^{\frac{k}{2}} (t_0 - t)^{\frac{k}{2}} K(x, x_0, t_0 - t) \tag{25}$$

for $0 < t < t_0$. Hamilton's computation [11] shows that if Σ_t are n-dimensional closed submanifolds evolving along the mean curvature flow in M then

$$\begin{aligned}\frac{d}{dt}\int_{\Sigma_t}\Phi_{(x_0,t_0)}d\mu_{\Sigma_t} &= -\int_{\Sigma_t}\left|\mathbf{H}-\frac{D^{\perp}\Phi_{(x_0,t_0)}}{\Phi_{(x_0,t_0)}}\right|^2\Phi_{(x_0,t_0)}d\mu_{\Sigma_t}\\ &\quad -\int_{\Sigma_t}g^{\alpha\beta}\left(D_\alpha D_\beta\Phi_{(x_0,t_0)}-\frac{D_\alpha\Phi_{(x_0,t_0)}D_\beta\Phi_{(x_0,t_0)}}{\Phi_{(x_0,t_0)}}+\frac{\Phi_{(x_0,t_0)}g_{\alpha\beta}}{2(t_0-t)}\right)d\mu_{\Sigma_t}\end{aligned} \tag{26}$$

where $\alpha,\beta = n+1,...,n+k$ denote the indices for a basis normal to Σ_t. Here $D^{\perp}\Phi_{(x_0,t_0)}$ denotes the normal component of the gradient $D\Phi_{(x_0,t_0)}$ along Σ in M.

Under the curvature assumption, Hamilton's matrix Harnack inequality [10] asserts that the last integral in (26) is nonnegative. We remark on that if $M = M_1\times\mathbb{R}^p$ with the product metric then under the same curvature assumption as above the last integral in (26) is still nonnegative. In fact, the heat kernel of M is just the product of the heat kernels of M_1 and $\mathbb{R}^p$, and both M_1 and $\mathbb{R}^p$ have the curvature property required in Hamilton's Harnack estimate. Then direct computation verifies the claimed nonnegativity by applying the heat operator to the product of the kernels then using Hamilton's result.

Therefore, if Σ moves by mean curvature flow in M for $t\in[0,t_0)$ then for any $x_0\in M$ the function $\int_{\Sigma_t}\Phi_{(x_0,t_0)}d\mu_t$ is decreasing in $t<t_0$ and

$$\Theta(\mathcal{M},x_0,t_0) := \lim_{t\nearrow t_0}\int_{\Sigma_t}\Phi_{(x_0,t_0)}d\mu_{\Sigma_t} \tag{27}$$

exists, where $\mathcal{M}$ denotes the mean curvature flow $\Sigma, t\in[0,t_0)$. We call $\Theta(\mathcal{M},x_0,t_0)$ the density of the mean curvature flow Σ_t at (x_0,t_0) and Θ the density function.

The upper-semicontinuity of the Gaussian density function for mean curvature flows in the Euclidean space was proven in [20]. The proof in [6] and [20], which will be presented below for the sake of completeness, carries over for Θ.

Proposition 3.10. *(Upper-semicontinuity of Θ) Let M be a smooth manifold with parallel Ricci curvature and nonnegative sectional curvature where M is either closed or the direct product $M_1\times\mathbb{R}^p$ with the product metric for some closed manifold M_1. Suppose that the closed submanifolds Σ evolve by mean curvature flow in M for $t\in[0,t_0)$. Let (x_j,t_j) be a sequence of points in $M\times[0,t_0)$ such that $\lim_{j\to\infty}(x_j,t_j)=(x_0,t_0)$ and $t_j\le t_{j+1}$. Then*

$$\Theta(\mathcal{M},x_0,t_0)\ge\limsup_{j\to\infty}\Theta(\mathcal{M},x_j,t_j).$$

Proof. For a fixed $t\in[0,t_0)$, there is some $j_0\in\mathbb{N}$ such that $t<t_j$ for all $j>j_0$. The integral $\int_\Sigma\Phi_{(x_j,t_j)}$ is decreasing in t for each (x_j,t_j), so for $j>j_0$

$$\int_{\Sigma_t}\Phi_{(x_j,t_i)}\ge\lim_{t\nearrow t_j}\int_\Sigma\Phi_{(x_j,t_j)}=\Theta(\mathcal{M},x_j,t_j).$$

The heat kernel $K(x,y,t)\in C^\infty(M\times M\times\mathbb{R}^+)$, so for a fixed $t<t_0$

$$\lim_{j\to\infty}\Phi_{(x_j,t_j)}(x,t)=\Phi_{(x_0,t_0)}(x,t)$$

holds for every $x\in M$. Integrating over the closed submanifold Σ and then taking lim sup with $j\to\infty$, we have

$$\int_\Sigma\Phi_{(x_0,t_0)}\ge\limsup_{j\to\infty}\Theta(\mathcal{M},x_j,t_j).$$

Then letting $t \nearrow t_0$

$$\Theta(\mathcal{M}, x_0, t_0) \geq \limsup_{j\to\infty} \Theta(\mathcal{M}, x_j, t_j)$$

and this completes the proof. □

Proposition 3.11. *Let M be a smooth manifold with parallel Ricci curvature and nonnegative sectional curvature where M is either closed or $M_1 \times \mathbb{R}^p$. Let Σ be closed submanifolds moving by mean curvature flow in M for $t \in [0, t_0)$. If there exist a sequence $(x_j, t_j) \in \Sigma_{t_j} \times [0, t_0)$ such that $x_j \to x_0$ and $t_j \nearrow t_0$ as $j \to \infty$, then*

$$\Theta(\mathcal{M}, x_0, t_0) \geq 1. \tag{28}$$

Proof. Recall the following well-known short-time asymptotic expansion of heat kernel: on a complete n-dimensional Riemannian manifold N there are smooth functions $\phi_i(x, y), i = 0, 1, 2, ...,$ defined on $(N \times N)\backslash\{(x, y) \in N \times N : x \text{ is in the cut locus of } y\}$ with

$$\phi_0 > 0, \quad \phi_0(x, x) = 1$$

and when $t \to 0$

$$K(x, y, t) \sim (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{r^2(x, y)}{4t}\right) \sum_{i=0}^{\infty} \phi_i(x, y)\, t^i \tag{29}$$

on any compact subset of $(N \times N)\backslash\{(x, y) \in N \times N : x \text{ is in the cut locus of } y\}$ uniformly, where $r(x, y)$ is the distance function on M. Note that x is in the cut locus of y if and only if y is in the cut locus of x (cf. [2], [13]).

