The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific and
Cultural Organization

IAEA
International Atomic Energy Agency

1950-4

School and Workshop on Dynamical Systems

30 June - 18 July, 2008

Ergodic theory, combinatorics, diophantine approximation - 3

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LECTURE 3: RATES OF DECAY OF MATRIX COEFFICIENTS AND KAZHDAN PROPERTY

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Preliminary version

In this lecture, we prove a quantitative version of Theorem **??**. A vector v is called K-finite if the span of Kv is finite. Then we set $d(v) = \dim \langle Kv \rangle$.

Theorem 0.1 (Howe, Tan, Oh)**.** *Let $G = \mathrm{SL}_d(\mathbb{R})$, $d \geq 3$, and π be a unitary representation of G on a Hilbert space $\mathcal{H}$ such that $\mathcal{H}$ contains no nonzero invariant vectors. Then for every K-finite vectors $v, w \in \mathcal{H}$,*

$$| \langle \pi(a)v, w \rangle | \leq c(\varepsilon) d(v)^{1/2} d(w)^{1/2} \|v\| \|w\| \left(\max_{i<j} \frac{a_i}{a_j} \right)^{-1/4+\varepsilon}$$

for every $a \in A^+$ and $\varepsilon > 0$.

Remark 0.2. With a little more work, the rate can be improved by a factor of two, but the aim of this notes is to give a proof with minimal technology.

We note that an analogue of this theorem fails for $\mathrm{SL}_2(\mathbb{R})$, and there is no uniform general rate in this case. This is related to the fact that $\mathrm{SL}_2(\mathbb{R})$ does not have Kazhdan property (see Corollary 0.12 below). Nonetheless, uniform rates have been established for some special families of representations of $\mathrm{SL}_2(\mathbb{R})$ of number-theoretic significance. This is related to the Selberg conjecture and property τ, which we don't have time to discuss in these lectures.

The proof of Theorem 0.1 uses copies of subgroups $\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ embedded in $\mathrm{SL}_d(\mathbb{R})$. In fact, the crucial step is the following

Proposition 0.3. *Let π be a unitary representation of the group $G = \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ on a Hilbert space $\mathcal{H}$. Assume that $\mathcal{H}$ contains no nonzero vectors fixed by $\mathbb{R}^2$. Then for v, w in a dense subset of $\mathcal{H}$ consisting of* $\mathrm{SO}(2)$*-eigenvectors, we have*

$$| \langle \pi(a_t)v, w \rangle | \leq c(v, w)\, t^{-1} \quad \text{for every } t > 1.$$

where $a_t = \mathrm{diag}(t, t^{-1})$.

Proof. We identify the space $\hat{\mathbb{R}^2}$ of unitary characters of $\mathbb{R}^2$ with $\mathbb{R}^2$ by setting $\chi_{u,v}(x,y) = e^{i(vx-uy)}$. The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\hat{\mathbb{R}^2}$ by $(g\cdot\chi)(v) = \chi(g^{-1}v)$, and under such identification, this is the standard action on $\mathbb{R}^2$ (note that $\mathrm{SL}_2(\mathbb{R})$ preserves the symplectic form $vx-uy$). Note also that these actions agree with the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}^2$ by conjugations as a subgroup of G.

Although the group of unitary operators $\pi(\mathbb{R}^2)$ may not have a basis consisting of eigenvectors as in the finite-dimensional case, there is a natural substitute — a projection-valued measure P_B, B is a Borel subset of $\mathbb{R}^2$, such that

$$\pi(r) = \int_{\mathbb{R}^2} \chi_z(r) dP_z, \quad r \in \mathbb{R}^2.$$

This equality means that for $v, w \in \mathcal{H}$,

$$\langle \pi(r)v, w\rangle = \int_{\mathbb{R}^2} \chi_z(r)\, d\,\langle P_z v, w\rangle\,.$$

We have equivarience relation:

$$\pi(g)P_B\pi(g)^{-1} = P_{gB} \quad \text{for Borel } B \subset \mathbb{R}^2. \tag{0.1}$$

For $s > 1$, we consider

$$\Omega_s = \{x \in \mathbb{R}^2 :\, s^{-1} \le \|x\| \le s\}.$$

Note that it follows from (0.1) that P_{Ω_s} commutes if $\pi(k_\theta)$. Since $\mathcal{H}$ has no vectors fixed by $\mathbb{R}^2$, $P_{\{(0,0)\}} = 0$, and by continuity of the measure, $P_{\Omega_s}v \to v$ as $s \to \infty$ for every $v \in \mathbb{H}$. Hence, it suffices to prove the claim for SO(2)-eigenvectors in $\cup_{s>1}\mathrm{Im}(P_{\Omega_s})$.

