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Open questions leading to a global perspective in dynamics

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J Palis

Abstract

We will address one of the most challenging and central problems in dynamical systems, meaning flows, diffeomophisms or, more generally, transformations, defined on a compact manifold without boundary or an interval on the real line: can we describe the behavior in the long run of typical trajectories for typical systems?

Poincaré was probably the first to point in this direction and stress its importance.

We shall consider finite-dimensional C^∞,C^r with $r \geq 1$, parameterized families of dynamics endowed with the C^∞, C^r topology. The concept of "typical" in our context is taken in terms of Lebesgue probability both in parameter and phase spaces.

Our purpose is to discuss a conjecture stating that for a typical dynamical system, almost all trajectories have only finitely many choices, of (transitive) attractors, where to accumulate upon in the future. Interrelated conjectures will also be discussed.

Main global conjecture

- There is a dense set D of dynamics such that any element of D has finitely many attractors whose union of basins of attraction has total probability.
- The attractors of the elements in D support a physical (SRB) measure and are stochastically stable in their basins of attraction.
- For generic finite-dimensional families of dynamics, with total probability in parameter space, the corresponding systems display attractors satisfying the properties above.

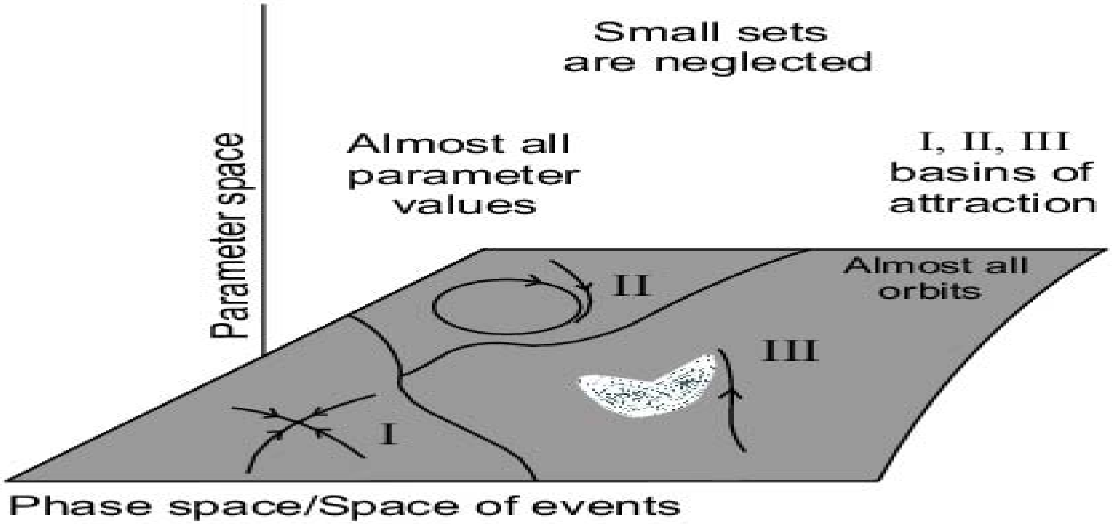


Figure 1: Parameter space × Phase space

Figure 1: Parameter space x Phase space

Remarkable recent positive results: consider the family

$$f_a(x) = ax(1-x), \quad f_a : [0,1] \circlearrowleft, \quad 0 < a \leq 4$$

Lyubich: the conjecture is true for families of quadratic transformations of the interval. He made use of work of Martens-Nowicki, Sullivan and McMullen.

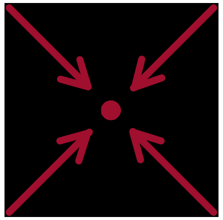
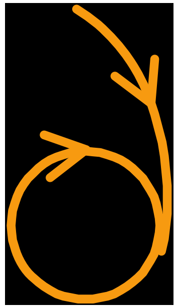
Avila-de Melo-Lyubich, Avila-Moreira: the conjecture is true for C^k families, $k \geq 2$, of unimodal maps.

