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From dynamics to group theory via examples

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List of Exercises

From dynamics to group theory via examples

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I. On left-orders on groups: algebraic aspects

(1) Given a left-orderable group G (of arbitrary cardinality), consider the topology on the set $\mathcal{LO}(G)$ having as a sub-basis the sets of the form $U_f = \{\preceq \text{ in } \mathcal{LO}(G) \colon f \succ id\}$, where $f \neq id$. Prove that, endowed with this topology, $\mathcal{LO}(G)$ is compact.

Hint. Use Tychonov Theorem. (See [MW] or [N2].)

(2) Prove that a group G is left-orderable if and only if for every finite family of elements $g_1, \ldots, g_n$ different from the identity, there exist $\varepsilon_i \in \{-1, +1\}$ so that id does not belong to the semigroup generated by $g_1^{\varepsilon_1}, \ldots, g_k^{\varepsilon_k}$.

Hint. Use the compactness of $\mathcal{LO}(G)$.

(3) In analogy to (2), show that G is bi-orderable if and only if for every finite family of elements $g_1, \ldots, g_n$ different from the identity, there exist $\varepsilon_i \in \{-1, +1\}$ so that id does not belong to the smallest semigroup satisfying the following properties:

– it contains the elements $g_1^{\varepsilon_1}, \ldots, g_k^{\varepsilon_k}$,

– for every g, h in the semigroup, the elements hgh^{-1} and $h^{-1}gh$ also belong to it.

(4) Recall that a left-ordering $\preceq$ on a group G is Conradian if for every positive f, g there exists $n \in \mathbb{N}$ such that $fg^n \succ g$. Show that for every Conradian ordering actually one has $fg^2 \succ g$ for all positive elements f, g. Conclude that the subespace of Conradian orders is closed (perhaps empty) in $\mathcal{LO}(G)$.

Hint. Suppose that $fg^2 \preceq g$ for two positive elements f, g, and show that $fh^n \preceq h$ for every $n \in \mathbb{N}$, where $h = fg$.

(5) Using (4), and in analogy to (3), show that a group G is Conrad orderable if and only if for every finite family of elements $g_1, \ldots, g_n$ different from the identity, there exist $\varepsilon_i \in \{-1, +1\}$ so that id does not belong to the smallest semigroup satisfying the following properties:

– it contains the elements $g_1^{\varepsilon_1}, \ldots, g_k^{\varepsilon_k}$,

– for every g, h in the semigroup, the element $h^{-1}gh^2$ also belong to it.

(6) Prove that if a group is locally indicable (that is, if every non-trivial finitely generated subgroup admits a non-trivial homomorphism into $(\mathbb{R}, +)$), then G is Conrad orderable. (Recall that the converse also holds, as it was proved in Lecture 2.)

Hint. Use the local indicability to check the Conrad orderability condition in (5). (See [N2].)

(7) Recall that an ordering is *Archimedean* if for every $f \succ id$ and g in G there exists $n \in \mathbb{N}$ such that $f^n \succ g$. Show that every left-invariant Archimedean ordering is bi-invariant. (See [Co].)

(8) Let G be a group having a bi-invariant Archimedean order $\preceq$, and let us fix a positive element $f \in G$. For each $g \in G$ and each $p \in \mathbb{N}$ consider the unique integer $q = q(p)$ such that $f^q \preceq g^p \prec f^{q+1}$. Prove that the sequence $q(p)/p$ converges to a real number as p goes to infinite.

Hint. Show that $q(p_1) + q(p_2) \leq q(p_1 + p_2) \leq q(p_1) + q(p_2) + 1$ and apply a well-known lemma on (almost) subaditive sequences (see [Ma] for this lemma).

(9) Prove that the map $\phi : G \to (\mathbb{R}, +)$ is a group homomorphism.