For $v, w \in \mathrm{Im}(P_{\Omega_s})$ and $a = \mathrm{diag}(t, t^{-1})$, we have

$$\begin{aligned}
\langle \pi(a)v, w\rangle &= \langle \pi(a)P_{\Omega_s}v, P_{\Omega_s}w\rangle = \langle P_{a\Omega_s}\pi(a)v, P_{\Omega_s}w\rangle \\
&= \langle \pi(a)v, P_{a\Omega_s}P_{\Omega_s}w\rangle = \langle P_{\Omega_s}\pi(a)v, P_{a\Omega_s\cap\Omega_s}w\rangle \\
&= \langle \pi(a)P_{a^{-1}\Omega_s}v, P_{a\Omega_s\cap\Omega_s}w\rangle = \langle \pi(a)P_{a^{-1}\Omega_s}P_{\Omega_s}v, P_{a\Omega_s\cap\Omega_s}w\rangle \\
&= \langle \pi(a)P_{a^{-1}\Omega_s\cap\Omega_s}v, P_{a\Omega_s\cap\Omega_s}w\rangle\,.
\end{aligned}$$

Hence, by the Cauchy-Schwartz inequality,

$$|\,\langle \pi(a)v, w\rangle\,| \le \|P_{a^{-1}\Omega_s\cap\Omega_s}v\|\|P_{a\Omega_s\cap\Omega_s}w\|. \tag{0.2}$$

Note that the sets $a_t^{-1}\Omega_s \cap \Omega_s$ are contained in the strip around the x-axis of size s/t, so we expect that the norm $\|P_{a^{-1}\Omega_s\cap\Omega_s}v\|$ decay as $t \to \infty$. We prove that this is the case when v is an eigenfunction of $\{k_\theta\}$. Let $\pi(k_\theta)v = e^{2\pi in\theta}v$. We decompose $\mathbb{R}^2$ into disjoint sectors S_1, $\dots$, S_m of equal size $\theta = 2\pi/m$. Then

$$\pi(k_\theta)P_{S_i}v = P_{k_\theta S_i}\pi(k_\theta)v = e^{2\pi in\theta}P_{S_{i+1}}v.$$

This shows that the vectors $P_{S_1}v, \dots, P_{S_m}v$ have the same norms. Since we have orthogonal decomposition

$$v = \sum_{i=1}^{m} P_{S_i}v.$$

It follows that $\|P_{S_i}v\| = \frac{1}{m} = \frac{\theta}{2\pi}$. This implies that

$$\|P_{a^{-1}\Omega_s\cap\Omega_s}v\| \le \frac{2\sin^{-1}(s^2/t)}{2\pi},$$

and a similar estimate holds for $\|P_{a\Omega_s\cap\Omega_s}w\|$. Now the proposition follows from (0.2). □

The main draw-back of Proposition 0.3 is that the bound is not explicit in terms of v and w. This problem is rectified in Proposition 0.10 below. The proof will be carried out in several steps. With the help of Proposition 0.3, we show that the tensor square of $\pi|_{\mathrm{SL}_2(\mathbb{R})}$ embeds in a sum of the regular representations, and for the regular representation, we establish explicit estimate in terms of the Harish-Chandra function.