On the closed submanifold Σ, by Li-Yau's heat kernel upper bound [18] for complete Riemannian manifold with nonnegative Ricci curvature, for any $r > 0$

$$\begin{aligned}
&\lim_{t\nearrow t_0} \int_{\Sigma\backslash B_{x_0}(r)} (4\pi(t_0 - t))^{\frac{k}{2}}\, K(x, x_0, t_0 - t) d\mu(x) \\
&\leq C \lim_{t\nearrow t_0} \int_{\Sigma\backslash B_{x_0}(r)} \frac{(t_0 - t)^{\frac{k}{2}}}{\mathrm{Vol}(B_{x_0}(\sqrt{t_0 - t}))} \exp\left(-\frac{d^2(x, x_0)}{5(t_0 - t)}\right) d\mu(x) \\
&\leq C_1 \lim_{t\nearrow t_0} \int_{\Sigma\backslash B_{x_0}(r)} (t_0 - t)^{-\frac{n}{2}} \exp\left(-\frac{r^2}{5(t_0 - t)}\right) d\mu(x) \\
&\leq C_1 \mathrm{Vol}(\Sigma_0) \lim_{t\nearrow t_0} (t_0 - t)^{-\frac{n}{2}} \exp\left(-\frac{r^2}{5(t_0 - t)}\right) = 0
\end{aligned} \tag{30}$$

where C and C_1 denote positive constants depending only on M. When deriving (30), we have used that the volume of $B_{x_0}(\sqrt{t_0 - t})$ is bounded from below by a constant multiple of $(t_0 - t)^{\frac{n+k}{2}}$ as $t_0 - t \to 0$ and the volume of Σ is decreasing in t along mean curvature flow. Choose r small enough so that $B_{x_0}(r)$ does not intersect the cut locus of x_0 in M. By the heat kernel expansion (29) for small time, we see from (30) that

$$
\begin{aligned}
\Theta(\mathcal{M}, x_0, t_0) &= \lim_{t \nearrow t_0} \int_{\Sigma} (4\pi(t_0 - t))^{\frac{k}{2}} K(x, x_0, t_0 - t) d\mu(x) \\
&= \lim_{t \nearrow t_0} \int_{\Sigma \cap B_{x_0}(r)} (4\pi(t_0 - t))^{\frac{k}{2}} K(x, x_0, t_0 - t) d\mu(x) \\
&= \lim_{t \nearrow t_0} \int_{\Sigma \cap B_{x_0}(r)} (4\pi(t_0 - t))^{-\frac{n}{2}} \exp\left(-\frac{d^2(x, x_0)}{4(t_0 - t)}\right) \phi_0(x, x_0) d\mu(x) \\
&= \lim_{t \nearrow t_0} \int_{\Sigma \cap B_{x_0}(r)} (4\pi(t_0 - t))^{-\frac{n}{2}} \exp\left(-\frac{d^2(x, x_0)}{4(t_0 - t)}\right) d\mu(x) \qquad (31)
\end{aligned}
$$

since ϕ_0 is continuous and $\phi(x_0, x_0) = 1$.

Along the mean curvature flow, Σ_{t_j} is immersed for each $t_j < t_0$. Then set

$$m_j = \lim_{\rho \to 0} \frac{\mathrm{Vol}(\Sigma_{t_j} \cap B_{x_j}(\rho))}{\omega_n \rho^n}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Note that (31) holds if x_0, t_0 are replaced by x_j, t_j respectively. Applying (31) at (x_j, t_j) and using the standard fact that the Gaussian density is equal to 1 on each of the m_j sheets of the immersion at x_j,

$$\Theta(\mathcal{M}, x_j, t_j) = m_j \geq 1$$

and $m_j = 1$ if Σ_{t_j} is embedded at x_j. Now the upper-semicontinuity of Θ yields

$$\Theta(\mathcal{M}, x_0, t_0) \geq \limsup_{j \to \infty} \Theta(\mathcal{M}, x_j, t_j) \geq 1$$

and we finish the proof. □

3.6. **Bounds on Area Ratio and Monotonicity Formula.** On a complete manifold with nonnegative Ricci curvature, the heat kernel has upper and lower bounds in terms of distance function between points and volume of geodesic balls, due to the well-known work of Li and Yau. In this section, we show that the heat kernel bounds and Hamilton's Harnack inequality imply an upper area bound for mean curvature flow, and the upper bound on heat kernel and the monotonicity formula for a time varying test function, which mimics Brakke's spherical shrinking test functions, produce a lower area bound for the flow. We observe that the upper bound, which involves the volume of geodesic balls of radius $\sqrt{t_0}$ arising from the heat kernel estimates, tends to 0 when t_0 grows to infinity, provided Σ has nontrivial Euclidean component in $M = M_1 \times \mathbb{R}^p$. The parabolic maximum principle applied to the coordinate functions in the $\mathbb{R}^p$-directions then shows that Σ must stay in a bounded region of M. So the positive lower area bound prevents the flow from admitting long time smooth solution.

Proposition 3.12. *(Upper Bound on Area Ratio) Assume that M is a smooth manifold of dimension $n + k$ with nonnegative sectional curvatures and parallel Ricci curvature and M is either closed or $M = M_1 \times \mathbb{R}^p$ as before. Let Σ be smooth mean curvature flow of immersed closed n-dimensional submanifolds in M. Then for all $x_0 \in M$ and $\rho \in (0, 2\sqrt{t_0})$ the estimate*

$$\sup_{[t_0 - 4\rho^2, t_0 - \rho^2)} \mathit{Vol}(\Sigma \cap B_{x_0}(\rho)) \leq C\rho^n \qquad (32)$$

holds for some positive constant C depending only on Σ_0, t_0 and M.

Proof. Under the curvature assumption, we have seen

$$\frac{d}{dt}\int_{\Sigma}\Phi_{(x_0,t_0)}\leq 0$$

and in particular

$$\int_{\Sigma}\Phi_{(x_0,t_0)}\leq\int_{\Sigma_0}\Phi_{(x_0,t_0)}$$

Since $Ric(M)\geq 0$, the heat kernel estimates in [18] read as

$$\frac{C_1\exp\left(-\frac{r^2(x,x_0)}{3t}\right)}{\mathrm{Vol}(B_{x_0}(\sqrt{t}))}\leq K(x,x_0,t)\leq\frac{C_2\exp\left(-\frac{r^2(x,x_0)}{5t}\right)}{\mathrm{Vol}(B_{x_0}(\sqrt{t}))}\tag{33}$$

for $t>0$ and some positive constants C_1, C_2 which only depend on the dimension of M. Therefore, for all ρ with $4\rho^2<t_0$ and $t\in[t_0-4\rho^2,t_0-\rho^2)$ on $\Sigma\cap B_{x_0}(\rho)$, the backward kernel can be estimated from below as

$$\Phi_{(x_0,t_0)}\geq\frac{C_1(4\pi)^k\rho^k}{\mathrm{Vol}(B_{x_0}(2\rho))}\exp\left(-\frac{r^2(x,x_0)}{3(t_0-t)}\right)\geq\frac{C_1(4\pi)^k\rho^k}{\mathrm{Vol}(B_{x_0}(2\rho))}e^{-\frac{1}{3}}.$$

