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Symmetries of turbulent state (Turbulence and Cascades)

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Abstract. The emphasis of this short course is on fundamental properties of developed turbulence, weak and strong. We shall be focused on the degree of universality and symmetries of the turbulent state. We shall see, in particular, which symmetries remain broken even when the symmetry-breaking factor goes to zero, and which symmetries, on the contrary, emerge in the state of developed turbulence. In particular, we describe how Schramm-Loewner Evolution (SLE) appears in turbulent inverse cascades.

1. Introduction

We define turbulence as a state of a physical system with many interacting degrees of freedom deviated far from equilibrium. This state is irregular both in time and in space and is accompanied by dissipation.

Turbulence is a state of a continuous medium (or a system with many degrees of freedom) deviated far from thermal equilibrium. That state is accompanied by dissipation and needs an external pumping to sustain it. Developed turbulence corresponds to the case when the scales of externally excited and effectively dissipated motions are vastly different. For example, a moving car leaves behind meter-size vortices while viscous friction is only effective for eddies smaller than a millimeter. Instabilities of large vortices, their breakdown and fragmentation bring energy from input to dissipation scales by a cascade: Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls, and so on to viscosity (Richardson, 1922).

Cascade must be a natural state of any nonlinear system where input and output are far away as long as the interaction is effectively local. Locality here means that effective energy exchange between different modes goes to zero with the ratio of their scales. Apart from energy, other quantities conserved by interaction can cascade too. For example, during ore pulverization (when colliding stones are broken) mass cascades towards smaller sizes, while in water droplet coagulation (say, in clouds) mass cascades towards larger sizes. The cascade towards small scales is usually called direct while that towards large scales is called inverse. If a system has more than one conservation law in the absence of input and dissipation, then input at some scale can generate both direct and inverse cascades simultaneously, as happens for two-dimensional vortex turbulence or wave turbulence on water surface. The interval of scales between input and output is called inertial interval (or transparency window).

Developed turbulence contains many excited degrees of freedom and requires a statistical description. Since most cases of turbulence involve many strongly interacting degrees of freedom, they can be neither described theoretically nor satisfactory modelled on computer. Therefore, a general symmetry aspects of turbulence statistics are of prime importance. Pumping and dissipation usually break symmetries (isotropy, scale invariance and time reversibility) and one asks if the symmetries are restored in the inertial interval (so that some of the information on pumping is forgotten). Scale invariance (or scaling) is particularly important to predict the properties of the scales unresolved by modelling. One calls probability density function of the velocity difference v measured at the distance r scale invariant if it is actually a function of a single variable (rather than two): $P(v,r) = f(v/r^a)/v$. One can use the cascade idea to guess the scaling properties of turbulence. For incompressible fluid, the energy flux (per unit mass) ϵ through the given scale r can be estimated via the velocity difference δv measured at that scale as the energy $(\delta v)^2$ divided by the time $r/\delta v$. That gives $(\delta v)^3 \sim \epsilon r$. Of course, δv is a fluctuating quantity and we ought to make statements on its moments or probability distribution $\mathcal{P}(\delta v, r)$. Energy flux constancy fixes the third moment, $\langle(\delta v)^3\rangle \sim \epsilon r$. It is a natural wish to have turbulence scale invariant in the inertial interval so that $\mathcal{P}(\delta v, r) = (\delta v)^{-1} f[\delta v/(\epsilon r)^{1/3}]$ is expressed via the dimensionless function f of a single variable. Initially, Kolmogorov made even stronger wish for the function f to be universal (i.e. pumping independent). Nature is under no obligation to grant wishes of even great scientists, particularly when it is in a state of turbulence. After hearing Kolmogorov talk, Landau remarked that the moments different from third are nonlinear functions of the input rate and must be sensitive to the precise statistics of the pumping. As we show below, the cascade idea can indeed be turned into an exact relation for the simultaneous correlation function which expresses the flux (third or fourth-order moment depending on the degree of nonlinearity). The relation requires the mean flux of the respective integral of motion to be constant across the inertial interval of scales. Is it enough to know just the flux i.e. the input rate of energy (or other quantity) in a statistical steady state? The answer on scale invariance and universality is "definitely no for direct cascades" and "probably yes for inverse cascades", as discussed in more details below.