Exercise 0.4. Prove the following formulas for the Haar measure on $G = \mathrm{SL}_2(\mathbb{R})$:

$$\int_G f(g)\, dm(g) = \int_{\mathbb{R}\times(0,\infty)\times[0,2\pi)} f(u_s a_t k_\theta)t^{-2}d\theta\frac{dt}{t}ds,$$
$$\int_G f(g)\, dm(g) = \int_{[0,2\pi)\times\times[1,\infty)\times[0,2\pi)} f(k_{\theta_1}a_t k_{\theta_2})(t^2-1/t^2)d\theta_1\frac{dt}{t}d\theta_2.$$

Exercise 0.5. Prove the formula for the Haar measure on $G = \mathrm{SL}_d(\mathbb{R})$:

$$\int_G f(g)\, dm(g) = \int_{N\times A\times K} f(nak)\Delta(a)dndadk$$

where Δ is the modular function on NA, and dn, da, dk denote Haar measures on corresponding subgroups.

A unitary representation π of a group G is called L^p here if for vectors v, w in a dense subspace, one has $\langle\pi(g)v, w\rangle \in L^p(G)$.

Exercise 0.6. Prove that under the assumptions of Proposition 0.3, the representation $\pi|_{\mathrm{SL}_2(\mathbb{R})}$ is L^p for $p > 2$.

We will need the following characterization of the L^2-representations:

Theorem 0.7 (Godement)**.** *A representation π of a group G is L^2 iff it can be embedded as a subrepresentation of the sum $\oplus_{n=1}^{\infty}\lambda_G$ where λ_G is the regular representation.*

Proof. (sketch) Suppose that there exists a dense subset $\mathcal{H}_0$ of $\mathcal{H}$ such that $\langle v, \pi(g)w\rangle \in L^2(G)$. Without loss of generality, we may assume that $\mathcal{H}_0$ is countable. Consider the map $T : \mathcal{H}_0 \to \oplus_{\mathcal{H}_0} L^2(G)$ defined by

$$Tw = \oplus_{v\in\mathcal{H}_0} \langle v, \pi(g)w\rangle\,, \quad w \in \mathcal{H}_0.$$

It is clear that it is injective, and satisfies the equivarience relation:

$$T \circ \pi(g) = (\oplus_{\mathcal{H}_0}\lambda(g)) \circ T, \quad g \in G.$$

To finish the proof, we need to show that the map T extends to the whole space $\mathcal{H}$... □

A remarkable property of semisimple groups is that the matrix coefficients of K-invariant vectors in the regular representation $L^2(G)$ can be bounded by an explicit decaying function — the *Harish-Chandra function* Ξ, which we now introduce. Recall that for $G = \mathrm{SL}_d(\mathbb{R})$, we have the Iwasawa decomposition $G = NAK$. For $g \in G$, we denote by $\mathbf{a}(g)$ its A-component. Let Δ be the modular function of the upper triangular group NA. The *Harish-Chandra function* is defined by

$$\Xi(g) = \int_K \Delta(\mathbf{a}(kg))^{-1/2}\, dk.$$

Note that Ξ is K-biinvariant.

Exercise 0.8. Prove that the Harish-Chandra function for $\mathrm{SL}_2(\mathbb{R})$ is given by

$$\Xi(a_t) = \frac{1}{2\pi}\int_0^{2\pi} (t^{-2}\cos^2\theta + t^2\sin^2\theta)^{-1/2} d\theta,$$

and for every $\varepsilon > 0$,

$$\Xi(a_t) \le c(\varepsilon) t^{-1+\varepsilon}, \quad t > 1.$$

We note that the asymptotic behaviour of the Harish-Chandra function is well-understood. It is known that $\Xi \in L^{2+\varepsilon}(G)$ for every $\varepsilon > 0$, and there are very sharp pointwise bounds on decay of Ξ.