It then follows

$$\begin{aligned}
\mathrm{Vol}(\Sigma\cap B_{x_0}(\rho)) &\leq e^{\frac{1}{3}}\frac{\mathrm{Vol}(B_{x_0}(2\rho))}{\rho^k}\int_{\Sigma}\Phi_{(x_0,t_0)}\\
&\leq e^{\frac{1}{3}}\frac{\mathrm{Vol}(B_{x_0}(2\rho))}{\rho^k}\int_{\Sigma_0}\Phi_{(x_0,t_0)}\\
&\leq C\frac{\mathrm{Vol}(B_{x_0}(2\rho))}{\rho^k}\int_{\Sigma_0}\frac{(4\pi t_0)^{\frac{k}{2}}}{\mathrm{Vol}(B_{x_0}(\sqrt{t_0}))}\exp\left(-\frac{r^2(x,x_0)}{5t_0}\right)\\
&\leq C\frac{\mathrm{Vol}(B_{x_0}(2\rho))}{\rho^k}\frac{t_0{}^{\frac{k}{2}}}{\mathrm{Vol}(B_{x_0}(\sqrt{t_0}))}\mathrm{Vol}(\Sigma_0) \qquad (34)\\
&\leq C(t_0,\Sigma_0,M)\rho^n
\end{aligned}$$

where we have used Bishop's volume comparison theorem for manifolds with nonnegative Ricci curvature to estimate

$$\mathrm{Vol}(B_{x_0}(2\rho))\leq\omega_{n+k}2^{n+k}\rho^{n+k}$$

where the right hand side equals the volume of the Euclidean ball of radius 2ρ. □

The weighted monotonicity formula for mean curvature flows in the Euclidean space (cf. [9] and [1]) generalizes to the Riemannian setting.

Proposition 3.13. *(Weighted Monotonicity Formula) Let M be a complete manifold of dimension $n+k$ with a Riemannian metric g. Let Σ be a closed n-dimensional submanifolds evolving by mean curvature flow in (M,g). Fix a point (x_0,t_0) in $M\times\mathbb{R}$. For any smooth function f defined on Σ, $t<t_0$*

$$\begin{aligned}
\frac{d}{dt}\int_{\Sigma}f\Phi_{(x_0,t_0)} &= \int_{\Sigma}\left(\frac{\partial f}{\partial t}-\Delta f\right)\Phi_{(x_0,t_0)}-\int_{\Sigma}\left|\mathbf{H}-\frac{D^{\perp}\Phi_{(x_0,t_0)}}{\Phi_{(x_0,t_0)}}\right|^2 f\Phi_{(x_0,t_0)}\\
&\quad -\int_{\Sigma}fg^{\alpha\beta}\left(D_\alpha D_\beta\Phi_{(x_0,t_0)}-\frac{D_\alpha\Phi_{(x_0,t_0)}D_\beta\Phi_{(x_0,t_0)}}{\Phi_{(x_0,t_0)}}+\frac{g_{\alpha\beta}\Phi_{(x_0,t_0)}}{2(t_0-t)}\right) \qquad (35)
\end{aligned}$$

where α, β are the indices for a basis normal to Σ. If the sectional curvature of (M, g) is nonnegative and the Ricci curvature is parallel and M is either closed or $M_1 \times \mathbb{R}^p$, then

$$\frac{d}{dt}\int_\Sigma f\Phi_{(x_0,t_0)} \leq \int_\Sigma \left(\frac{\partial f}{\partial t} - \Delta f\right)\Phi_{(x_0,t_0)} - \int_\Sigma \left|\mathbf{H} - \frac{D^\perp \Phi_{(x_0,t_0)}}{\Phi_{(x_0,t_0)}}\right|^2 f\Phi_{(x_0,t_0)} \tag{36}$$

Proof. Since Σ satisfies the mean curvature flow equation (**??**) and $K(x, x_0, t_0 - t)$ satisfies the backward heat equation, one uses the chain rule to compute

$$\begin{aligned}
\frac{d\Phi_{(x_0,t_0)}}{dt} &= \frac{\partial \Phi_{(x_0,t_0)}}{\partial t} + D\Phi_{(x_0,t_0)} \cdot \mathbf{H} \\
&= -k\pi \left(4\pi(t_0 - t)\right)^{\frac{k}{2}-1} K(x, x_0, t_0 - t) \\
&\quad - \left(4\pi(t_0 - t)\right)^{\frac{k}{2}} \Delta_M K(x, x_0, t_0 - t) + D\Phi_{(x_0,t_0)} \cdot \mathbf{H} \\
&= -\frac{k}{2(t_0 - t)}\Phi_{(x_0,t_0)} - \Delta_M \Phi_{(x_0,t_0)} + D\Phi_{(x_0,t_0)} \cdot \mathbf{H}.
\end{aligned}$$

Recall that the area element $d\mu_t$ of Σ evolves along mean curvature flow according to the equation

$$\frac{\partial}{\partial t} d\mu_t = -\left|\mathbf{H}\right|^2 d\mu_t.$$

We can then write

$$\begin{aligned}
\frac{d}{dt}\int_\Sigma f\Phi_{(x_0,t_0)} &= \int_\Sigma \left(\frac{\partial f}{\partial t} - \Delta f\right)\Phi_{(x_0,t_0)} + \int_\Sigma \Phi_{(x_0,t_0)}\Delta f - \int_\Sigma \left|\mathbf{H}\right|^2 f\Phi_{(x_0,t_0)} \\
&\quad -\frac{k}{2(t_0 - t)}\int_\Sigma f\Phi_{(x_0,t_0)} - \int_\Sigma f\Delta_M \Phi_{(x_0,t_0)} + \int_\Sigma fD\Phi_{(x_0,t_0)} \cdot \mathbf{H}.
\end{aligned} \tag{37}$$