Multimodal case for maps of the interval??

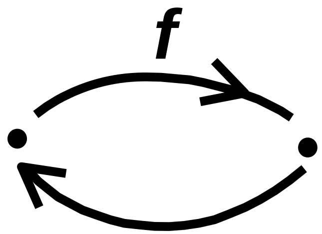
Very hopeful, upon results of Kozlovski-Shen-van Strien on the density of hyperbolicity.

Attractors

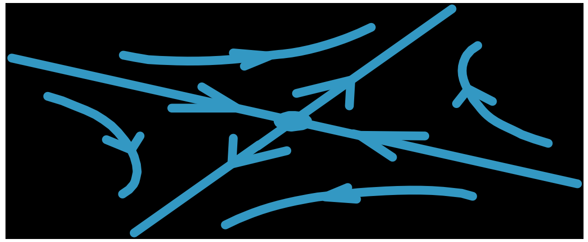
Invariant, transitive, positive limit set of orbits starting at positive measure set (Lebesgue), full neighborhood.

Attractors: point,  

circle

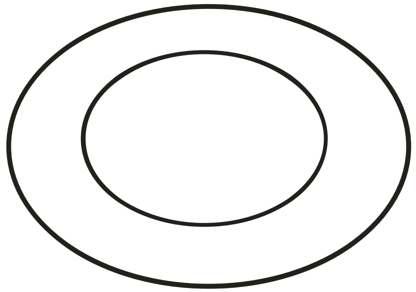
finite set 

Not



whole space of events: $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$: $\quad R^2 \longrightarrow R^2$

torus



Other attractors:

Lorenz´s “butterfly”, 63

Hénon, 75’: expansion and folding
segment x fractal
strange attractor / chaotic

Pioneer work of Kolmogorov in fluid dynamics
Later, in the early 70’, May in the context of population growth.

Persistent attractor. Exists with positive probability (Lebesgue) in parameter space

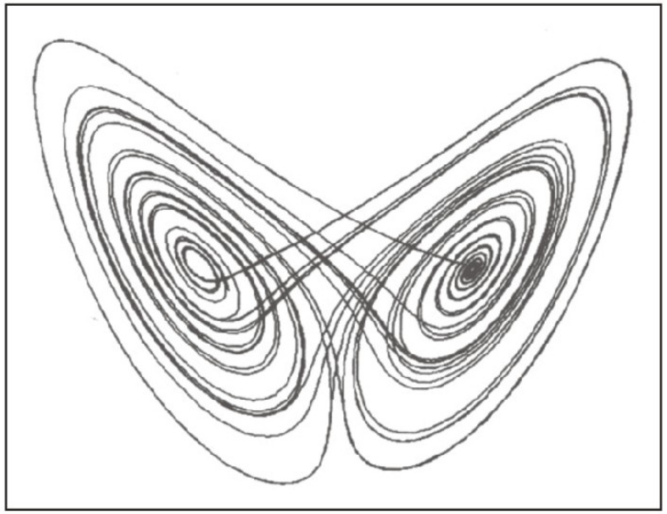
1dim quadratic transformation, Jacobson

2dim quadratic diffeomorphism, Hénon, Benedicks-Carleson, Mora-Viana

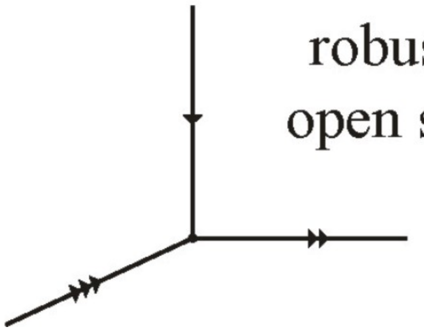
3dim flow, Lorenz-Rovella

Algebraically Simple Quadratic - Tucker (Viana: Math. Intelligencer, Springer, 2000)