Theorem 0.9. *Let $G = \mathrm{SL}_d(\mathbb{R})$ and $\phi, \psi \in L^2(G)$ be K-invariant functions. Then*

$$|\langle\lambda(g)\phi, \psi\rangle| \le \|\phi\|_2\|\psi\|_2\, \Xi(g), \quad g \in G.$$

Proof. The proof is based on the Herz's "principe de majoration". Clearly, we only need to prove the estimate for $g = a \in A$. Using

Exercise 0.5 and Fubini theorem, we obtain

$$\begin{aligned}\langle \lambda(g)\phi, \psi\rangle &= \int_G \phi(ga)\psi(g)dm(g)\\ &= \int_{N\times A\times K} \phi(nbka)\psi(nb)\Delta(b)dndbdk\\ &\le \int_K \left(\int_{N\times A} \phi^2(nbka)\Delta(b)dndb\right)^{1/2} \left(\int_{N\times A} \psi^2(nbk)\Delta(b)dndb\right)^{1/2} dk,\end{aligned}$$

where the last estimate is the Cauchy-Schwarz inequality in $L^2(NA)$. Since ψ is K-invariant,

$$\int_{N\times A} \psi^2(nbk)\Delta(b)dndb = \int_{N\times A\times K} \psi^2(nbk)\Delta(b)dndbdk = \|\psi\|_2^2.$$

In terms of the Iwasawa decomposition, $ka = \mathbf{n}(ka)\mathbf{a}(ka)\mathbf{k}(ka)$, and

$$nbka = nb\mathbf{n}(ka)\mathbf{a}(ka)\mathbf{k}(ka) = (nb\mathbf{n}(ka)b^{-1})(b\mathbf{a}(ka))\mathbf{k}(ka).$$

Then using invariance of the measures, we get

$$\begin{aligned}\int_{N\times A} \phi^2(nbka)\Delta(b)dndb &= \int_{N\times A} \phi^2((nb\mathbf{n}(ka)b^{-1})(b\mathbf{a}(ka)))\Delta(b)dndb\\ &= \Delta(\mathbf{a}(ka))^{-1}\int_{N\times A} \phi^2(nb)\Delta(b)dndb\\ &= \Delta(\mathbf{a}(ka))^{-1}\|\phi\|^2.\end{aligned}$$

This implies the theorem. □

Now we can prove a more explicit form of Proposition 0.3 for SO(2)-finite vectors:

Proposition 0.10. *Let notation be as in Proposition ??. Then for every $v, w \in \mathcal{H}$ which are* SO(2)*-finite,*

$$|\langle \pi(a_t)v, w\rangle| \le c(\varepsilon)d(v)^{1/2}d(w)^{1/2}\|v\|\|w\|\, t^{-\frac{1}{2}+\varepsilon}, \quad t > 1, \tag{0.3}$$

for every $\epsilon > 0$.

Proof. First, we reduce the proof to the case when v and w are SO(2)-eigenfunctions. Every SO(2)-finite vector v can be written as $v = \sum_{i=1}^n v_i$ where v_i's are orthogonal SO(2)-eigenfunctions. Then by the Cauchy–Schwartz inequality,

$$\sum_{i=1}^n \|v_i\| = d(v)^{1/2}\left(\sum_{i=1}^n \|v_i\|^2\right)^{1/2} = d(v)^{1/2}\|v\|.$$

Hence, estimate (0.3) follows from the estimate for SO(2)-eigenfunctions by linearity.

It follows from the estimate in Proposition 0.3 and Exercise **??** that the matrix coefficients $\langle \pi(g)v, w\rangle$ belong to $L^4(\mathrm{SL}_2(\mathbb{R}))$ for v, w in a dense subspace $\mathcal{H}_0$ of $\mathcal{H}$.

Let $\rho = \pi|_{\mathrm{SL}_2(\mathbb{R})}$. Consider the tensor square $\rho \otimes \rho$ of the representation ρ. For $v_1, v_2, w_1, w_2 \in \mathcal{H}_0$, we have

$$\langle (\rho \otimes \rho)(g)(v_1 \otimes v_2), (w_1 \otimes w_2)\rangle = \langle \rho(g)v_1, w_1\rangle \langle \rho(g)v_2, w_2\rangle ,$$

and it follows from the Cauchy-Schwarz inequality that this expression is in $L^2(G)$. Since linear combinations of vectors $v_1 \otimes v_2$ with $v_1, v_2 \in \mathcal{H}_0$ form a dense subspace of $\mathcal{H} \otimes \mathcal{H}$, we conclude that $\rho \otimes \rho$ is an L^2 representation. Hence, by Theorem 0.7, $\rho \otimes \rho$ is a subrepresentation of a direct sum of regular representation. Applying Theorem 0.9 (it is easy to check that it extends to direct sums), we get that for every SO(2)-invariant vectors $v, w \in \mathcal{H}$,