The ambient Laplacian Δ_M and the induced Laplacian Δ on Σ are related by

$$\Delta_M \Phi_{(x_0,t_0)} = \Delta \Phi_{(x_0,t_0)} + g^{\alpha\beta} D_\alpha D_\beta \Phi_{(x_0,t_0)} - D\Phi_{(x_0,t_0)} \cdot \mathbf{H}$$

where as before α, β denote the indices for a basis of $T\Sigma^\perp$. We also notice that

$$\left|\mathbf{H} - \frac{D^\perp \Phi_{(x_0,t_0)}}{\Phi_{(x_0,t_0)}}\right|^2 \Phi_{(x_0,t_0)} = \left|\mathbf{H}\right|^2 \Phi_{(x_0,t_0)} - 2D\Phi_{(x_0,t_0)} \cdot \mathbf{H} + \frac{g^{\alpha\beta} D_\alpha \Phi_{(x_0,t_0)} D_\beta \Phi_{(x_0,t_0)}}{\Phi_{(x_0,t_0)}}$$

and since Σ has no boundary by Green's formula

$$\int_\Sigma f\Delta \Phi_{(x_0,t_0)} = \int_\Sigma \Phi_{(x_0,t_0)}\Delta f.$$

Substituting these formulas into (37) yields (35). Now (36) follows from Hamilton's matrix Harnack estimate [10] because under the curvature assumption on the metric and the assumption on M the integrand of the last integral in (35) is pointwise nonnegative. □

Proposition 3.14. *(Lower Bound on Area Ratio) Let Σ_t be n-dimensional closed submanifolds evolving by mean curvature flow in an $(n + k)$-dimensional manifold M for $t \in [0, t_0)$. Assume that M is either a closed manifold or a direct product of a closed manifold with some Euclidean space and M has nonnegative sectional curvature and parallel Ricci curvature. If there are $x_j \in \Sigma_{t_j}$ and $x_j \to x_0$ as $t_j \to t_0$, then for any $\alpha \in (0, 1)$ there exists a positive constant $C(\alpha, n, M)$ such that for all ρ^2 in $[0, \min\{r_0^2, \frac{1+c\alpha}{\alpha}t_0, \frac{1+c\alpha}{\alpha}\tau(M)\})$,*

$$Vol(\Sigma_{t_0 - \frac{\alpha}{1+c\alpha}\rho^2} \cap B_{x_0}(\rho)) \geq C(\alpha, n, M)\rho^n \tag{38}$$

where c is a positive constant which depends on n, k, M and $r_0, \tau(M)$ are positive constants depending on M.

Proof. We seek a time dependent test function which is a nonnegative subsolution to the heat equation on Σ and has support in geodesic balls. Let us first recall some standard results about distance function on Riemannian manifolds (see [15] for example). For each x_0 let $r(x, x_0)$ be the distance function and define

$$\varphi(x, x_0) = r^2(x, x_0).$$

Assume the sectional curvature of M satisfies

$$-a^2 \leq K \leq b^2$$

for some nonnegative constants a, b in the geodesic ball $B_{x_0}(r_1)$ where $r_1 < \frac{\pi}{2b}$ if $b > 0$. Then φ is smooth on $B_{x_0}(r_1)$ and its Hessian satisfies

$$br\frac{\cos(br)}{\sin(br)} \leq \frac{1}{2}\nabla d\varphi(v, v) \leq ar\frac{e^{ar} + e^{-ar}}{e^{ar} - e^{-ar}}$$

for any unit tangent vector $v \in T_x M$ where $r = r(x, x_0) < r_1$. Therefore there exists a positive number $r_0 < r_1$, which may depend on a and b, such that for all $r < r_0$ the following inequalities hold on $B_{x_0}(r_0)$

$$1 - 2b^2 r^2 \leq \frac{1}{2}\nabla d\varphi(v, v) \leq 1 + 2a^2 r^2. \tag{39}$$

Now we modify Brakke's spherical shrinking test function in the Euclidean case [1] by setting

$$f(x, x_0, t, t_0, \sigma) = \begin{cases} \left(1 - \frac{\varphi(x,x_0)+c(t-t_0)}{\sigma^2}\right)^3 & \text{if it is nonnegative} \\ 0 & \text{otherwise} \end{cases}$$

where c is a positive constant which will be determined later. Note that $f \in C_0^2$ with support in the ball $B_{x_0}(\sqrt{\sigma^2 + c(t_0 - t)})$ which we require to be contained in $B_{x_0}(r_0)$ in order to have (39).

Along mean curvature flow,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + Df \cdot \mathbf{H}$$

and when f is restricted to any immersed submanifold Σ_t

$$\Delta_M f = \Delta_{\Sigma_t} f + g^{\alpha\beta} D_\alpha D_\beta f - Df \cdot \mathbf{H}$$

So we are led to

$$\begin{aligned} \left(\frac{d}{dt} - \Delta_{\Sigma_t}\right) f &= \frac{\partial f}{\partial t} - \Delta_M f + g^{\alpha\beta} D_\alpha D_\beta f \\ &= \frac{3}{\sigma^2} f^{\frac{2}{3}} \left(-c + \Delta_M \varphi - g^{\alpha\beta} D_\alpha D_\beta \varphi\right). \end{aligned}$$

The bounds on the Hessian of φ in (39) imply

$$\begin{aligned} \Delta_M \varphi - g^{\alpha\beta} D_\alpha D_\beta \varphi &\leq (n + k)(2 + 4a^2 r^2) - k(2 - 4b^2 r^2) \\ &= 2n + 4r^2((n + k)a^2 + kb^2). \end{aligned}$$

Then by setting

$$c = 2n + 4r_0^2\left((n + k)a^2 + kb^2\right)$$

we see that

$$\left(\frac{d}{dt} - \Delta_{\Sigma_t}\right) f \leq 0.$$

Applying the weighted monotonicity formula (36) to the function f defined above, we have

$$\frac{d}{dt}\int_{\Sigma} f\Phi_{(x_0,t_0)} \leq 0.$$

Thus

$$\lim_{t\nearrow t_0}\int_{\Sigma} f(x,t)\Phi_{(x_0,t_0)}d\mu(x)$$

exists. For any $\alpha \in (0,1)$ and $\alpha\sigma^2 < t_0$, there is $j_0 \in \mathbb{N}$ such that $t_j > t_0 - \alpha\sigma^2$ for all $j > j_0$. Integrating the above differential inequality over $(t_0 - \alpha\sigma^2, t_j)$, we see

$$\int_{\Sigma_{t_j}} f\Phi_{(x_0,t_0)} \leq \int_{\Sigma_{t_0-\alpha\sigma^2}} f\Phi_{(x_0,t_0)} \tag{40}$$

Note that the nonnegative continuous function f is defined globally on M. By the arguments in the proof of Proposition 3.11,

$$f(x_j,t_j) \leq f(x_j,t_j)\, m_j = \lim_{t\nearrow t_j}\int_{\Sigma} f(x,t)\Phi_{(x_j,t_j)} \tag{41}$$

where m_j is the volume density of Σ_{t_j} at x_j. For each pair (x_j,t_j), we also have

$$\frac{d}{dt}\int_{\Sigma} f\Phi_{(x_j,t_j)} \leq 0.$$

Thus for any fixed $t < t_j$ the inequality below holds and the limit therein exists

$$\lim_{t\nearrow t_j}\int_{\Sigma} f(x,t)\Phi_{(x_j,t_j)} \leq \int_{\Sigma_t} f(x,t)\Phi_{(x_j,t_j)}. \tag{42}$$