2. Burgers turbulence

Consider arguably the simplest hydrodynamic system. In the reference moving with the sound velocity, weakly compressible 1d flows ($u \ll c$) are described by the Burgers equation (Landau and Lifshits 1987, E et al 1997, Frisch and Bec 2001):

$$u_t + uu_x - \nu u_{xx} = 0 \; . \tag{1}$$

This equation can be also written for the potential h defined by $u = \nabla h$, then it can be considered in multi-dimensional versions as well when it describes surface growth, directed polymer etc. Such form

$$h_t + (\nabla h)^2/2 - \nu\Delta h = \xi \tag{2}$$

is called Kardar-Parisi-Zhang (KPZ) equation when the driving forces is white both in time and in space: $\langle\xi(\mathbf{x},t)\xi(0,0)\rangle = T\delta(t)\delta(\mathbf{x})$. Such a small-scale driven state has a simple Gaussian equilibrium (Gibbs) single-point probability distribution: $P(h) \sim \exp(-U/T)$ where the energy $U = \int h_x^2 dx/2$ is conserved by the nonlinear term in 1d. Indeed, the Fokker-Planck equation for the pdf,

$$\partial P(h,t) = \int dx \frac{\delta}{\delta h(x)}\left[T\frac{\delta P}{\delta h(x)} - P\left(\frac{\delta U}{\delta h(x)} + \frac{h_x^2}{2}\right)\right] \tag{3}$$

has such a solution since $\int dx h_x^2 \delta U/\delta h = \int dx h_x^2 h_{xx} = 0$ and $\int dx \delta h_x^2/\delta h(x) = 0$. Gaussian statistics is completely determined by the second moment which behaves in a diffusive way:

$$\langle[h(x) - h(0)]^2\rangle = \int_{-\infty}^{\infty}[\exp(iqx) - 1]\frac{T}{q^2}\frac{dq}{2\pi} = T|x| \ . \tag{4}$$

Consider now the Burgers turbulence driven by a large-scale force or appearing from a large-scale initial distribution (so that the Reynolds number is large). Note that (1) has a propagating shock-wave solution $u = 2v\{1 + \exp[v(x - vt)/\nu]\}^{-1}$ with the energy dissipation rate $\nu\int u_x^2\, dx$ independent of ν. The shock width ν/v is a dissipative scale and we consider acoustic turbulence produced by a pumping correlated on much larger scales (for example, pumping a pipe from one end by frequencies much less than cv/ν). After some time, it will develop shocks at random positions. Here we consider the single-time statistics of the Galilean invariant velocity difference $\delta u(x,t) = u(x,t) - u(0,t)$. The moments of δu are called structure functions $S_n(x,t) = \langle[u(x,t) - u(0,t)]^n\rangle$. Quadratic nonlinearity relates the time derivative of the second moment to the third one:

$$\frac{\partial S_2}{\partial t} = -\frac{\partial S_3}{3\partial x} - 4\epsilon + \nu\frac{\partial^2 S_2}{\partial x^2} \ . \tag{5}$$

Here $\epsilon = \nu\langle u_x^2\rangle$ is the mean energy dissipation rate. Equation (5) describes both a free decay (then ϵ depends on t) and the case of a permanently acting pumping which generates turbulence statistically steady at scales less than the pumping length. In the first case, $\partial S_2/\partial t \simeq S_2 u/L \ll \epsilon \simeq u^3/L$ (where L is a typical distance between shocks) while in the second case $\partial S_2/\partial t = 0$ so that $S_3 = 12\epsilon x + \nu\partial S_2/\partial x$.

