The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific and
Cultural Organization

IAEA
International Atomic Energy Agency

1950-9

School and Workshop on Dynamical Systems

30 June - 18 July, 2008

From dynamics to group theory via examples

A. Navas
Universidad de Chile, Santiago, Chile

From dynamics to group theory via examples

Andrés Navas

Univ. of Santiago, Chile

July 2008

A folklore principle

A countable group is orderable if and only if it admits a faithful action by orientation preserving homeomorphisms of the real line.

A folklore principle

A countable group is orderable if and only if it admits a faithful action by orientation preserving homeomorphisms of the real line.

– Given $G \subset \mathrm{Homeo}_+(\mathbb{R})$ we may fix a dense sequence (x_n) of points in the real line and define $f \prec g$ if and only if the first $n \geq 1$ for which $f(x_n) \neq g(x_n)$ is such that $f(x_n) < g(x_n)$ (a "dynamical lexicographic ordering").

A folklore principle

A countable group is orderable if and only if it admits a faithful action by orientation preserving homeomorphisms of the real line.

– Given $G \subset \mathrm{Homeo}_+(\mathbb{R})$ we may fix a dense sequence (x_n) of points in the real line and define $f \prec g$ if and only if the first $n \geq 1$ for which $f(x_n) \neq g(x_n)$ is such that $f(x_n) < g(x_n)$ (a "dynamical lexicographic ordering").

– Given an ordering $\preceq$ on a countable group G, let $t \colon G \to \mathbb{R}$ be any order preserving map (with $t(id) = 0$). Define the action of G on the set $t(G)$ by letting $g(t(h)) = t(gh)$. This action may be extended continuously to the whole line... (dynamical realization).

Many actions on the line do not appear as dynamical realizations

Many actions on the line do not appear as dynamical realizations

Exercise: this is the case of the actions of non-Abelian groups of piecewise affine homeomorphisms of the closed interval.

Many actions on the line do not appear as dynamical realizations

Exercise: this is the case of the actions of non-Abelian groups of piecewise affine homeomorphisms of the closed interval.

In fact, none of the preceding procedures is canonical. However, this is not a "problem". In fact, this may be used to create many new orderings on a given orderable group !

Many actions on the line do not appear as dynamical realizations

Exercise: this is the case of the actions of non-Abelian groups of piecewise affine homeomorphisms of the closed interval.

In fact, none of the preceding procedures is canonical. However, this is not a "problem". In fact, this may be used to create many new orderings on a given orderable group !

Example. Two "generic" homeomorphisms of the real line generate a free group. Thus, F_2 is left-orderable...

Example: $\mathcal{LO}(F_n)$ is a Cantor set

Example: $\mathcal{LO}(F_n)$ is a Cantor set

– Given an ordering $\preceq$ on F_n, let us consider the corresponding dynamical realization.

Example: $\mathcal{LO}(F_n)$ is a Cantor set

– Given an ordering $\preceq$ on F_n, let us consider the corresponding dynamical realization.

– Perturb slightly the homeomorphisms corresponding to the generators of F_n, and induce a new ordering on the group generated by the new homeomorphisms via the “dynamically lexicographical” procedure.

Example: $\mathcal{LO}(F_n)$ is a Cantor set

– Given an ordering $\preceq$ on F_n, let us consider the corresponding dynamical realization.

– Perturb slightly the homeomorphisms corresponding to the generators of F_n, and induce a new ordering on the group generated by the new homeomorphisms via the "dynamically lexicographical" procedure.

– In general, the new group is still free (generically, two homeomorphisms satisfy no non trivial relation).

Example: $\mathcal{LO}(F_n)$ is a Cantor set

– Given an ordering $\preceq$ on F_n, let us consider the corresponding dynamical realization.

– Perturb slightly the homeomorphisms corresponding to the generators of F_n, and induce a new ordering on the group generated by the new homeomorphisms via the "dynamically lexicographical" procedure.

– In general, the new group is still free (generically, two homeomorphisms satisfy no non trivial relation).

– Therefore, the new ordering "lives" on F_n. Clearly, if the topological perturbation was small then the new ordering is very close to the original one.

Example: $\mathcal{LO}(F_n)$ is a Cantor set

– Given an ordering $\preceq$ on F_n, let us consider the corresponding dynamical realization.

– Perturb slightly the homeomorphisms corresponding to the generators of F_n, and induce a new ordering on the group generated by the new homeomorphisms via the "dynamically lexicographical" procedure.

– In general, the new group is still free (generically, two homeomorphisms satisfy no non trivial relation).