Lorenz, 1963

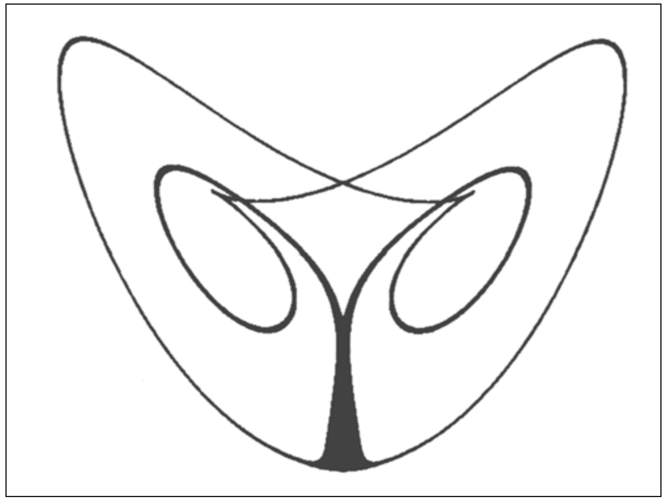


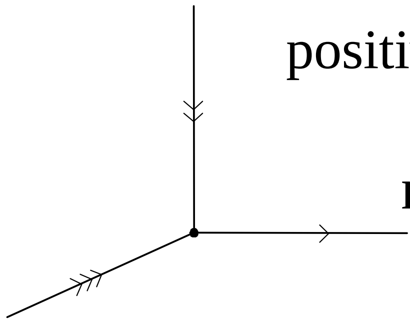
$$\begin{cases} \dot{x} = 10x + 10y \\ \dot{y} = 28x - y - xz \\ \dot{z} = -\frac{8}{3}z + xy \end{cases}$$



robust
open set

Rovella, 1992





positive probability
persistent
not robust

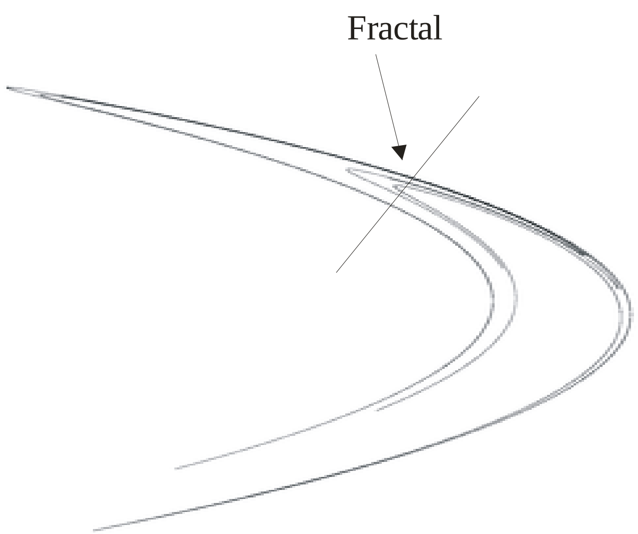
Hénon Transformation

1D : Feigenbaum, Coullet-Tresser period, doubling.

Jacobson 1D, Benedicks-Carleson, Mora-Viana

Diaz-Rocha-Viana: saddle-node cycles.

