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Analysis in Complex Geometry

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An important question in complex geometry is to characterize those topological manifolds that admit a complex structure. Once a complex structure is found, one wants to search for the existence of algebraic or geometric structures that are compatible with the complex structure.

Most geometric structures are given by Hermitian metrics or connections that are compatible with the complex structure. In most cases, we look for connections with special holonomy group.

A connection may have torsion. The torsion tensor is not well-understood. Much more research need to be done especially for those Hermitian connections with special holonomy group.

Karen Uhlenbeck

Simon Donaldson

By using the theorem of Donaldson-Uhlenbeck-Yau, it is possible to construct special holonomy connections with torsion. Their significance in relation to complex or algebraic structure need to be explored.

Kähler metrics have no torsion and their geometry is very close to that of algebraic geometry. Yet, as was demonstrated by Voisin, there are Kähler manifolds that are not homotopy equivalent to any algebraic manifolds. The distinction between Kähler and algebraic geometry is therefore rather delicate.

Erich Kähler

Algebraic geometry is a classical subject and there are several natural equivalences of algebraic manifolds: birational equivalence, biregular equivalence, and arithmetic equivalence.

In this talk, we shall explore several important questions that need to be solved.

I. Finding the necessary and sufficient conditions for a smooth even dimensional manifold to admit an almost complex structure (up to deformation).

This is a problem in algebraic topology. One needs to understand how to lift the classifying map from M^{2n} to $SO(2n)$ to a map from M^{2n} to $U(n)$. For n small, this was solved by the method of Postnikov tower construction. But no general condition for arbitrary n has been found.

II. What are the conditions for a manifold to have an integrable almost complex structure?

Once an almost complex structure J is found, it may not be deformable to an integrable J. There are plenty of examples of such manifolds for $n = 2$. In this dimension, the Hirzebruch-Riemann-Roch gives very strong constraints. I found four-dimensional manifold with trivial tangent bundle which admits no integrable complex structure. My example have large fundamental group. It will be nice to find simply connected examples.

Friedrich Hirzebruch

For $n > 2$, the Hirzebruch-Riemann-Roch formula is not powerful enough. I conjectured that every almost complex manifold admits an integrable almost complex structure (which may not be deformable to the original one).

Potentially, one can try to deform an almost complex structure to an integrable one by certain parabolic equation. In the process of deformation, it is quite likely that certain bubbling process may occur and the final obtained integrable complex structure is not necessarily homotopic to the initial almost complex structure.

III. Moduli space of integrable complex structure

It is a difficult problem to determine the moduli space of integrable complex structure over a fixed smooth manifold. Potentially there can be an infinite number of components of such structures. Perhaps one should fix the Chern classes of such complex structures. In that case, one knows that the moduli space of algebraic manifolds of general type is a bounded family. Hence, there are only a finite number of components and each component form a quasi-projective manifold. This follows from the works of D. Gieseker and E. Viehweg.

IV. Classification of complex surfaces

Complex surfaces were classified by Kodaira, generalizing the work of the Italian algebraic geometers. His basic tool came from the Atiyah-Singer index formula.

Kunihiko Kodaira

Most complex surfaces are deformable to algebraic surfaces. All surfaces are classified except class VII_0. It was conjectured that class VII_0 surfaces are those that were constructed by Bombieri and Inoue.

Enrico Bombieri

For class VII_0 surfaces that admit no holomorphic curves, this was proved to be true by the works of Bogomolov, and Li-Yau-Zheng, based on different arguments. It is likely that the method of Li-Yau-Zheng, which is based on Hermitian-Yang-Mills connections, can be generalized to cover the case of class VII_0 with curves. If this can be done, it will give a complete classification of complex surfaces that cannot support Kähler metrics.

V. Classification of higher dimension complex manifolds

For complex manifolds with dimension greater than two, the space of integrable compelx structures are much more flexible. Many examples were constructed. A large class of complex manifolds are obtained from the twistor space of anti-self-dual four-manifold (constructed by Taubes) and the Clemens-Friedman construction of complex three-folds by blowing down rational curves and smoothing.

Based on the Clemens-Friedman construction, Reid speculated that every Calabi-Yau threefold can be deformed to each other through non-Kähler complex manifolds. This speculation demonstrates the potential role of non-Kähler complex manifolds.

Eugenio Calabi

In order to understand non-Kähler complex manifolds, we need to construct geometric structures. Naturally, we have Hermitian $(1,1)$-forms ω which may not be closed.

There are many possible conditions we can impose on ω so that the Riemannian structure given by ω is more compatible with the complex structure.