Hint. Let g_1, g_2 be arbitrary elements in G such that $g_1 g_2 \preceq g_2 g_1$ (the case where $g_2 g_1 \preceq g_1 g_2$ is analogous). Show that, if $f^{q_1} \preceq g_1^p \prec f^{q_1+1}$ and $f^{q_2} \preceq g_2^p \prec f^{q_2+1}$, then

$$f^{q_1+q_2} \preceq g_1^p g_2^p \preceq (g_1 g_2)^p \preceq g_2^p g_1^p \prec f^{q_1+q_2+2},$$

and conclude that

$$\phi(g_1) + \phi(g_2) = \lim_{p\to\infty} \frac{q_1 + q_2}{p} \leq \phi(g_1 g_2) \leq \lim_{p\to\infty} \frac{q_1 + q_2 + 1}{p} = \phi(g_1) + \phi(g_2).$$

(10) Prove that the homomorphism ϕ is one-to-one.

Hint. Show first that if $g_1 \preceq g_2$ then $\phi(g_1) \leq \phi(g_2)$. Now let h be a positive element in G such that $\phi(h) = 0$. Take $n \in \mathbb{N}$ so that $h^n \succeq f$ and conclude that $0 = n\phi(h) = \phi(h^n) \geq \phi(f) = 1$, which is absurd.

II. On left orders on groups: a dynamical aspect (Hölder Theorem)

(1) Show that if G is a group acting freely by homeomorphisms of the real line, then G admits a bi-invariant Archimedean ordering.

Hint. Consider the order relation $\preceq$ on G such that $g \prec h$ if $g(x) < h(x)$ for some (equivalently, for all) $x \in \mathbb{R}$.

(2) Let G be a non-trivial group acting freely by homeomorphisms of the real line. Endow G with the order in (1), and consider the corresponding embedding ϕ from G into $(\mathbb{R}, +)$ constructed in I. Show that if $\phi(G)$ is isomorphic to $(\mathbb{Z}, +)$, then the action of G is conjugate to the action of the latter group by translations. In the other case the group $\phi(G)$ is dense in $(\mathbb{R}, +)$, an for each point x in the line define

$$\varphi(x) = \sup\{\phi(h) \in \mathbb{R} : h(0) \leq x\}.$$

Show that φ: $\mathbb{R} \to \mathbb{R}$ is a continuous map which satisfies the equality $\varphi(h(x)) = \varphi(x) + \phi(h)$ for all $x \in \mathbb{R}$ and all $h \in G$.

Hint. For the continuity, notice that in the opposite case the set $\mathbb{R} \setminus \varphi(\mathbb{R})$ would be a non empty open set invariant by the translations of $\phi(G)$, which is impossible.

(3) Conclude that if G is a group acting freely on the line, then its action is semiconjugate to an action by translations. Prove that an analogous statement holds for free actions on the circle. More precisely, prove that if G is a group acting freely by circle homeomorphisms, then it is Abelian and semiconjugate to a subgroup of $\mathrm{SO}(2, \mathbb{R})$.

(4) Prove that if G is a finitely generated subgroup of $\mathrm{Homeo}_+(\mathrm{S}^1)$ all of whose elements are torsion, then G is finite.

(5) Show that if G is a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ all of whose elements are elliptic, then G is conjugate to a group of rotations.

III. Thurston's stability theorem

Let us consider a finitely generated subgroup G of $\mathrm{Diff}_+^1([0,1])$. Suppose that for all the elements in G, the derivative at the origin equals 1. (If this is not the case, then the map $g \to \log(g'(0))$ provides a non-trivial homomorphism from G into $(\mathbb{R},+)$). Let $\Gamma = \{h_1, \ldots, h_k\}$ be a finite family of generators for G. For each $f \in G$ let us define the function Δ_f by $\Delta_f(x) = f(x) - x$. Notice that $(\Delta_f)'(0) = 0$ for every $f \in G$.

(1) Prove that for every $x \geq 0$ and every f, g in G, there exist y, z (near the origin if x is near 0) such that

$$\Delta_{fg}(x) = \Delta_f(x) + \Delta_g(x) + (\Delta_f)'(y)\ \Delta_g(x),$$

$$\Delta_{f^{-1}(x)} = -\Delta_f(x) - (\Delta_f)'(z)\ \Delta_{f^{-1}}(x).$$

(2) Fixing an strictly decreasing sequence of points x_n converging to the origin and which are not fixed by G, for each $n \in \mathbb{N}$ let us choose $i_n \in \{1, \ldots, k\}$ so that $|\Delta_{h_{i_n}}(x_n)| \geq |\Delta_{h_j}(x_n)|$ for every $j \in \{1, \ldots, k\}$. Passing to a subsequence if necessary, we may assume that i_n is constant (say, equal to 1 after reordering the indexes), and that each one of the k sequences $(\Delta_{h_i}(x_n)/\Delta_{h_1}(x_n))$ converges to a limit ϕ_i (less than or equal to 1) as n goes to the infinite. From the equalities in (1) show that the map $h_i \longmapsto \phi_i$ extends into a normalized homomorphism from G into $(\mathbb{R}, +)$.