Fractal

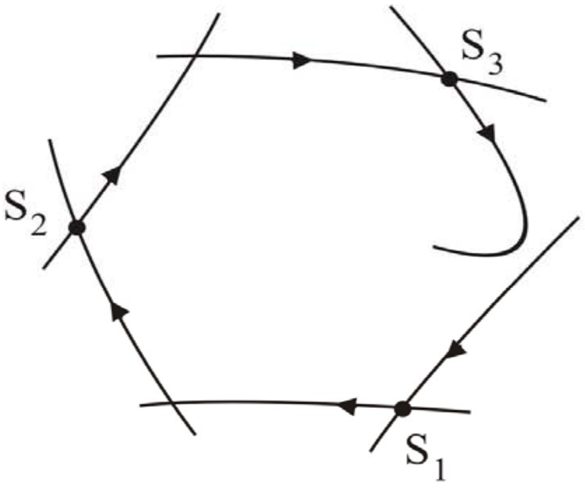


probability

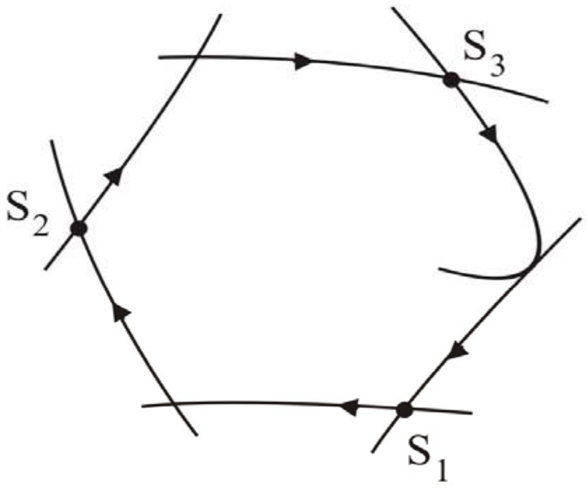
persistent

not robust

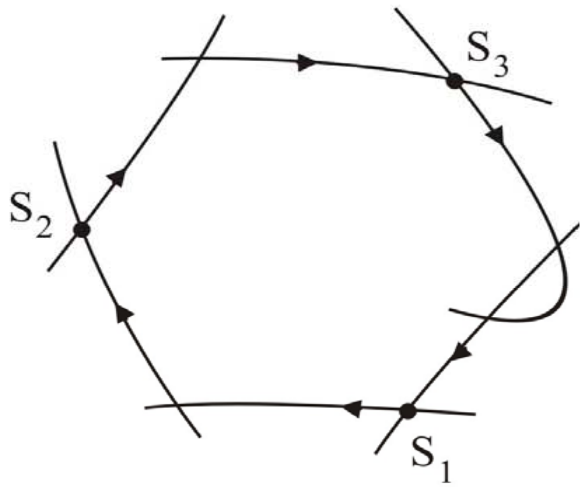
$(x,y) \quad (1-ax^2+y,\ bx)$



No cycle

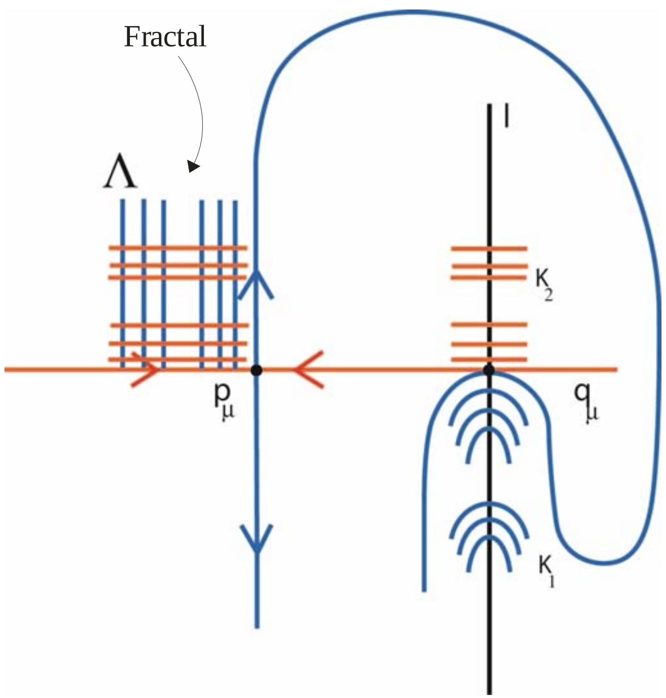


Creating an unstable cycle

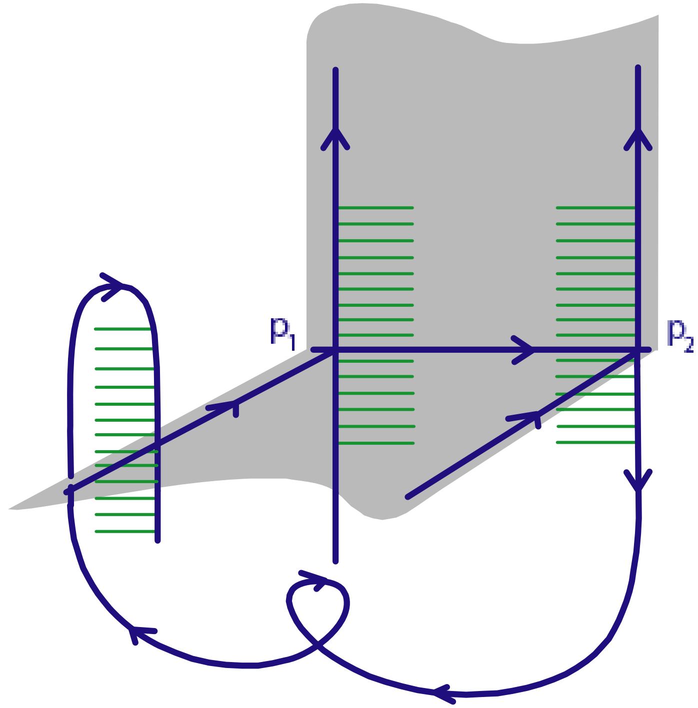


A stable cycle

Cycles



Homoclinic tangency



Heterodimensional

Strategic dichotomy for main conjecture

Three Parts:

- Robust absence of homoclinic tangencies or heteroclinic cycles implies robust presence of some form of hyperbolicity – uniform, partial, dominated decomposition.

Proof of the main conjecture:
- in the presence of some form of hyperbilicity
- near unfolding of homoclinic tangencies or heterodimensional cycles.

Related Conjectures

Conjecture 1: any dynamical system can be C^r approximated by a hyperbolic one or one with homoclinic tangency or heterodim. cycle , $r \geq 1$.

Conjecture 2 (weaker relevant version): C^r approximation by a Morse-Smale one or one with transversal homoclinic tangency.

Pujals-Sambarino solved (1) in dim. 2,Corvisier solved (2) in any dim., r=1 in both cases. Also, good contribution by Wen in any dim. Relevant results are due to Diaz-Rocha and Bonatti-Diaz concerning heterodim. cycles