Consider now limit $\nu \to 0$ at fixed x (and t for decaying turbulence). Shock dissipation provides for a finite limit of ϵ at $\nu \to 0$ then

$$S_3 = -12\epsilon x \ . \tag{6}$$

It is thus the flux constancy that fixes $S_3(x)$ which is universal that is determined solely by ϵ and depends neither on the initial statistics for decay nor on the pumping for steady turbulence. On the contrary, other structure functions $S_n(x)$ are not given by $(\epsilon x)^{n/3}$. Indeed, the scaling of the structure functions can be readily understood for any dilute set of shocks (that is when shocks do not cluster in space) which seems to be the case both for smooth initial conditions and large-scale pumping in Burgers turbulence. In this case, $S_n(x) \sim C_n|x|^n + C'_n|x|$ where the first term comes from the regular (smooth) parts of the velocity (the right x-interval in Fig. 1) while the second comes from $O(x)$ probability to have a shock in the interval x. The scaling exponents, $\xi_n = d\ln S_n/d\ln x$, thus behave as follows: $\xi_n = n$ for $n \le 1$ and $\xi_n = 1$ for $n > 1$. That means that the probability density function (PDF) of the velocity

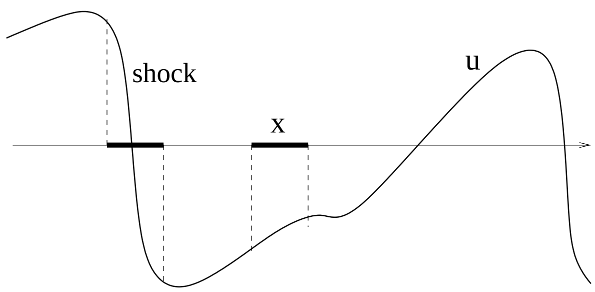

Figure 1. Typical velocity profile in Burgers turbulence.

difference in the inertial interval $P(\delta u, x)$ is not scale-invariant, that is the function of the re-scaled velocity difference $\delta u/x^a$ cannot be made scale-independent for any a. Simple bi-modal nature of Burgers turbulence (shocks and smooth parts) means that the PDF is actually determined by two (nonuniversal) functions, each depending of a single argument: $P(\delta u, x) = \delta u^{-1} f_1(\delta u/x) + x f_2(\delta u/u_{rms})$. Breakdown of scale invariance means that the low-order moments decrease faster than the high-order ones as one goes to smaller scales, i.e. the smaller the scale the more probable are large fluctuations and the statistics is getting more and more non-Gaussian. In other words, the probability of strong fluctuations increases with the resolution. When the scaling exponents ξ_n do not lie on a straight line, this is called an anomalous scaling since it is related again to the symmetry (scale invariance) of the PDF broken by pumping and not restored even when $x/L \to 0$. When the scaling exponents ξ_n do not lie on a straight line, this is called an anomalous scaling. The term "anomaly" in theoretical physics means that the effect of symmetry breaking stays finite when the symmetry-breaking factor goes to zero.

As an alternative to the description in terms of structures (shocks), one can relate the anomalous scaling in Burgers turbulence to the additional integrals of motion. Indeed, the integrals $E_n = \int u^{2n}\, dx/2$ are all conserved by the inviscid Burgers equation. Any shock dissipates the finite amount of E_n at the limit $\nu \to 0$ so that similarly to (6) one denotes $\langle \dot{E}_n \rangle = \epsilon_n$ and obtains $S_{2n+1} = -4(2n+1)\epsilon_n x/(2n-1)$ for integer n. We thus conclude that the statistics of velocity differences in the inertial interval depends on the infinitely many pumping-related parameters, the fluxes of all dynamical integrals of motion.

Note that $S_2(x) \propto |x|$ corresponds to $E(k) \propto k^{-2}$, since every shock gives $u_k \propto 1/k$ at $k \ll v/\nu$, that is the energy spectrum is determined by the type of structures (shocks) rather than by energy flux constancy. That is Burgers turbulence demonstrates the universality of a different kind: the type of structures that dominate turbulence (here, shocks) is universal while the statistics of their amplitudes depends on pumping.