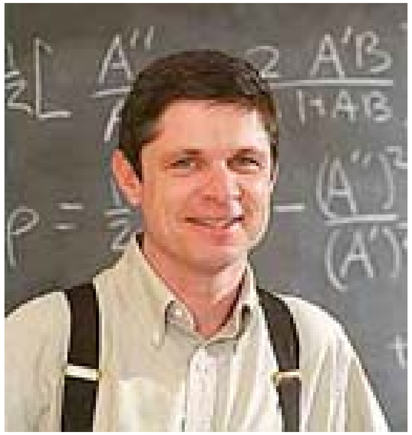
We can require the following conditions:

$$\mathbf{A_k}: \qquad \partial\bar{\partial}(\omega^k) = 0 \qquad \text{for } k < \dim\ M\ ;$$

$$\mathbf{B_k}: \qquad d(\omega^k) = 0 \qquad \text{for } k < \dim\ M\ .$$

Condition A_{n-1} is the Gauduchon metric condition. It was proved by Gauduchon that any Hermitian metric can be deformed conformally to such a metric. Jost and I studied A_{n-2} where harmonic map arguments work well. There are obstructions to the existence of A_{n-2} metrics.

Condition B_{n-1} is the balanced metric condition and was studied by Michelsohn. Its existence is invariant under birational change. Balanced metrics have become more popular because of the consideration of supersymmetry in string theory.



Andrew Strominger

In 1986, Strominger made a proposal for supersymmetric compactification in the theory of the heterotic string.

It requires the manifold to admit a non-vanishing holomorphic n-form Ω and the Hermitian form ω is required to satisfy

$$d(\|\Omega\|_\omega \, \omega^{n-1}) = 0 \ .$$

The existence of supersymmetry is an important and useful tool. It would be highly desirable to explore the consequence of supersymmetry.

The existence of balanced manifolds with trivial canonical line bundle is not clearly understood. The construction of Calabi-Eckman suggested many such examples that are torus bundles over Calabi-Yau manifolds. These appeared in physics in the works of Dasgupta-Rajesh-Sethi, and Becker-Dasgupta. On such manifolds, Fu and I were able to construct solutions for the heterotic compactifaction proposed by Strominger.

In order to understand the conjecture of Reid and to study tunneling between Calabi-Yau manifolds of different topological type, Fu-Li-Yau proved that when we smooth out the blown down of rational curves in a Calabi-Yau manifold, the resulting manifold also admits a balanced metric.

The concept of a balanced manifold with a trivial canonical line bundle can be considered as a natural generalization of the Calabi-Yau manifold. We shall consider the case of $n = 3$.

The Hermitian metric ω can be required to satisfy the following equations.

1. The above equation $\quad d(\|\Omega\|_\omega\, \omega^2) = 0$.

2. The tangent bundle admits a Hermitian-Yang-Mills connection with respect to the polarization $\|\Omega\|_\omega\, \omega^2$ so that the curvature F satisfies the condition

$$F \wedge \omega^2 = 0 \ ,$$

$$F^{2,0} = F^{0,2} = 0 \ .$$

3. $$\sqrt{-1}\partial\bar{\partial}\omega = \mathrm{tr}(R \wedge R) - \mathrm{tr}(F \wedge F)$$

This is the Strominger system of equations that is standardly solved by taking the Yang-Mills bundle to be the tangent bundle. In general, we can take a holomorphic bundle different from the tangent bundle in the above equations with the requirement that the second Chern form of the bundle is equal to the second Chern form of the manifold within the $\partial\bar{\partial}$ cohomology.

It is important to find out whether the above equations determine ω uniquely or not (as long as we fix the cohomology class of $\|\Omega\|_\omega \, \omega^2$).

Would the existence of ω follow from some stability condition of the manifold, besides the obvious requirement that the holomorphic bundle has to be stable with respect to $\|\Omega\|_\omega \, \omega^2$?

One may try to use the fixed point argument for this problem.

The Strominger system is more general than what we just described. It consists of a coupled system of holomorphic bundles with Hermitian metrics. Hence, uniqueness is more complicated.

It would be useful to find interesting quantities that are invariant under deformation of complex structures in the Strominger system. Such invariants would be very useful for studying Calabi-Yau manifolds.

For complex three manifolds, it should be interesting to consider Hermitian metrics ω such that $\sqrt{-1}\partial\bar{\partial}\omega$ equals to some combination of Chern forms and delta function forms supported on cycles of algebraic curves.

VI. Kähler metrics

Kähler metric is of course one of the most beautiful metric in geometry. The fact that it is parametrized by the Kähler cone in $H^{1,1}(M, \mathbb{R})$ plus a function enables us to write many difficult geometric systems in terms of some scalar (non-linear) elliptic equation. This is precisely the reason why the Einstein metric in Kähler geometry is much easier to handle than in general Riemannian geometry.