Combining (41) with (42), we obtain

$$f(x_j,t_j) \leq \int_{\Sigma_t} f(x,t)\Phi_{(x_j,t_j)}.$$

Now by letting $t_j \nearrow t_0$ in the inequality above, we conclude

$$f(x_0,t_0) \leq \lim_{t_j\nearrow t_0}\int_{\Sigma_{t_j}} f\Phi_{(x_0,t_0)} \tag{43}$$

By (40), (43) and the fact

$$f(x_0,x_0,t_0,t_0,\sigma) = 1$$

we have

$$1 \leq \int_{\Sigma_{t_0-\alpha\sigma^2}} f\Phi_{(x_0,t_0)}. \tag{44}$$

Since M is compact or $M = M_1 \times \mathbb{R}^p$, there exist positive constants $C_1(M)$ and $\tau(M)$ depending only on M such that

$$\mathrm{Vol}(B_x(\tau)) \geq C_1(M)\tau^{n+k} \tag{45}$$

for all $x \in M$ and $\tau \leq \tau(M)$. Then the heat kernel upper bound yields

$$\Phi_{(x_0,t_0)}(x, t_0 - \alpha\sigma^2) \leq \frac{C(\alpha\sigma^2)^{\frac{k}{2}}}{\mathrm{Vol}(B_{x_0}(\sqrt{\alpha\sigma^2}))} \leq \frac{C}{C_1\alpha^{\frac{n}{2}}\sigma^n}$$

for $\sqrt{\alpha}\sigma < \tau(M)$. Applying the estimate above to (44) and noting that

$$0 \le f \le (1+c\alpha)^3$$

and f is supported in $B_{x_0}(\sqrt{\sigma^2 + c\alpha\sigma^2})$ at $t = t_0 - \alpha\sigma^2$, we obtain

$$\frac{C_1 \alpha^{\frac{n}{2}} \sigma^n}{C(1+c\alpha)^3} \le \mathrm{Vol}(\Sigma_{t_0 - \alpha\sigma^2} \cap B_{x_0}(\sqrt{1+c\alpha}\sigma)).$$

To simplify the expression, we set $\rho = \sqrt{1+c\alpha}\sigma$. It follows that $\alpha\sigma^2 = \frac{\alpha}{1+c\alpha}\rho^2$. Finally, we recall all restrictions on ρ:

$$\begin{array}{rlll} (i) & \alpha\sigma^2 < t_0 & \text{implies} & \rho^2 < \frac{1+c\alpha}{\alpha} t_0 \\ (ii) & \alpha\sigma^2 < \tau(M) & \text{implies} & \rho^2 < \frac{1+c\alpha}{\alpha}\tau(M) \\ (iii) & (1+c\alpha)\sigma^2 < r_0 & \text{implies} & \rho^2 < r_0 \end{array}$$

This means $\rho^2 < \min\{r_0, \frac{1+c\alpha}{\alpha} t_0, \frac{1+c\alpha}{\alpha}\tau(M)\}$. □

3.7. **Mean Value Inequality.** Let $\Sigma = F(\Sigma, t)$ be immersed submanifolds moving by mean curvature flow in M for $t \in [0, T)$. There are two ways to obtain continuous functions f on Σ. One is by restricting continuous functions on M to Σ, so the functions are defined extrinsically. The other one is by taking continuous functions on $\Sigma \times [0, T)$ and at the points where Σ is immersed but not embedded the functions may take different values to ensure continuity, so the functions are defined intrinsically. To be more precise, if $F(p,t) = x_0 = F(q,t)$ for $p \ne q$ there exist neighborhoods D_p and D_q of p and q in Σ respectively such that $F(\cdot, t)$ embeds D_p and D_q into M. When f is regarded as a function on Σ, its continuity and its value at x_0 are determined by the neighborhoods D_p and D_q. Extrinsically or intrinsically defined functions arise naturally when we study mean curvature flow. On an immersed submanifold Σ_{t_0}, if $x_0 \in \Sigma_{t_0}$ is an immersed point, then there exist at most m_{x_0} points p in Σ satisfying $F(p, t_0) = x_0$ where m_{x_0} is the volume density of Σ_{t_0} at x_0 and it counts the number of sheets containing x_0 inside a small ball.

The arguments in the proof of Proposition 3.11 let us to conclude

Lemma 3.15. *Let $\Sigma = F(\Sigma, t)$ for $t \in [0, T)$ and M be as in Proposition 3.12. Let f be a continuous function on $\Sigma \times [0, T]$. Then for any $t_0 \in (0, T)$ and any $x_{t_0} \in \Sigma_{t_0}$*

$$f(p_1, t_0) + \dots + f(p_{m_{t_0}}, t_0) = \lim_{t \nearrow t_0} \int_\Sigma f(x,t)\Phi_{(x_{t_0}, t_0)} d\mu(x) \tag{46}$$

holds, where m_{t_0} is the volume density of Σ_{t_0} at $x_{t_0} = F(p_j, t_0)$ for $j = 1, \dots, m_{t_0}$.

Proposition 3.16. *(Mean Value Inequality) Let M be a smooth Riemannian manifold with parallel Ricci curvature and nonnegative sectional curvature. Assume that M is either closed or a direct product $M_1 \times \mathbb{R}^p$ of a closed manifold M_1 with a Euclidean space $\mathbb{R}^p$. Let $\Sigma = F(\Sigma, t)$ be closed submanifolds evolving by smooth mean curvature flow in M for $t \in [0, T)$. Let f be a function on Σ which is a subsolution of the heat operator on Σ*

$$\left(\frac{d}{dt} - \Delta_\Sigma\right) f \le 0$$

for all $t \in [0, T)$. Then there is a positive constant $\tau(M)$ depending only on M such that for all $\rho \in (0, \min\{\tau(M), \sqrt{t_0}\})$ and $t_0 < T$ such that

$$\sum_{i=1}^{m_{x_0}} f^2(p_i, t_0) \le \frac{C(M)}{\rho^{n+2}} \int_{t_0-\rho^2}^{t_0} \int_{\Sigma \cap B_{x_0}(\rho)} f^2 \tag{47}$$

where p_i are distinct points in Σ with $F(p_i, t_0) = x_0$ and m_{x_0} is the volume density of Σ_{t_0} at x_0.