3. 3d Navier-Stokes turbulence

Incompressible fluid flow is described by the Navier-Stokes equation

$$\partial_t \mathbf{v}(\mathbf{r},t) + \mathbf{v}(\mathbf{r},t)\cdot\nabla\mathbf{v}(\mathbf{r},t) - \nu\nabla^2\mathbf{v}(\mathbf{r},t) = -\nabla p(\mathbf{r},t)\,, \quad \mathrm{div}\,\mathbf{v} = 0\,. \tag{7}$$

We are again interested in the structure functions $S_n(\mathbf{r},t) = \langle[(\mathbf{v}(\mathbf{r},t) - \mathbf{v}(0,t))\cdot\mathbf{r}/r]^n\rangle$ and consider distance r smaller than the force correlation scale for a steady case and smaller than the size of turbulent region for a decay case. Similar to (5), one can derive the Karman-Howarth relation between S_2 and S_3 (see Landau and Lifshits 1987):

$$\frac{\partial S_2}{\partial t} = -\frac{1}{3r^4}\frac{\partial}{\partial r}(r^4 S_3) + \frac{4\epsilon}{3} + \frac{2\nu}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial S_2}{\partial r}\right)\,. \tag{8}$$

Here $\epsilon = \nu\langle(\nabla\mathbf{v})^2\rangle$ is the mean energy dissipation rate. Neglecting time derivative (which is zero in a steady state and small comparing to ϵ for decaying turbulence) one can multiply (8) by r^4 and integrate: $S_3(r) = -4\epsilon r/5 + 6\nu dS_2(r)/dr$. Kolmogorov considered the limit $\nu \to 0$ for fixed r and *assumed* nonzero limit for ϵ which gives the so-called 4/5 law (Kolmogorov 1941, Landau and Lifshits 1987, Frisch 1995):

$$S_3 = -\frac{4}{5}\epsilon r\,. \tag{9}$$

Similar to (6), this relation means that the kinetic energy has a constant flux in the inertial interval of scales (the viscous scale η is defined by $\nu S_2(\eta) \simeq \epsilon\eta^2$). Let us stress that this flux relation is built upon the assumption that the energy dissipation rate ϵ has a nonzero limit at vanishing viscosity. Since the input rate can be independent of viscosity, this is the assumption needed for an existence of a steady state at the limit: no matter how small the viscosity, or how high the Reynolds number, or how extensive the scale-range participating in the energy cascade, the energy flux is expected to remain equal to that injected at the stirring scale. Unlike compressible (Burgers) turbulence, here we do not know the form of the specific singular structures that are supposed to provide non-vanishing dissipation in the inviscid limit (as shocks waves do). Experimental data show, however, that the dissipation rate is indeed independent of the Reynolds number when $Re \gg 1$. Historically, persistence of the viscous dissipation in the inviscid limit (both in compressible and incompressible turbulence) is the first example of what is now called "anomaly" in theoretical physics: a symmetry of the equation (here, time-reversal invariance) remains broken even as the symmetry-breaking factor (viscosity) becomes vanishingly small (see e.g. Falkovich and Sreenivasan 2006). If one screens a movie of steady turbulence backwards, we can tell that something is indeed wrong!

