The Abdus Salam International Centre for Theoretical Physics

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School and Workshop on Dynamical Systems

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Ergodic theory, combinatorics, diophantine approximation

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WEEK 2

VITALY BERGELSON AND ALEXANDER GORODNIK

1. Lecture plan

(1) Three principles of Ramsey Theory.
(2) Furstenberg's proof of Szemeredi's theorem. Combinatorial and number-theoretical applications of multiple recurrence.
(3) Combinatorial and dynamical applications of ultrafilters.
(4) Ergodic Ramsey Theory and nilpotent groups.
(5) Open problems and conjectures.

Recommended novice-friendly literature which can be downloaded from

http://www.math.ohio-state.edu/∼vitaly/

(1) Ergodic Ramsey Theory – an update, Ergodic Theory of Z^d-actions (edited by M. Pollicott and K. Schmidt), London Math. Soc. Lecture Note Series 228 (1996), 1-61
(2) Combinatorial and Diophantine Applications of Ergodic Theory (with appendices by A. Leibman and by A. Quas and M. Wierdl), Handbook of Dynamical Systems, vol. 1B, B. Hasselblatt and A. Katok, eds., Elsevier (2005), pp. 745-841
(3) The multifarious Poincare recurrence theorem, Descriptive set theory and dynamical systems (edited by M. Foreman, A. Kechris, A. Louveau, B. Weiss), London Math. Soc. Lecture Note Series 277 (2000), 31-57
(4) Ergodic theory and Diophantine problems, Topics in symbolic dynamics and applications (edited by F. Blanchard, A. Maass and A. Nogueira), London Math. Soc. Lecture Note Series 279 (2000), 167-205
(5) Minimal idempotents and ergodic Ramsey theory, Topics in Dynamics and Ergodic Theory 8-39, London Math. Soc. Lecture Note Series 310, Cambridge Univ. Press, Cambridge, 2003
(6) Multiplicatively large sets and ergodic Ramsey theory, Israel Journal of Mathematics 148 (2005), 23-40

2. Problems

(1) Let A be a matrix in $\mathrm{SL}_d(\mathbb{Z})$ and assume that all of its eigenvalues have absolute value 1. Prove that this implies that these eigenvalues are actually roots of unity.

(2) Let $a > 0$ and let B_n, $n = 1, 2, \ldots$, be measurable subsets in a probability space $(X, \mathcal{B}, \mu)$ satisfying $\mu(B_n) \geq a$. Prove that there exists a subsequence B_{n_k}, $k = 1, 2, \ldots$, such that:
   (a) the intersection of any finite collection of sets from this subsequence has positive measure,
   (b) $\limsup_{N\to\infty} \frac{1}{N} |\{k : n_k \leq N\}| > 0$.

(3) Let $R \subset \mathbb{N}$ be a set of recurrence, which means that for any measure-preserving transformation of a probability space $(X, \mathcal{B}, \mu)$ and $B \in \mathcal{B}$ with $\mu(B) > 0$, there exists $n \in R$ such that $\mu(B \cap T^{-n}B) > 0$. Prove that for any finite partition of $R = R_1 \sqcup \cdots \sqcup R_s$, one of R_i's is a set of recurrence.

(4) (a) Prove that any set of differences in $\mathbb{N}$ (that is, any set of the form $n_i - n_j$, $i > j$, where $\{n_i\}$ is a sequence in $\mathbb{N}$) is a set of recurrence.
   (b) Prove that there exists a set of differences $n_i - n_j$, $i > j$, such that the set of squares along this set, $(n_i - n_j)^2$, do not form a set of recurrence.
   (c) Prove that if a set R contains arbitrarily long progressions of the form $n, 2n, \ldots, kn$, then R is a set of recurrence.
   (d) Prove that 2^n, $n = 1, 2, \ldots$, is not a set of recurrence.

(5) Let α be an irrational number. Prove that the two-parameter sequence $\{mn\alpha : m, n \in \mathbb{N}\}$, is uniformly distributed modulo 1. That is, for any $f \in C(\mathbb{R}/\mathbb{Z})$ and $x \in \mathbb{R}/\mathbb{Z}$,

$$\frac{1}{N^2} \sum_{m,n=1}^{N} f(mn\alpha + x) \to \int_{\mathbb{R}/\mathbb{Z}} f(y)dy.$$

(6) Let T be a measure-preserving transformation of a probability space $(X, \mathcal{B}, \mu)$. Prove that the following properties are equivalent:
   (a) There are no non-trivial eigenfunctions for the corresponding operator $U_T : L^2(X) \to L^2(X)$.
   (b) The action of $T \times T$ on $X \times X$ is ergodic.
   (c) For every $B_0, B_1 \in \mathcal{B}$,

$$\frac{1}{N} \sum_{n=0}^{N-1} |\mu(B_0 \cap T^{-n}(B_1)) - \mu(B_0)\mu(B_1)| \to 0$$

as $N \to \infty$.

(d) For every $B_0, B_1, B_2 \in \mathcal{B}$,

$$\frac{1}{N}\sum_{n=0}^{N-1} \mu(B_0 \cap T^{-n}(B_1) \cap T^{-2n}(B_2)) \to \mu(B_0)\mu(B_1)\mu(B_2)$$

as $N \to \infty$.