(3) Extend the above proof to groups of germs of C^1 diffeomorphisms of the line fixing the origin.

IV. On Kazhdan group actions on S^1

(1) If g is a $C^{1+\alpha}$ circle diffeomorphism g prove that, for every $x \neq y$ on S^1,

$$\left| \frac{\sqrt{g'(x)g'(y)}}{g(x) - g(y)} - \frac{1}{x - y} \right| \leq C|x - y|^{\alpha - 1}.$$

Conclude that, for $\alpha > 1/2$, the function

$$(x, y) \mapsto \left| \frac{\sqrt{g'(x)g'(y)}}{g(x) - g(y)} - \frac{1}{x - y} \right|$$

belongs to $\mathcal{L}^2(\mathrm{S}^1 \times \mathrm{S}^1)$. (See [N1].)

(2) Let ν be a non-atomic measure on $\mathrm{S}^1 \times \mathrm{S}^1$ satisfying:

– $\nu([a, b] \times [c, d]) < \infty$ for $a < b < c < d < a$,

– $\nu(]a, b[\times]b, c[) = \infty$ for $a < b < c < a$.

Prove that, if a circle homeomorphism g fixes 3 points and $g \times g$ preserves ν, then g is the identity.

Hint. Suppose that $g \neq Id$, and let x be any point contained in some connected component $]a, b[$ of the complement of the set of its fixed points. Let $c \in]b, a[$ be another fixed point of g. Changing g by its inverse if necessary, one may assume that $a < x < g(x) < b$. Show that $\nu([x, g(x)] \times]c, a[) = 0$. By iterating g, conclude that $\nu(]a, b[\times]c, a[) = 0$, thus contradicting the hypothesis. (See [N1].)

V. Entropy and stationary measure

(1) Given $0 < a < b < 1$, consider a non-trivial homeomorphism $h \colon [a, b] \to [a, b]$. Let $f \colon [0, 1] \to [0, 1]$ be a homeomorphism such that $f(x) < x$ for every $x \in]0, 1[$ and $f(a) = b$. Define $g \colon [0, 1] \to [0, 1]$ by letting $g(x) = h^{2^n}(x)$ if $x \in f^n([a, b])$, and $g(x) = x$ otherwise. Prove that $h_{top}(\langle f, g \rangle) \geq 2$, where the system of generators considered is $\Gamma = \{f, g\}$. (See [GLW].)

(2) Given Lipschitz homeomorphisms $f_1, \ldots, f_k$ of a compact manifold, M denote by $L(f_1, \ldots, f_k)$ the minimum value among the Lipschitz constants of $f_1, \ldots, f_k$ and their inverses. Prove that

$$h_{top}(\langle f_1, \ldots, f_k \rangle) \leq dim(M)\ L(f_1, \ldots, f_k).$$

By slightly adapting the argument, conclude that the maps f, g in (1) cannot be simultaneously conjugate to maps having Lipschitz constant smaller than 2. (Compare [DKN, Th. D].)

References

[Co] Conrad, P. Right-ordered groups. *Michigan Math. Journal* (1959).

[DKN] Deroin, B., Kleptsyn, V., & Navas, A. Sur la dynamique unidimensionnelle en régularité intermédiaire. *Acta Math.* (2007).

[GLW] Ghys, E., Langevin, R. & Walzac, P. Entropie geométrique des feuilletages. *Acta Math.* (1988).

[Ma] Mañé, R. *Introdução à Teoria Ergodica.* Projeto Euclides (1983).

[MW] Morris-Witte, D. Amenable groups that act on the line. *Alg. Geom. and Top.* (2007).

[N1] Navas, A. Actions de groupes de Kazhdan sur le cercle. *Annales de l'ENS* (2002).

[N2] Navas, A. On the dynamics of left-orderable groups. Preprint on arxiv (2007).