The law (9) shows that the third-order moment is universal, i.e. it does not depend on the details of the turbulence production but is determined solely by the mean energy dissipation rate. The rest of the structure functions have never been derived. Kolmogorov (1941) and also Heisenberg, von Weizsacker and Onsager *presumed* the pair correlation function to be determined only by ϵ and r which would give $S_2(r) \sim (\epsilon r)^{2/3}$ and the energy spectrum $E_k \sim \epsilon^{2/3} k^{-5/3}$. Experiments suggest that $\zeta_n = d\ln S_n/d\ln r$ lie on a smooth concave curve sketched in Fig. 2. While ζ_2 is close to 2/3 it has to be a bit larger because experiments show that the slope at zero $d\zeta_n/dn$ is larger than 1/3 while $\zeta(3) = 1$ in agreement with (9). Like in Burgers, the PDF of velocity differences in the inertial interval is not scale invariant in the 3d incompressible turbulence. So far, nobody was able to find an explicit relation between the anomalous scaling for 3d Navier-Stokes turbulence and either structures or additional integrals of motion. We understand qualitatively the breakdown of scale invariance in Navier-Stokes turbulence and in a related problem of passive scalar turbulence in terms of *statistical Lagrangian integrals of motion* (as opposite to dynamical integrals in the Burgers turbulence), see Section 5 below. Namely, it is believed that the correlation functions are determined by persistent structures. For example, the second velocity moment must have a scaling (close but not equal to 2/3) of the statistically conserved quantity build out of velocity vectors of two fluid particles and the distance between them: this scaling is determined by the law of de-correlation of two vectors convected by the flow (rather than energy flux constance which determines only the third moment).

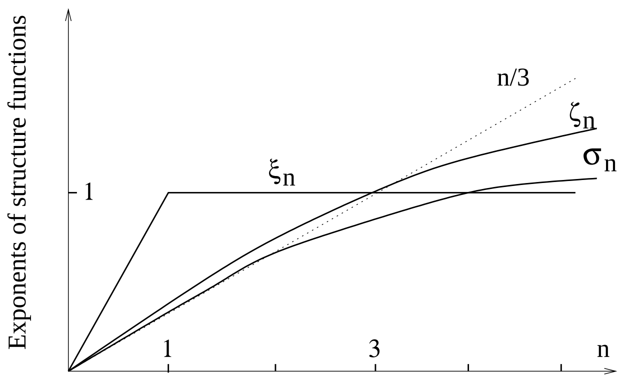

Figure 2. The scaling exponents of the structure functions ξ_n for Burgers, ζ_n for 3d Navier-Stokes and σ_n for the passive scalar. The dotted straight line is $n/3$.

While not exact, the Kolomogorov's approximation $S_2(\eta) \simeq (\epsilon\eta)^{2/3}$ can be used to estimate the viscous scale: $\eta \simeq LRe^{-3/4}$. The number of degrees of freedom involved into 3d incompressible turbulence can thus be roughly estimated as $N \sim (L/\eta)^3 \sim Re^{9/4}$. That means, in particular, that detailed numerical simulation of water or oil pipe flows ($Re \sim 10^4 \div 10^7$) or turbulent cloud ($Re \sim 10^6 \div 10^9$) is beyond the reach of today (and possibly tomorrow) computers. To calculate correctly at least the large-scale part of the flow, it is desirable to have some theoretical model to parameterize the small-scale motions. Here, the main obstacle is our lack of qualitative understanding and quantitative description of how turbulence statistics changes with the scale. This breakdown of scale invariance in the inertial range is another example of anomaly (effect of pumping scale does not disappear even at the limit $r/L \to 0$). Such an anomalous (or multi-fractal) scaling, is an important feature of turbulence, and sets it apart from the usual critical phenomena: one needs to work out the behavior of moments of each order independently rather than get it from dimensional analysis. Anomalous scaling in turbulence is such that $\zeta_{2n} < n\zeta_2$ so that S_{2n}/S_2^n for $n > 2$ increases as $r \to 0$. The relative growth of high moments means that strong fluctuations become more probable as the scales become smaller. Its practical importance is that it limits our ability to produce realistic models for small-scale turbulence.

Since we know neither the structures nor the extra conservation laws that are responsible for an anomalous scaling in the 3d incompressible turbulence, then, to get some qualitative understanding of this very complicated problem, we now pass to another (no less complicated) problem of 2d turbulence. That latter problem will motivate us to consider passive scalar turbulence, which will, in particular, teach us a new concept of statistical conservation laws that will shed some light on 3d turbulence too.