Proof. We observe that for any smooth function ϕ on $M \times (t_0 - \rho^2, t_0)$

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right)(f^2\phi^2) &= \phi^2\left(\frac{d}{dt} - \Delta\right) f^2 + f^2\left(\frac{d}{dt} - \Delta\right)\phi^2 - 8f\phi\nabla\phi\cdot\nabla f \\ &\leq -2\phi^2 |\nabla f|^2 + f^2\left|\left(\frac{d}{dt} - \Delta\right)\phi^2\right| + 8\phi f |\nabla\phi| |\nabla f| \\ &\leq f^2\left|\left(\frac{d}{dt} - \Delta\right)\phi^2\right| + 8f^2|\nabla\phi|^2\end{aligned}$$

by using the fact that f is a subsolution and Young's inequality. Choose ϕ such that

$$\begin{array}{rll} 0 \leq \phi \leq 1 & \text{in} & M \times (t_0 - \rho^2, t_0) \\ \phi \equiv 1 & \text{in} & B_{x_0}(\frac{\rho}{2}) \times (t_0 - \frac{\rho^2}{4}, t_0) \\ \phi \equiv 0 & \text{in} & M\backslash B_{x_0}(\rho) \times (t_0 - \rho^2, t_0) \end{array}$$

and

$$\rho|D\phi| + \rho^2\left|D^2\phi\right| + \rho^2\left|\frac{\partial\phi}{\partial t}\right| \leq C_0(M)$$

for some constant $C_0(M)$ which depends on (M, g). For ϕ so chosen, we have

$$\left(\frac{d}{dt} - \Delta\right)(f^2\phi^2) \leq \frac{C_0}{\rho^2} f^2 \tag{48}$$

in $M \times (t_0 - \rho^2, t_0)$ and

$$\left(\frac{d}{dt} - \Delta\right)(f^2\phi^2) \equiv 0 \tag{49}$$

in $B_{x_0}(\frac{\rho}{2}) \times (t_0 - \frac{\rho^2}{4}, t_0)$.

Since $Ric(M) \geq 0$, it follows from the heat kernel upper estimate

$$K(x, x_0, t_0 - t) \leq \frac{C(n+k)}{\sqrt{\mathrm{Vol}(B_{x_0}(\sqrt{t_0 - t}))}} \exp\left(-\frac{r^2(x, x_0)}{5(t_0 - t)}\right)$$

that as long as $\rho \leq \tau(M)$, where $\tau(M)$ is defined in (52), and $t_0 - t > \rho^2/4$, then

$$K(x, x_0, t, t_0) \leq \frac{C(M)}{\rho^{n+k}} \tag{50}$$

holds for some positive constant $C(M)$ depending only on M.

We now estimate the backward kernel by decompose $B_{x_0}(\rho)\times(t_0-\rho^2, t_0)\backslash B_{x_0}(\frac{\rho}{2})\times(t_0-\frac{\rho^2}{4}, t_0)$ into two disjoint regions. First, when $\rho^2/4 < t_0 - t < \rho^2$ it follows directly from the heat kernel upper estimate (50) that

$$\Phi_{(x_0,t_0)}(x, t) = (4\pi(t_0 - t))^{\frac{k}{2}} K(x, x_0, t, t_0) \leq \frac{C(M)}{\rho^n}.$$

Second, when $0 < t_0 - t \leq \rho^2/4$,

$$\begin{aligned}\Phi_{(x_0,t_0)}(x,t) &\leq \frac{C(t_0-t)^{\frac{k}{2}}}{\mathrm{Vol}(B_{x_0}(\sqrt{t_0-t}))}\exp\left(-\frac{r^2(x,x_0)}{5(t_0-t)}\right)\\ &\leq \frac{C(M)}{(t_0-t)^{\frac{n}{2}}}\exp\left(-\frac{\rho^2}{20(t_0-t)}\right)\\ &\leq \frac{C(M)}{\rho^n}\end{aligned}$$

because $\sqrt{t_0-t} < \rho/2 < \tau(M)$. In the last step above, we have applied the following elementary fact to $y = \rho^2/(t_0-t)$: for any $c > 0$, there exists a positive constant $C(n,c)$ such that for all $y \geq 0$ the inequality below holds

$$y^n \leq C(n,c)e^{cy}.$$

Recall $f^2\phi^2$ is supported in $M\backslash B_{x_0}(\rho) \times [t_0 - \frac{\rho^2}{4}, t_0)$. By the weighted monotonicity formula (36) together with (48), (49) and the estimate on $\Phi_{(x_0,t_0)}$ above, we have

$$\frac{d}{dt}\int_\Sigma f^2\phi^2\Phi_{(x_0,t_0)} \leq \frac{C_0(M)C(M)}{\rho^{n+2}}\int_{\Sigma\cap B_{x_0}(\rho)} f^2.$$

Noting $\phi(x, t_0-\rho^2) = 0$ for all $x \in M$, integrating the inequality above over $(t_0-\rho^2, t)$ yields

$$\int_\Sigma f^2\phi^2\Phi_{(x_0,t_0)} \leq \frac{C_0(M)C(M)}{\rho^{n+2}}\int_{t_0-\rho^2}^{t}\int_{\Sigma\cap B_{x_0}(\rho)} f^2.$$

Since $\phi(x_0,t_0) = 1$, by Lemma 3.15 we obtain that for any $p \in \Sigma$ with $F(p,t_0) = x_0$

$$\begin{aligned}\sum_{i=1}^{m_{x_0}} f^2(p_i,t_0) &= \sum_{i=1}^{m_{x_0}} f^2(p,t_0)\phi^2(x_0,t_0)\\ &= \lim_{t\nearrow t_0}\int_\Sigma f^2\phi^2\Phi_{(x_0,t_0)}\\ &\leq \frac{C_0(M)C(M)}{\rho^{n+2}}\int_{t_0-\rho^2}^{t_0}\int_{\Sigma\cap B_{x_0}(\rho)} f^2.\end{aligned}$$

The proof is now complete. □

4. A Maximum Principle for evolution equations

We discuss a useful maximum principle for evolution equations on a complete Riemannian manifold with time dependent metrics. The theorem below is due to Ecker-Huisken [8] which is based on an earlier work of Liao-Tam [16] on the maximum principle on complete noncompact manifolds with time independent metric.

Theorem 4.1. *Let M be a manifold with Riemannian metrics $g(t)$. Suppose that*

$$\left|\frac{d}{dt}g\right|_g \leq \alpha < \infty. \tag{51}$$

and the following volume growth condition holds:

$$vol(B_r(p), g(t)) \leq e^{c(1+r^2)} \tag{52}$$

for some uniform constant $c > 0$ and some $p \in M$, $B(r,p)$ is the geodesic ball at the t. Let $f \in C^0(M \times [0,T]) \cap C^\infty(M \times (0,T])$. Assume

$$\frac{\partial}{\partial t}f \leq \Delta_{g(t)}f + \mathbf{a}\cdot\nabla f + Bf \tag{53}$$

where $|\mathbf{a}| < \beta < \infty$ and $|B| \leq \gamma < \infty$ on $M \times [0,T]$. Assume $f \leq 0$ on $M \times \{0\}$ and

$$\int_0^T \int_M e^{-\delta r^2_{g(t)}(x,p)}|\nabla f|^2(x)d\mu_{g(t)}dt < \infty \tag{54}$$

for some $\delta > 0$. Then $f \leq 0$ on $M \times [0,T]$.