– Therefore, the new ordering "lives" on F_n. Clearly, if the topological perturbation was small then the new ordering is very close to the original one.

– On the other hand, the new ordering does not coincide with the original one if the dynamical realization is "non structurally stable" (which holds for free group actions).

Archimedean orders and free actions

Definition. An ordering $\preceq$ on a group G is *Archimedean* if for every $f \succ id$ and g in G there exists $n \in \mathbb{N}$ such that $f^n \succ g$.

Theorem (Hölder). Every group endowed with an Archimedean ordering is order isomorphic to a subgroup of $(\mathbb{R}, +)$.

Archimedean orders and free actions

Definition. An ordering $\preceq$ on a group G is *Archimedean* if for every $f \succ id$ and g in G there exists $n \in \mathbb{N}$ such that $f^n \succ g$.

Theorem (Hölder). Every group endowed with an Archimedean ordering is order isomorphic to a subgroup of $(\mathbb{R}, +)$.

– Dynamical realizations of Archimedean orderings are free actions (*i.e.*, no non-trivial element has fixed points).

Archimedean orders and free actions

Definition. An ordering $\preceq$ on a group G is *Archimedean* if for every $f \succ id$ and g in G there exists $n \in \mathbb{N}$ such that $f^n \succ g$.

Theorem (Hölder). Every group endowed with an Archimedean ordering is order isomorphic to a subgroup of $(\mathbb{R}, +)$.

– Dynamical realizations of Archimedean orderings are free actions (*i.e.*, no non-trivial element has fixed points).

Theorem (Hölder). Every free action by homeomorphisms of the real line is topologically semiconjugate to an action by translations.

Bi-invariant orderings and essentially free actions

– The action of a group G of orientation preserving homeomorphisms of the real line is *essentially free* if for each $f \neq id$ one has either

$$f(x) \geq x \text{ for every } x \in \mathbb{R}$$

or

$$f(x) \leq x \text{ for every } x \in \mathbb{R}.$$

Bi-invariant orderings and essentially free actions

– The action of a group G of orientation preserving homeomorphisms of the real line is *essentially free* if for each $f \neq id$ one has either

$$f(x) \geq x \text{ for every } x \in \mathbb{R}$$

or

$$f(x) \leq x \text{ for every } x \in \mathbb{R}.$$

– The action of the dynamical realization of a bi-invariant ordering is essentially free.

– Every subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ whose action is essentially free is bi-orderable.

Conradian orderings and 1-dimensional horseshoes

The Conrad property

– An ordering $\preceq$ on a group G satisfy the Conrad property if for every $f \succ id$ and $g \succ id$ there exists $n \in \mathbb{N}$ such that $fg^n \succ g$.

Conradian orderings and 1-dimensional horseshoes

The Conrad property

– An ordering $\preceq$ on a group G satisfy the Conrad property if for every $f \succ id$ and $g \succ id$ there exists $n \in \mathbb{N}$ such that $fg^n \succ g$.

Exercise. For a Conradian ordering $\preceq$ one has $fg^2 \succ g$ for all positive elements f, g.

Conradian orderings and 1-dimensional horseshoes

The Conrad property

– An ordering $\preceq$ on a group G satisfy the Conrad property if for every $f \succ id$ and $g \succ id$ there exists $n \in \mathbb{N}$ such that $fg^n \succ g$.

Exercise. For a Conradian ordering $\preceq$ one has $fg^2 \succ g$ for all positive elements f, g.

– Dynamical realizations of Conradian orderings have no 1-dimensional horseshoes. Dynamical lexicographic orderings induced from actions without 1-dimensional horseshoes are Conradian.

Regular invariant measures

– For a non-trivial finitely generated group of homeomorphisms of the real line either there exists a discrete orbit which is unbounded (from both sides), or it is semiconjugate to a group of translations.

Regular invariant measures

– For a non-trivial finitely generated group of homeomorphisms of the real line either there exists a discrete orbit which is unbounded (from both sides), or it is semiconjugate to a group of translations.

– In particular, such a group preserves a measure on the line which is finite on compact sets.

Group homomorphisms for Conrad-orderable groups

Theorem (Conrad). Every finitely generated group with a Conradian ordering has a non-trivial homomorphism into $(\mathbb{R}, +)$.

Group homomorphisms for Conrad-orderable groups

Theorem (Conrad). Every finitely generated group with a Conradian ordering has a non-trivial homomorphism into $(\mathbb{R}, +)$.

– One can also characterize the Conrad orderability for a group in this way... (Brodskii, Rhemtulla-Rolfsen, N).