Entropy approximation and large deviations

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Abstract

Let Ω be a compact metric space, let τ be a representation of $\mathbb{Z}^d$ (resp. $\mathbb{Z}_+^d$) by homeomorphisms (resp. continuous maps) on Ω, let f be an element of the set $C(\Omega)$ of real-valued continuous functions on Ω, let (ν_α) be a net of Borel probability measures on the space $\mathcal{M}(\Omega)$ of Borel probability measures on Ω, and let $t_\alpha \downarrow 0$. We show how well-known results in large deviation theory allow to prove that (ν_α) satisfies the large deviation upper-bounds with powers (t_α) and rate function

$$I^f(\mu) = \begin{cases} P^\tau(f) - \mu(f) - h_\mu^\tau & \text{if } \mu \in \mathcal{M}^\tau(\Omega) \\ \\ +\infty & \text{if } \mu \in \mathcal{M}(\Omega) \backslash \mathcal{M}^\tau(\Omega) \end{cases}$$

(where $P^\tau(\cdot)$, $h_\cdot^\tau$, $\mathcal{M}^\tau(\Omega)$ denote respectively the pressure map, the entropy map, and the set of invariant elements of $\mathcal{M}(\Omega)$) if and only if

$$\overline{\lim}\ t_\alpha \log \int_{\mathcal{M}(\Omega)} e^{\mu(g)/t_\alpha} \nu_\alpha(d\mu) \leq P^\tau(f+g) - P^\tau(f) \quad \text{for all } g \in C(\Omega). \tag{1}$$

Assume that there is a vector space $V \subset C(\Omega)$ such that any invariant measure can be approximated weakly and in entropy by a net of measures, each one being the unique equilibrium state for some function in V. Then the large deviation principle holds if and only if the upper limit in (1) is a limit and the equality holds. In fact, we get a strong form of the large deviation principle in the sense that for each convex open G containing some invariant measure, $\lim t_\alpha \log \nu_\alpha(G)$ exists and satisfies

$$\lim t_\alpha \log \nu_\alpha(G) = \lim t_\alpha \log \nu_\alpha(\overline{G}) = -\inf_{\mu \in \overline{G}} I^f(\mu) = -\inf_{\mu \in G \cap \bigcup_{k \in V} \mathcal{M}_k^\tau(\Omega)} I^f(\mu), \tag{2}$$

where $\mathcal{M}_k^\tau(\Omega)$ denotes the set of equilibrium states for k. Moreover, f is not required to have a unique equilibrium state, as it is the case with usual techniques. The above entropy approximation property is well-known for the iteration of hyperbolic rational maps; it also has been proved recently for the multidimensional full shift. In both cases, we obtain (2) for various kinds of nets (ν_α). In the case of rational maps, this allows us to strengthen known results where only upper bounds had been proved. In the case of the multidimensional full shift, the classical results concerning Gibbs fields are improved in various ways.

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