Proof. From (51), $g(t)$ are uniformly equivalent to $g(0)$: $\exists\, C_1, C_2$ depending on T such that

$$C_2\, g(0) \leq g(t) \leq C_1\, g(0). \tag{55}$$

Following Liao-Tam, fix η with $0 < \eta < \min\{T, 1/64c, 1/32\alpha, 1/32\delta\}$. Set

$$h(y,s) = -\frac{\theta r^2_{g(s)}(y,p)}{4(2\eta - s)},\ 0 < s < \eta$$

where θ is yet to determined and $r_{g(s)}(y,p)$ is the distance between points y and p in $g(s)$. Then

$$\begin{aligned}\frac{dh}{ds} &= -\frac{\theta r^2_{g(s)}(y,p)}{4(2\eta-s)^2} - \frac{\theta r_{g(s)}}{2(2\eta-s)}\frac{dr_{g(s)}}{ds}\\ &= -\theta^{-1}|\nabla h|^2 - \frac{\theta r_{g(s)}}{2(2\eta-s)}\frac{dr_{g(s)}}{ds}.\end{aligned}$$

For any fixed curve with length $l(s)$ measured in $g(s)$, we have

$$\left|\frac{dl(s)}{ds}\right| = \left|\frac{d}{ds}\int\sqrt{g(C'(\tau),C'(\tau))}d\tau\right| \leq \frac{1}{2}\alpha l(s)$$

where we have used (51)

$$\frac{d}{ds}g(s)(C'(\tau),C'(\tau)) \leq \alpha g(s)(C'(\tau),C'(\tau)).$$

In particular,

$$\left|\frac{d}{ds}r_s\right| \leq \frac{1}{2}\alpha r_s.$$

Therefore, for $\theta = \frac{1}{4}, \eta \leq \frac{1}{4\alpha}$,

$$\frac{d}{ds}h \leq -\theta^{-1}|\nabla h|^2 + \theta^{-1}\alpha|\nabla h|^2(2\eta - s) \leq -2|\nabla h|^2. \tag{56}$$

As in [16], for $K > 0$ define $f_K = \max\{\min(f, K), 0\}$ and take a smooth time independent function ϕ with compact support. For $0 < \epsilon < \eta$, by (53) we have

$$0 \leq \int_\epsilon^\eta e^{-bs} \int_M \phi^2 e^h f_K \left(\Delta f - \frac{\partial f}{\partial s}\right) + \beta \int_\epsilon^\eta e^{-bs} \int_M \phi^2 e^h f_K |\nabla f| + \gamma \int_\epsilon^\eta e^{-bs} \int_M \phi^2 e^h f_K f$$

where b will be chosen later and we have used $f_K f \geq 0$. Integrating by parts,

$$\begin{aligned}
\int_\epsilon^\eta e^{-bs} \int_M \phi^2 e^h f_K \Delta f &= -\int_\epsilon^\eta e^{-bs} \int_M \phi^2 e^h \nabla f_K \nabla f \\
&\quad - \int_\epsilon^\eta e^{-bs} \int_M \phi^2 e^h f_K \nabla h \nabla f - 2\int_\epsilon^\eta e^{-bs} \int_M \phi e^h f_K \nabla\phi\nabla f \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}
\end{aligned}$$

Observe

$$\begin{aligned}
\mathrm{I} &= -\int_\epsilon^\eta e^{-bs} \int_M \phi^2 e^h |\nabla f_K|^2 \\
\mathrm{II} &\leq \frac{1}{4}\int_\epsilon^\eta e^{-bs} \int_M \phi^2 e^h |\nabla f|^2 + \int_\epsilon^\eta e^{-bs} \int_M \phi^2 e^h f_K^2 |\nabla h|^2 \\
\mathrm{III} &\leq \frac{1}{2}\int_\epsilon^\eta e^{-bs} \int_M \phi^2 e^h |\nabla f|^2 + 2\int_\epsilon^\eta e^{-bs} \int_M e^h f_K^2 |\nabla \phi|^2.
\end{aligned}$$

Note

$$f_K = \begin{cases} K, & \text{if } f \geq K \\ f, & \text{if } 0 < f < K \\ 0, & \text{if } f \leq 0 \end{cases}$$

Hence

$$\frac{\partial f_K}{\partial s}(f_K - f) = 0 \tag{57}$$

whenever $\partial f_k / \partial s$ exists, and we have by using (57)

$$\begin{aligned}
-e^h f_K \frac{\partial f}{\partial s} &= -e^h f_K \frac{\partial f_K}{\partial s} + \frac{\partial}{\partial s}\{e^h f_K (f_K - f)\} - \frac{\partial e^h}{\partial s} f_K (f_K - f) \\
&\leq -e^h f_K \frac{\partial f_K}{\partial s} + \frac{\partial}{\partial s}\{e^h f_K (f_K - f)\}
\end{aligned} \tag{58}$$

Note f_K is uniformly Lipschitz continuous on compact subsets of $M \times [0, T]$. Note

$$\left|\frac{\partial}{\partial s}\sqrt{g}\right| \leq n\alpha\sqrt{g}$$

where $\sqrt{g}=\sqrt{\det g}$. Therefore, from (58)

$$
\begin{aligned}
-\int_\epsilon^\eta e^{-bs}\int_M \phi^2 e^h f_K\frac{\partial f}{\partial s} &\le -\frac{1}{2}\int_\epsilon^\eta e^{-bs}\int_M \phi^2 e^h\frac{\partial f_K^2}{\partial s}+\int_\epsilon^\eta\frac{\partial}{\partial s}\left(e^{-bs}\int_M\phi^2 e^h f_K(f_K-f)\right)\\
&\quad +b\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h f_K(f_K-f)-\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h f_K(f_K-f)\frac{\partial_s\sqrt{g}}{\sqrt{g}}\\
&= -\frac{1}{2}\int_\epsilon^\eta\frac{\partial}{\partial s}\left(e^{-bs}\int_M\phi^2 e^h f_K^2\right)-\frac{b}{2}\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h f_K^2+\frac{1}{2}\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h f_K^2\frac{\partial_s\sqrt{g}}{\sqrt{g}}\\
&\quad +\frac{1}{2}\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h f_K^2\frac{\partial h}{\partial s}+e^{-bs}\int_M\phi^2 e^h f_K(f_K-f)\Big|_{s=\epsilon}^{s=\eta}\\
&\quad +b\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h f_K(f_K-f)-\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h f_K(f_K-f)\frac{\partial_s\sqrt{g}}{\sqrt{g}}\\
&\le -\frac{1}{2}e^{-bs}\int_M\phi^2 e^h f_K^2\Big|_{s=\epsilon}^{s=\eta}+\frac{1}{2}(n\alpha-b)\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h f_K^2\\
&\quad +\frac{1}{2}\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h f_K^2\frac{\partial h}{\partial s}+e^{-bs}\int_M\phi^2 e^h f_K(f_K-f)\Big|_{s=\epsilon}^{s=\eta}\\
&\quad +(b-n\alpha)\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h f_K(f_K-f)\\
&\le -\frac{1}{2}e^{-bs}\int_M\phi^2 e^h f_K^2\Big|_{s=\epsilon}^{s=\eta}-\frac{b}{4}\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h f_K^2\\
&\quad +\frac{1}{2}\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h f_K^2\frac{\partial h}{\partial s}+e^{-bs}\int_M\phi^2 e^h f_K(f_K-f)\Big|_{s=\epsilon}^{s=\eta}
\end{aligned}
$$