We thus conclude that in direct cascades we have at least two anomalies:

- finite third moment means time-irreversibility even when $\nu \to 0$,
- scale invariance is not restored even when $x/L \to 0$

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4. 2d Turbulence and passive scalar

Large-scale motions in shallow fluid can be approximately considered two-dimensional. When the velocities of such motions are much smaller than the velocities of the surface waves and the velocity of sound, such flows can be considered incompressible. Their description is important for understanding atmospheric and oceanic turbulence at the scales larger than the atmosphere height and the ocean depth. Vorticity $\omega = curl\,\mathbf{v}$ is a scalar in a two-dimensional flow. It is advected by the velocity field and dissipated by viscosity. Taking *curl* of the Navier-Stokes equation one gets

$$d\omega/dt = \partial_t\omega + (\mathbf{v}\cdot\nabla)\omega = \nu\nabla^2\omega \ . \tag{10}$$

Two-dimensional incompressible inviscid flow just transports vorticity from place to place and thus conserves spatial averages of any function of vorticity, $\Omega_n \equiv \int \omega^n d\mathbf{r}$. In particular, we now have the second quadratic inviscid invariant (in addition to energy) which is called enstrophy: $\Omega_2 = \int \omega^2\, d\mathbf{r}$. The spectral density of the energy is $|\mathbf{v}_k|^2/2$ while that of the enstrophy is $|\mathbf{k}\times\mathbf{v}_k|^2$. Pumping (at some k_f) generally provides for an input of both E and Ω_2. If there are two inertial intervals (at $k \gg k_f$ and $k \ll k_f$), then there should be two cascades. Indeed, absorbing finite amount of Ω_2 at $k_d \to \infty$ corresponds to an absorption of an infinitely small E. It is thus clear that the flux of E has to go in opposite direction that is to large scales. A so-called inverse cascade with the constant flux of E can thus be realized at $k \ll k_f$, as was suggested by Kraichnan (1967). What about other Ω_n? The intuition developed so far might suggest that the infinity of dynamical conservation laws must bring about anomalous scaling. As we shall see, turbulence never fails to defy intuition.

Passive Scalar Turbulence. Before discussing vorticity statistics in two-dimensional turbulence, we describe a similar yet somewhat simpler problem of passive scalar turbulence which allows one to introduce the necessary notions of Lagrangian description of the fluid flow. Consider a scalar quantity $\theta(\mathbf{r},t)$ which is subject to molecular diffusion and advection by the fluid flow but has no back influence on the velocity (i.e. passive):

$$d\theta/dt = \partial_t\theta + (\mathbf{v}\cdot\nabla)\theta = \kappa\nabla^2\theta \ . \tag{11}$$

Here κ is molecular diffusivity. The examples of passive scalar are smoke in the air, salinity in the water and temperature when one can neglect thermal convection. Without viscosity and diffusion, ω and θ behave in the same way in the same 2d flow — they are both Lagrangian invariants satisfying $d\omega/dt = d\theta/dt = 0$. Note however that vorticity is related to velocity while the passive scalar is not.

Let us now consider passive scalar turbulence. For that we add random source of fluctuations φ:

$$\partial_t\theta + (\mathbf{v}\cdot\nabla)\theta = \kappa\nabla^2\theta + \varphi \ . \tag{12}$$

If the source φ produces the fluctuations of θ on some scale L then the inhomogeneous velocity field stretches, contracts and folds the field θ producing progressively smaller and smaller scales — this is the mechanism of the scalar cascade. If the rms velocity gradient is Λ then molecular diffusion is substantial at the scales less than the diffusion scale $r_d = \sqrt{\kappa/\Lambda}$. For scalar turbulence, the ratio $Pe = L/r_d$, called Peclet number, plays the role of the Reynolds number. When $Pe \gg 1$, there is an inertial interval with a constant flux of θ^2:

$$\langle(\mathbf{v}_1\cdot\nabla_1 + \mathbf{v}_2\cdot\nabla_2)\theta_1\theta_2\rangle = 2P \ , \tag{13}$$

