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From dynamics to group theory via examples

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General principle

Theorem (Thurston): The group $\mathcal{G}er^1_+(\mathbb{R}, 0)$ is locally indicable.

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If a group is too complicated and acts (smoothly) on a compact manifold, then the dynamics of the action should be relatively simple (perhaps trivial).

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Question. Can an irrational rotation be distorted inside some finitely generated group of C^2 circle diffeomorphisms ?

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Theorem (N). Every finitely generated group of $C^{3/2+}$ circle diffeomorphisms satisfying Kazhdan's property (T) is finite.

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– Kazhdan groups do not admit non-trivial homomorphisms into $(\mathbb{R}, +)$.

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– Generic random groups with density bigger than $1/3$.