by noting $f_K(f_K-f)\le 0$ and taking $b\ge 2n\alpha+4\gamma$. It then follows from (56)

$$
\begin{aligned}
-\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h f_K\frac{\partial f}{\partial s} &\le -\frac{1}{2}e^{-bs}\int_M\phi^2 e^h f_K^2\Big|_{s=\epsilon}^{s=\eta}-\gamma\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h f_K^2\\
&\quad -\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h f_K^2|\nabla h|^2+e^{-bs}\int_M\phi^2 e^h f_K(f_K-f)\Big|_{s=\epsilon}^{s=\eta} \qquad (59)
\end{aligned}
$$

We also esitmate

$$
\beta\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h f_K|\nabla f|\le\frac{1}{4}\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h|\nabla f|^2+\beta^2\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h f_K^2.
$$

We therefore have

$$
\begin{aligned}
0 &\le \int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h f_K\left(\Delta f-\frac{\partial f}{\partial s}\right)+\beta\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h f_K|\nabla f|+\gamma\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h f_K f\\
&= \mathrm{I}+\mathrm{II}+\mathrm{III}-\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h f_K\frac{\partial f}{\partial s}+\frac{1}{4}\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h|\nabla f|^2+\beta^2\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h f_K^2\\
&\quad +\gamma\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h f_K|\nabla f|\\
&\le -\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h|\nabla f_K|^2+\int_\epsilon^\eta e^{-bs}\int_M\phi^2 e^h|\nabla f|^2\\
&\quad +2\int_\epsilon^\eta e^{-bs}\int_M e^h|\nabla\phi|^2 f_K^2-\frac{e^{-bs}}{2}\int_M\phi^2 e^h f_K^2\Big|_{s=\epsilon}^{s=\eta}
\end{aligned}
$$

Letting $\epsilon \to 0$, using $f_K = 0$ at $s = 0$ because $f \leq 0$ at $s = 0$ and $f_K(f_K - f) \leq 0$, taking ϕ to be the cut-off function which is 1 in $B^0_R(p)$, 0 outside $B^0_{R+1}(p)$ and $0 \leq \phi \leq 1, |\nabla\phi|^0 \leq 2$, we have

$$(60)\quad \frac{e^{-b\eta}}{2}\int_{B^0_R(p)} e^h f_K^2\Big|_{s=\eta} \leq \int_0^\eta e^{-bs}\int_{B^0_{R+1}(p)} e^h(|\nabla f|^2 - |\nabla f_K|^2) + 2C_1\int_0^\eta e^{-bs}\int_{B^0_{R+1}(p)\setminus B^0_R(p)} e^h f_K^2$$

Since $0 < \eta < \min(T, 1/64c, 1/32\alpha, 1/32\delta)$, one checks $h(y,s) \leq -2cr_s^2(y,p)$ and $h(y,s) \leq -\delta r_s^2(y,p)$ for all $0 < s < \eta$. Since $f_K^2 \leq K^2$, we have, for each fixed $K > 0$, from the volume growth condition (52) that

$$(61)\quad \lim_{R\to\infty}\int_0^\eta e^{-bs}\int_{B^0_{R+1}(p)\setminus B^0_R(p)} e^h f_K^2 = 0$$

Note $0 \leq |\nabla f|^2 - |\nabla f_K|^2 \leq |\nabla f|^2$ and let $R \to \infty$ in (60)

$$(62)\quad \frac{e^{-b\eta}}{2}\int_M e^h f_K^2\Big|_{s=\eta} \leq \int_0^\eta e^{-bs}\int_M e^h\left(|\nabla f|^2 - |\nabla f_K|^2\right) < \infty$$

by (61) and the assumption (54).

Letting $K \to \infty$, we see $f_K^2 \to (f^+)^2$, and $|\nabla f_K|^2 \to |\nabla f|^2$ for all s. By the dominated convergence theorem,

$$\int_M e^h (f^+)^2|_{s=\eta} \leq 0$$

hence $f^+ = 0$ at $t = \eta$. Since η is arbitrary with $0 < \eta < \min(T, 1/64K, 1/32\delta)$ we conclude $f \leq 0$ on $M \times [0,T]$, by an inductive argument. □

We now apply this maximum principle to MCF.

Theorem 4.2. *Let $F_0 : M^n \to \mathbb{R}^{n+1}$ be a smooth MCF of a complete hypersurface with bounded $C^{2,\alpha}$-norm. Suppose the initial hypersurface $F_0(M)$ has nonnegative mean curvature. Then the smooth solution to MCF from F_0 over $[0,T]$ has nonnegative mean curvature, where T depends on n and the initial curvature bound.*

Proof. The equation we deal with is

$$\left(\frac{\partial}{\partial t} - \Delta\right) H = |A|^2 H.$$

Because $C^{2,\alpha}$-norm of F_0 is bounded, the initial surface has bounded curvature and $\sup_{M_0}|A| \leq c_0$. One can bound $\sup_{M_t}|A|$ in terms of c_0 on some small interval $[0,T]$. Since $Ric_{M_t} \geq -2|A|^2$, the uniform volume growth condition (52) holds on $[0,T]$ as Ricci curvature has a lower bound on this interval. From the parabolic theory, we have

$$\sup_{M_t} t^{1-\alpha}|\nabla H|^2 \leq C(n, T, c_0, \|F_0\|_{C^{2,\alpha}})$$

which in turn, together with the volume growth condition, implies (54). Recall

$$\frac{\partial}{\partial t} g_{ij} = -2Hh_{ij}.$$

So (51) holds. Now with $b = |A|^2$ in (53), Theorem 4.1 implies the desired result. □

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Department of Mathematics, The University of British Columbia, Vancouver, B.C. V6T 1Z2, Canada
E-mail address: jychen@math.ubc.ca