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From dynamics to group theory via examples

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Γ: finite system of generators

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$s(n, \varepsilon)$: maximal cardinality for an (n, ε)- separated set.

$$h_{top}(G, \varepsilon) = \limsup_{n \to \infty} \frac{\log(s(n, \varepsilon))}{n}.$$

$$h_{top}(G) = \lim_{\varepsilon \to 0} h_{top}(G, \varepsilon).$$

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Question. $h_{top}(G) = h_{top}(G|_{\Omega G})$?

Stationnary measure

μ: probability measure on G; $\quad \Sigma = (G^{\mathbb{N}}, \mu^{\mathbb{N}})$.

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Theorem. Stationary measures always exist.

An application

Theorem. Every homeomorphism of a compact manifold is topologically conjugate to a homeomorphism which is absolutely continuous with respect to the Lebesgue measure.

Groups of circle homeomorphisms

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Theorem (Antonov, Deroin-Kleptsyn-N). The stationary measure (with respect to μ) is unique.

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Question. If the action is minimal and smooth enough, is it possible to choose a probability μ on G so that the corresponding stationnary measure is absolutely continuous with respect to the Lebesgue measure ?

Theorem (Guivarch-Le Jan, Deroin-Kleptsyn-N). For every probability measure on $\mathrm{PSL}(2, \mathbb{Z})$ with finite support, the corresponding stationary measure on the circle is singular.

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Corollary. For ν-almost every point $x \in \mathrm{S}^1$, one has

$$\lambda_{max}(x) > 0.$$