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## Ergodic theory, combinatorics, diophantine approximation - 2

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# LECTURE 2: DECAY OF MATRIX COEFFICIENTS

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*Preliminary version*

A measure-preserving action of a group $G$ on a probability space $(X, \mu)$ is called mixing if for every $f_1, f_2 \in L^2(X)$, we have

$$\langle \pi_X(g) f_1, f_2 \rangle \to \left( \int_X f_1 \, d\mu \right) \left( \int_X f_2 \, d\mu \right)$$

as $g \to \infty$ ("$g \to \infty$" means that the sequence $g$ has no accumulation points).

**Exercise 0.1.** (1) Consider the measure-preserving transformation $T$ of a probability space $(X, \mu)$. If $L^2(X)$ has non-constant eigen-functions, then the action is not mixing. In particular, an ergodic rotation on the torus is not mixing.
(2) Let $T$ be a hyperbolic matrix in $\mathrm{SL}_2(\mathbb{Z})$. Then the corresponding $\mathbb{Z}$-action on the torus $\mathbb{R}^2/\mathbb{Z}^2$ is mixing. (Hint: consider what happens for characters first.)
(3) Generalise the previous example to automorphisms of $d$-dimensional torus.

The following theorem implies any ergodic action of the group $G = \mathrm{SL}_d(\mathbb{R})$ is mixing.

**Theorem 0.2** (Howe, Moore). *Consider a unitary representation $\pi$ of $G = SL_d(\mathbb{R})$ on a Hilbert space $\mathcal{H}$ such that $\mathcal{H}$ contains no nonzero vectors fixed by $G$. Then for every $v, w \in \mathcal{H}$,*

$$\langle \pi(g) v, w \rangle \to 0 \quad \text{as } g \to \infty.$$

We will use the following notation:

$$K = \mathrm{SO}(d) = \text{ the orthogonal group},$$

$$A = \{\mathrm{diag}(a_1, \ldots, a_d) : \prod_i a_i = 1, a_i > 0\},$$

$$A^+ = \{\mathrm{diag}(a_1, \ldots, a_d) : \prod_i a_i = 1, a_1 \geq \cdots \geq a_d > 0\},$$

$$N = \text{ the group of unipotent upper triangular matrices.}$$

We also set

$$k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad a_t = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

The following decompositions hold:

$$G = NAK \quad \text{(Iwasawa decomposition)},$$
$$G = KA^+K \quad \text{(Cartan decomposition)}.$$

**Exercise 0.3.** Check these decompositions. (Hint: you already know them from a linear algebra course. Review the Gramm-Schmidt orthogonalization and canonical forms of quadratic forms.)

*Proof.* We prove the theorem by contradiction. Suppose that for some sequence $g_n \to \infty$ and $v, w \in \mathcal{H}$, we have $\langle \pi(g_n)v, w\rangle \nrightarrow 0$. We write $g_n = k_n a_n l_n$ in terms of the Cartan decomposition. Then clearly, $a_n \to \infty$, and after passing to a subsequence, we may assume that $l_n v \to v'$ and $k_n^{-1} w \to w'$ for some $v', w' \in \mathcal{H}$. It is easy to check that

$$\langle \pi(g_n)v, w\rangle - \langle \pi(a_n)v', w'\rangle \to 0.$$

Hence, $\langle \pi(a_n)v', w'\rangle \nrightarrow 0$.

Let's consider $G = \mathrm{SL}_2(\mathbb{R})$. Our intermediate aim is to show that *there exists a nonzero vector fixed by* $\{u_s\}$. From the previous paragraph, we have vectors $v, w \in \mathcal{H}$ and $t_n \to +\infty$ such that $\langle \pi(a_{t_n})v, w\rangle \nrightarrow 0$. After passing to a subsequence, we may assume that $\lim \pi(a_{t_n})v = v'$, where the limit is in the weak! topology. Then $\langle \pi(a_{t_n})v, w\rangle \to \langle v', w\rangle$, so $v' \neq 0$. We have

$$\pi(u_s)v' = \lim \pi(u_s a_{t_n})v = \lim \pi(a_{t_n} u_{s/t_n^2})v.$$

Since

$$\|\pi(a_{t_n} u_{s/t_n^2})v - \pi(a_{t_n})v\| \to 0,$$

it follows that

$$\pi(u_s)v' = \lim \pi(a_{t_n})v = v',$$

which proves the claim.

