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From dynamics to group theory via examples

A. Navas
Universidad de Chile, Santiago, Chile

Strada Costiera 11, 34014 Trieste, Italy - Tel. +39 040 2240 111; Fax +39 040 224 163 - sci_info@ictp.it, www.ictp.it

From dynamics to group theory via examples

Andrés Navas

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Orderable groups

– An order relation $\prec$ on a group G is *left-invariant* if for every $f \prec g$ and h in G one has $hf \prec hg$.

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P^{+} is a *semigroup* : $\quad gh \in P^{+}$ for every g, h in P^{+}

– P^{+} is invariant under congugacy, that is $gP^{+}g^{-1} = P^{+}$ for every $g \in G$, if and only if the corresponding left-ordering is a bi-ordering.

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– Bi-orderable groups have the *unique root property*: for $n \in \mathbb{N}$,

$$f^n = g^n \quad \Longrightarrow \quad f = g$$

Examples

– Torsion free Abelian groups (bi-orderable)

$$\mathbb{Z}^n \subset (\mathbb{R}, +) \text{ with the induced ordering}$$

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– Thompson group F (bi-orderable)

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– Braid groups (Dehornoy, Thurston (Nielsen))

– Fundamental groups of some 3-manifolds
(Boyer,Calegari,Dunfield,Rolfsen,Rourke,Wiest...)

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– The subespace of bi-invariant orderings is closed (perhaps empty).

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Question: Is $HD(\mathcal{LO}(F_n)) > 0$?

The action of G on $\mathcal{LO}(G)$

– G acts on $\mathcal{LO}(G)$ by conjugacy (equivalently, by right multiplication): given an ordering $\preceq$, its image under $f \in G$ is the ordering $\preceq_f$ defined by

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Problem. Find a criterion to ensure that an ordering is accumulated by its conjugates.

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– Every action of G by homeomorphisms of a compact metric space admits an invariant probability measure.

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Exercise: for a μ-generic point $\preceq$, the following property holds:

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(right-recurrent ordering)