$$|\langle \rho(g)v, w\rangle|^2 \le \|v\|^2 \|w\|^2 \Xi(g).$$

Now the proposition follows from Exercise 0.8. $\square$

Proof of Theorem 0.1. We use various embedded copies of $\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ embedded in G. For simplicity, we carry out the computation for the subgroup

$$\begin{pmatrix} \mathrm{SL}_2(\mathbb{R}) & \mathbb{R}^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Note that $\mathcal{H}$ contains no nonzero $\mathbb{R}^2$-invariant vectors because of Theorem **??**. Hence, Proposition 0.10 applies. For $a = \mathrm{diag}(a_1, \dots, a_d) \in A^+$, we write $a = a'a''$ where

$$a' = \mathrm{diag}\left(\left(\frac{a_1}{a_2}\right)^{1/2}, \left(\frac{a_2}{a_1}\right)^{1/2}, 1, \dots, 1\right),$$
$$a'' = \mathrm{diag}((a_1a_2)^{1/2}, (a_1a_2)^{1/2}, a_3, \dots, a_d).$$

Note that a'' commutes with $\mathrm{SL}_2(\mathbb{R})$. In particular, $\pi(a'')v$ is SO(2)-finite, and

$$\dim \langle \mathrm{SO}(2)\pi(a'')v\rangle = \dim \langle \mathrm{SO}(2)v\rangle \le d(v).$$

It is also clear that $\dim \langle \mathrm{SO}(2)w\rangle \le d(w)$. By Proposition 0.10,

$$|\langle \pi(a)v, w\rangle| = |\langle \pi(a')\pi(a'')v, w\rangle| \le c(\varepsilon)d(v)^{1/2}d(w)^{1/2}\|v\|\|w\| \left(\frac{a_1}{a_2}\right)^{-1/4+\varepsilon}$$

for every $\varepsilon > 0$. Using similar estimate for other copies of $\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$, we finally deduce that

$$|\langle \pi(a)v, w\rangle| \le c(\varepsilon) d(v)^{1/2} d(w)^{1/2} \|v\| \|w\| \left(\max_{i<j} \frac{a_i}{a_j}\right)^{-1/4+\varepsilon}$$

for every $\varepsilon > 0$, as required. □

Corollary 0.11. *There exists $p(d) > 0$ such that every unitary representation of $\mathrm{SL}_d(\mathbb{R})$, $d \ge 3$, without fixed vectors is L^p for every $p > p(d)$.*

Proof. This is just a computation. The main ingredient is the estimate from Theorem 0.1. □

A group G is called *Kazhdan group* if every unitary representation of G which contains almost invariant vectors also contains invariant vectors.

This should be compared with the notion of amenable groups. Note that it follows from Theorem **??** that if a group is both amenable and has Kazhdan property, then it is compact.

Corollary 0.12. *The group $\mathrm{SL}_d(\mathbb{R})$, $d \ge 3$, has Kazhdan property.*

Proof. Let π be a representation of G on a Hilbert space $\mathcal{H}$ which has no invariant vectors, but has a sequence v_n of almost invariant vectors, namely,

$$\|\pi(g)v_n - v_n\| \to 0$$

uniformly on compact sets. Let $w_n = \int_K \pi(k)v_n \, dk$. Then

$$\|w_n - v_n\| \le \int_K \|\pi(k)v_n - v_n\| \, dk \to 0.$$

This implies that the sequence w_n is also almost invariant and $\|w_n\| \to 1$, but w_n's are K-invariant, so we can apply the estimate from Theorem 0.1 to get a contradiction. For $a \in A^+$,

$$\|\pi(a)w_n - w_n\|^2 = 2 - 2\mathrm{Re}\,\langle \pi(a)w_n, w_n\rangle \ge 2 - 2\sigma(a)\|w_n\|_2^2$$

σ is an explicit decaying function. Since $\|\pi(a)w_n - w_n\| \to 0$ uniformly on compact sets, this is impossible. □