**Exercise 0.4.** Generalize this argument to $\mathrm{SL}_d(\mathbb{R})$. Namely, show that there exists a nonzero vector fixed by the subgroups of the form $\begin{pmatrix} id & * \\ 0 & id \end{pmatrix}$.

Now we fix a nonzero vector $v$ which is invariant under $N = \{u_s\}$, and consider the function $\phi(g) = \langle \pi(g)v, v\rangle$, which $N$-biinvariant. Consider the action of $G$ on $\mathbb{R}^2$. Since $\mathrm{Stab}_G(e_1) = N$ and the action is trivial, we have identification

$$\mathbb{R}^2 - \{0\} = G/N,$$

and $\phi$ can be consider as an $N$-invariant function on $\mathbb{R}^2 - \{0\}$. Note the $N$-orbits in $\mathbb{R}^2$ are the horizontal lines, except the $x$-axis, and single points on the $x$-axis. The function $\phi$ is constant on horizontal lines $y = c$, $c \neq 0$. Hence, by continuity, it is constant on the line $y = 0$. We have

$$\langle \pi(a_t)v, v\rangle = \phi(t, 0) = \phi(0, 0) = \|v\|^2.$$

By the equality case, of the Cauchy-Schwart inequality, $\pi(a_t)v = v$. This implies that the function $\phi$ is biinvariant under $AN$. Since $AN$ has a dense orbit in $\mathbb{R}^2$, $\phi$ is constant, i.e., $\langle \pi(g)v, v\rangle = \|v\|^2$. As above, this implies that $v$ is $G$-invariant, which is a contradiction.

**Exercise 0.5.** Complete the proof for $\mathrm{SL}_d(\mathbb{R})$: using Exercise 0.4, deduce that there exists a nonzero vector which is invariant under copies of $\mathrm{SL}_2(\mathbb{R})$ which generate $\mathrm{SL}_d(\mathbb{R})$.

□

Now we observe that decay of matrix coefficient implies a mean ergodic theorem:

**Corollary 0.6.** *Consider an ergodic action of $G = \mathrm{SL}_d(\mathbb{R})$ on a probability space $(X, \mu)$. Let $B_n$ be a sequence of Borel subsets of $G$ such that $0 < m(B_n) < \infty$ and $m(B_n) \to \infty$. Then for every $f \in L^2(X)$,*

$$\frac{1}{m(B_n)} \int_{B_n} f(g^{-1}x)\, dm(g) \to \int_X f\, d\mu \quad \text{as } n \to \infty$$

*in $L^2$-norm.*

*Proof.* It suffices to consider a function $f$ with $\int_X f\, d\mu = 0$.

Let $\varepsilon > 0$ and $Q$ be a compact subset of $G$ such that

$$|\langle \pi_X(g)f, f\rangle| < \varepsilon \quad \text{for all } g \notin Q.$$

Then we have

$$\begin{aligned}\left\|\frac{1}{m(B_n)} \int_{B_n} \pi_X(g)f\, dm(g)\right\|^2 &= \frac{1}{m(B_n)^2} \int_{B_n\times B_n} \left\langle \pi_X(g_2^{-1}g_1)f, f\right\rangle \\ &\leq \frac{(m\otimes m)(\{(g_1, g_2) \in B_n \times B_n :\ g_2^{-1}g_1 \in Q\})}{m(B_n)^2}\|f\|^2 + \varepsilon.\end{aligned}$$

Making a change of variable $(g_1, g_2) \mapsto (g_1, g_2^{-1}g_1)$, we deduce that

$$(m\otimes m)(\{(g_1, g_2) \in B_n \times B_n :\ g_2^{-1}g_1 \in Q\}) \leq m(B_n)m(Q).$$

Since $m(B_n) \to \infty$, the corollary follows. □