Theorem: Any diffeo can be C^1 approximated by one that is essentially hyperbolic or it exhibits a homoclinic tangency or heterodim cycle (Crovisier-Pujals). A diffeo is essentially hyp if it has a finite number of hyp attractors such that the union of their basins of attraction is open and dense in the phase space.
The C^1 restriction is due to Pugh´s closing lemma or Hayashi´s connecting lemma. We advocate that these questions for C^r, $r > 1$, may be more tractable in the context of this program Lyubich and Martens are pursuing this worthy line for dissipative Henon family of maps.

Concerning the main conjecture in the context of a weak form of hyperbolicity, we have:

Theorem [Tsujii]: Partially hyperbolic surface endomorphisms of class C^∞ ($C^{r,}$, $r \geq 19$), generically (residually) carry finitely many ergodic SRB measures whose union of basins of attraction has total Lebesgue probability.

This question can be considered in higher dimensions in the context of parameterized families of maps, like in the main global conjecture.

Flows

Conjecture 3. In any dimension, every flow can be C^r approximated by a hyperbolic one or by one displaying a homoclinic tangency, a singular cycle or a heterodimensional cycle.

Conjecture 4. Every three-dimensional flow can be C^r approximated by a hyperbolic one or by one displaying a singular cycle or a Lorenz-like attractor or repeller, $r \geq 1$.

