Planar Embeddings with a globally attracting fixed point

Begoña Alarcón Cotillas, Víctor Guíñez and Carlos Gutierrez

Abstract

We study the appearing of a globally attracting fixed point for planar embeddings using techniques of topological dynamic. The motivation of our work is the Discrete Markus-Yamabe Question.

1 Introduction

We study the appearing of a globally attracting fixed point for planar embeddings. Our work is motivated by the following.

DMY Question (Discrete Markus-Yamabe Question) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1-map such that $f(0) = 0$ and $\mathrm{Spec}(f) \subset B(0,1)$. Is 0 a global attractor for the discrete dynamical system generated by f?.

To state our results, we shall need the following definitions:

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-map. Denote by $Spec(f)$ the set of eigenvalues of Df_p , for all $p \in \mathbb{R}^n$.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-map. We say that γ is an $f-$invariant curve from 0 to ∞ with no self-intersections if γ is a smooth embedded curve starting at 0 and going to infinity such that $f(\gamma) \subseteq \gamma$. It is assumed that $0 \in \gamma$.

Let D be a topological disk of $\mathbb{R}^2$ and $f \in C^0(\mathbb{R}^2)$. We say that ∞ is a $D-$repellor of f if:

(1) $f(D) \subseteq Int(D)$

(2) $\forall\, p \notin D,\ \exists n \in \mathbb{N}$ such that $f^n(p) \in D$

We say that $0 \in \mathbb{R}^2$ is an attractor/local attractor (resp. a repellor/local repellor) for an embedding $f : \mathbb{R}^2 \to \mathbb{R}^2$, if there exists a compact disc D, contained in the domain of definition of f (resp. contained in the domain of definition of f^{-1}), which is a neighborhood of 0 such that $f(D) \subset Int(D)$ (resp. $f^{-1}(D) \subset Int(D)$) and $\cap_{n=1}^{\infty} f^n(D) = \{0\}$ (resp. $\cap_{n=1}^{\infty} f^{-n}(D) = \{0\}$. If besides these conditions we have that, for all $p \in \mathbb{R}^2$, $\omega(p) = \{0\}$, we shall say that 0 is a global attractor for f.

If $f : \mathbb{R}^2 \cup \{\infty\} \to \mathbb{R}^2 \cup \{\infty\}$ is a homeomorphism of the Riemann Sphere, with $f(\infty) = \infty$, we may similarly define when ∞ is either an atractor or a repellor.

There is known that the DMY Question had a positive answer for polynomial diffeomorphisms of $\mathbb{R}^2$ but was false for rational diffeomorphism of $\mathbb{R}^2$. The corresponding example exhibited there does not have ∞ as a repellor. Therefore, we wondered if the DMY Question had a positive answer, for smooth diffeomorphisms of $\mathbb{R}^2$ under the additional assumption

(a) ∞ is a repellor.

We prove that this assumption is not good enough either:

Theorem 1. There exists a smooth diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ having an order four periodic point and such that ∞ is a repellor, $f(0) = 0$ and $\text{Spec}(f) \subset B(0, 1)$.

As every global diffeomorphism f of $\mathbb{R}^2$, having 0 as a global attractor, satisfies

(b) f has an $f-$invariant curve from 0 to ∞ with no self-intersections

we have studied the MYD Question under the additional assumptions (a) and (b) obtaining the following results:

Theorem 2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1-orientation preserving embedding (resp. diffeomorphism) such that $f(0) = 0$ and $\text{Spec}(f) \cap [1, 1 + \epsilon) = \emptyset$, for some $\epsilon > 0$.

If there exists an f-invariant curve γ from 0 to ∞ with no self-intersections, then $\Omega(f) \subset \gamma$ and for all $p \in \mathbb{R}^2$ either $\omega(p) = 0$ or $\omega(p) = \emptyset$.

Theorem 3. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1-orientation preserving embedding (resp. diffeomorphism) such that $f(0) = 0$ and $\text{Spec}(f) \subset B(0, 1)$. If there exists an f-invariant curve γ from 0 to ∞ with no self-intersections, then $\Omega(f) = \{0\}$ and 0 is a local attractor.

Theorem 4. Let D be a topological disk of $\mathbb{R}^2$ containing the origin and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving embedding verifying:

(i) 0 is the only fixed point of f.

(ii) ∞ is a $D-$repellor.

(iii) There exists a $f-$invariant curve γ from 0 to ∞ without self-intersections.

Then, for all $p \in \mathbb{R}^2$, $\omega(p) = 0$.

Theorem 5. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1-orientation-preserving embedding such that $f(0) = 0$ verifying:

(i) There exist real numbers $R > 0$ and $0 < \alpha < 1$ such that $||Df_p \cdot p|| < \alpha\, ||p||$, for every $||p|| > R$.

(ii) $Spec(f) \subset B(0, 1)$.

(iii) There exists a $f-$invariant curve γ from 0 to ∞ with no self-intersections.

Then 0 is a global attractor for f.