A fundamental question is to solve my conjecture that a Kähler manifold admits a Kähler metric with a constant scalar curvature in the Kähler class if the manifold is stable with respect to this class. The precise definition of stability should come from geometric invariant theory and presumably is given by Donaldson's stability. Donaldson has verified this statement for toric surfaces.

Another fundamental question is to understand the structure of the Kähler cone. There is an important theorem of Demailly on the cohomological characterization of the Kähler class. However, it is important to understand the following question on the boundary of the Kähler cone:

When will the class on the boundary of the Kähler cone support a non-negative smooth $(1,1)$-form ω?

Naturally the degeneracy of ω will be important and will be reflected by studying ω^k for $k > 0$.

The study of such forms are important for understanding the boundary of the moduli space of Kähler metrics with constant scalar curvature. In the case when the manifold is Calabi-Yau, the Kähler cone should be mirrored to the Teichmüller space of complex structures on the mirror manifold, perhaps with quantum corrections. Hence, it is rather interesting to understand the Kähler cone.

There is a natural geometric flow given by Calabi which may deform Kähler metrics to those with constant scalar curvature. Unfortunately, it is an equation of higher order. Not much is known about them.

Richard Hamilton

In 1981, when Hamilton started his research on Ricci flow, I suggested my two students Bando and Cao to study the Kähler analogue.

Bando studied the flow for three dimensional Kähler manifolds with non-negative bisectional curvature. He proved that the Ricci flow preserves the curvature condition. His result was generalized by Mok-Zhong to higher dimension.

I suggested Cao to prove the Frenkel conjecture based on Ricci flow. Despite some beautiful work of Perelman, this task has not been achieved. Much more work and new ideas are needed in this subject.

VII. Birational geometry

As mentioned earlier, Voisin gave a counterexample to the conjecture of Kodaira that every Kähler manifold can be deformed into a projective algebraic manifold. Nevertheless, it is still a nice problem to find the conditions so that a Kähler manifold can be be deformable to a projective one.

It is interesting to note that for manifolds of dimension ≥ 3, birational geometry is much richer than that for surfaces.

Chen-Yu Chi and I have initiated a program to study problems in birational geometry. This approach will be more geometric than other more algebraic approaches. Most of the arguments can be phrased in a purely algebraic manner, however, it is quite likely that some of them can be applied to deal with the geometry over different fields.

Given a projective variety M, we have studied the geometric information that can be provided by the pluricanonical space $H^0(M, mK_M)$. The minimal model program led by Mori, Kawamata, Kollár and others had achieved great success.

Shigefumi Mori

While earlier workers had solved the problem for three-folds completely, the spectacular finite generation question was recently solved by several people, using different approaches: the analytic approach of Siu and the algebraic approach of Birkar, Cascini, Hacon and McKernan.

Yum-Tong Siu

In our approach, instead of using the full canonical ring, we have focused our study on the pluricanonical space for a fixed m.

Ideally, we would like to determine the birational type of our algebraic variety based on the information on this space only. Any birational transformations of algebraic manifolds will induce a linear map between the corresponding pluricanonical spaces (for each fixed m).

The pluigenera are of course invariant under the birational transformations. But more importantly, there are other finer invariants that are preserved by these transformations.

The most important one are the natural norm-like functions (called "norms") induced by integrating over M the m-th root of the product of a m-pluricanonical form and its conjugate. The norm defines an interesting geometry which has not been explored extensively before. We have initiated a program to study this geometry.

The first major questions we addressed is the following Torelli type theorem:

Given two algebraic varieties M and M', suppose there is a linear map that defines an isometry (with respect to the norm mentioned above) between the two normed vector spaces $H^0(M, mK_M)$ and $H^0(M', mK_{M'})$. We claim that with a few exceptional cases of M and M', the linear isometry is induced by a birational map between M and M'. This can be considered as a Torelli type theorem for birational geometry.

We call this kind of theorem a Torelli type theorem because the classical Torelli theorem says that the periods of integrals determine an algebraic curve, except for hyper-elliptic curves. This remarkable theorem was generalized to higher dimensional algebraic varieties.

Ilya Piatetski-Shapiro Igor Shafarevich

The most notable one was the work of Piatetski-Shapiro and Shafarevich for algebraic K3 surfaces, which was generalized to Kähler K3's by Burns-Rappaport, where they proved the injectivity of period maps.

The surjectivity of period maps for K3 surfaces was shown using Ricci flat metrics by Siu and Todorov following the work of Kulikov and of Perrson and Pinkham. This phenomenon of surjectivity is known to be rather generic, and in many cases the period map can be proved to have degree one for hypersurfaces (as in Donagi's work).