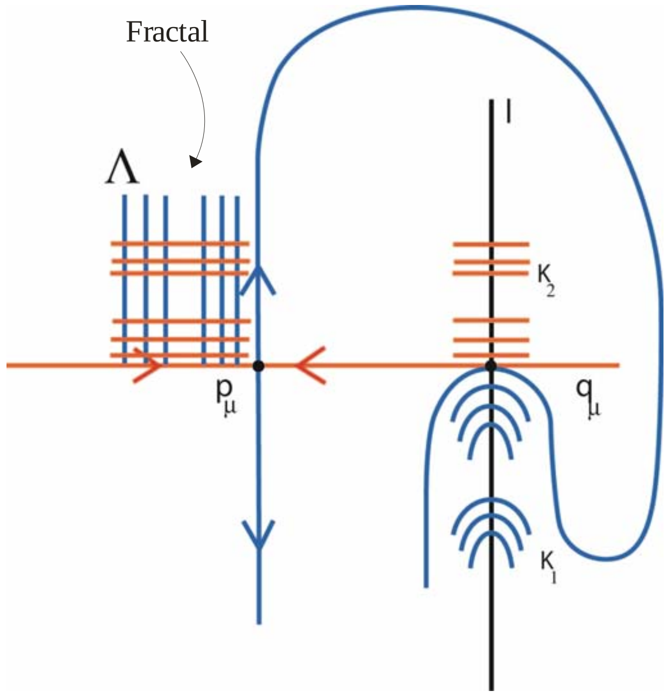
In dimension 3 Arroyo, Rodriguez-Hertz solved (3) and Morales, Pacifico, Pujals showed that a robustly transitive singular set is a Lorenz-like attractor or repeller.

Remark. For dissipative evolution equations like Euler, Navier–Stokes, there are outstanding conjectures concerning the existence of complete solutions and that they often approach a finite-dimensional space in the future.

Such global solutions would then typically approach finitely many stochastically stable finite-dimensional attractors.

Unfolding Homoclinic -Heteroclinic Tangencies

Main Goal: Finitely many attractors for typical parameter in neighborhood in phase space.



Question: Zero density of attractors at initial bifurcating parameter value.

Newhouse: In dim 2, residually in some intervals in parameter line, say I, there are infinitely many sinks.

Palis,Viana: Similar in higher dimensions.

Also, Henon-like attractors:Benedicks-Carleson, Mora-Viana, Colli, Leal.

Conjecture 5. In the unfolding of homoclinic tangencies for generic 1-dim parameter families of C^2 surface diffeos, with total probability in the parameter line,corresponding diffeos do not display infinitely many attractors, in particular sinks, in a small neighbourhood of the closure of the homoclinic orbit.

Some progress by Gorodetski, Kaloshin.

Conjecture 6. In the same setting as (5), we have total prevalence of no-attractors.

In dim 2, Newhouse,Palis,Takens proved total prevalence of hyperbolicity if dim $\Lambda < 1$ and Moreira,Palis,Yoccoz showed this is not true if dim $\Lambda > 1$. Yet if dim $\Lambda > 1$ but not much bigger, there is total prevalence of no-attractors.

In higher dimensions, Moreira,Palis,Viana are showing total prevalence of hyperbolicity if and only if dim $\Lambda < 1$.

References:

J. Palis, 2000 A global view of dynamics and a conjecture on the denseness of finitude of attractors, *Astérisque* 261 335–47

J. Palis, Open Questions Leading to a Global Perspective in Dynamics, *Invited paper, Nonlinearity 21 (2008) T37-T43*

Bonatti, Diaz and Viana, Dynamics Beyond Uniform Hyperbolicity: A Global Geometric and Probabilistic Perspective, Springer, 2004

A N N E X

SRB – invariant probability measures

Let *A* be an attractor for *f* , and μ an *f*-invariant measure on *A* – attracts sets of positive Lebesgue probability.
(*f*, A, μ) is a SRB measure if for any g **continuous**

$$\lim_{n\to\infty} \frac{1}{n}\sum g\left(f^{i}(x)\right) = \int g\,d\mu$$

$$x \in E \subset B(A)$$

, B basin of attraction for $\boldsymbol{A}$

$$m(E) > 0$$

***m* denotes Lebesgue measure**

Stochastic Stability. Let (f, A, μ), $\mu \in$ SRB, finitely many-parameter families of maps. Random Lebesgue choice of parameters gives rise to maps f_j

Let $z_j = f_j \circ \ldots \circ f_1 (z_0)$, $z_0 \in B(A)$

f_j ε - C^r near f, $\varepsilon > 0$

(f, A, μ) is stochastically stable if given a neighborhood V of μ in the weak topology, the weak limit of

$$\frac{1}{n}\sum_{j=0}^{n-1}\delta_{Z_j}$$

is in ***V*** for a.a. $(z_0, f_1, f_2, \ldots)$ if ε is small, where δ stands for Dirac measure.

Hyperbolic Diffeomorphism *f*

hyperbolic limit set *L*

$$T_L M = E^s \oplus E^u$$

$df \,|\, E^s, \; df^{-1} \,|\, E^u$ contractions

Flow $X_t, \; t \in R$

$$\| dX_t \mid E^s \|, \; \| dX_{-t} \mid E^u \| \le C\, e^{\lambda t}, \; t \in R$$

$C, \quad 0 < \lambda < 1$

Stability Conjecture (Palis-Smale)

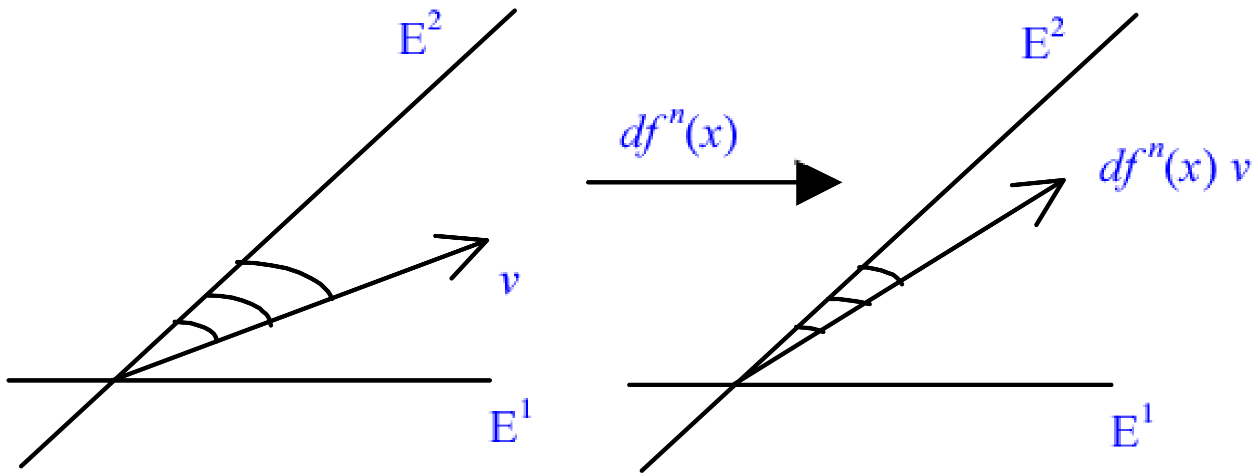
$$\text{hyperbolicity} \overset{C^r}{\longleftrightarrow} \text{structural stability}$$

$\longrightarrow$ Anosov, Palis-Smale, Robbin, de Melo, Robinson

$\longleftarrow$ Mañé, Hayashi - C^1

Other relevant contributions by Liao, Sannami, Pliss, Doering, Hu, Wen

Dominated Decomposition



Partial hyperbolicity

f, Λ invariant $T_\Lambda M$ = sum E^s , E^c , E^u .

E^s uniform contracting, E^u uniform expending,

E^u dominates the sum of E^s , E^c ,

E^s is dominated by the sum of E^c , E^u .