where $P = \kappa\langle(\nabla\theta)^2\rangle = \langle\varphi\theta\rangle$ and subscripts denote the spatial points. In considering the passive scalar problem, the velocity statistics is presumed to be given. Still, the correlation function (13) mixes $\mathbf{v}$ and θ and does not generally allow one to make a statement on any correlation function of θ. The proper way to describe the correlation functions of the scalar at the scales much larger than the diffusion scale is to employ the Lagrangian description that is to follow fluid trajectories. Indeed, if we neglect diffusion, then the equation (12) can be solved along the characteristics $\mathbf{R}(t)$ which are called Lagrangian trajectories and satisfy $d\mathbf{R}/dt = \mathbf{v}(\mathbf{R},t)$. Presuming zero initial conditions for θ at $t \to -\infty$ we write (see also Sect. 1.2.3 in the Gawędzki course)

$$\theta\Big(\mathbf{R}(t),t\Big) = \int_{-\infty}^{t} \varphi\Big(\mathbf{R}(t'),t'\Big)\,dt' \ . \tag{14}$$

In that way, the correlation functions of the scalar $F_n = \langle\theta(\mathbf{r}_1,t)\ldots\theta(\mathbf{r}_n,t)\rangle$ can be obtained by integrating the correlation functions of the pumping along the trajectories that satisfy the final conditions $\mathbf{R}_i(t) = \mathbf{r}_i$. We consider a pumping which is Gaussian, statistically homogeneous and isotropic in space and white in time:

$$\langle\varphi(\mathbf{r}_1,t_1)\varphi(\mathbf{r}_2,t_2)\rangle = \Phi(|\mathbf{r}_1 - \mathbf{r}_2|)\delta(t_1 - t_2)$$

where the function Φ is constant at $r \ll L$ and goes to zero at $r \gg L$. The pumping provides for symmetry $\theta \to -\theta$ which makes only even correlation functions F_{2n} nonzero. The pair correlation function is as follows:

$$F_2(r,t) = \int_{-\infty}^{t} \Phi\Big(R_{12}(t')\Big)\, dt' \ . \tag{15}$$

Here $R_{12}(t') = |\mathbf{R}_1(t') - \mathbf{R}_2(t')|$ is the distance between two trajectories and $R_{12}(t) = r$. The function Φ essentially restricts the integration to the time interval when the distance $R_{12}(t') \le L$. Simply speaking, the stationary pair correlation function of a tracer is $\Phi(0)$ (which is twice the injection rate of θ^2) times the average time $T_2(r,L)$ that two fluid particles spent within the correlation scale of the pumping. The larger r the less time it takes for the particles to separate from r to L and the less is $F_2(r)$. Of course, $T_{12}(r,L)$ depends on the properties of the velocity field. A general theory is available only when the velocity field is spatially smooth at the scale of scalar pumping L. This so-called Batchelor regime happens, in particular, when the scalar cascade occurs at the scales less than the viscous scale of fluid turbulence (Batchelor 1959, Kraichnan 1974, Falkovich et al 2001). This requires the Schmidt number ν/κ (called Prandtl number when θ is temperature) to be large, which is the case for very viscous liquids. In this case, one can approximate the velocity difference $\mathbf{v}(\mathbf{R}_1,t) - \mathbf{v}(\mathbf{R}_2,t) \approx \hat\sigma(t)\mathbf{R}_{12}(t)$ with the Lagrangian strain matrix $\sigma_{ij}(t) = \nabla_j v_i$. In this regime, the distance obeys the linear differential equation

$$\dot{\mathbf{R}}_{12}(t) = \hat\sigma(t)\mathbf{R}_{12}(t) \ . \tag{16}$$

The theory of such equations is well-developed and is related to what is called Lagrangian chaos and multiplicative large deviations theory. Fluid trajectories separate exponentially as typical for systems with dynamical chaos (see, e.g. Antonsen and Ott 1991, Falkovich et al 2001): At t much larger than the correlation time of the random process $\hat\sigma(t)$, all moments of R_{12} grow exponentially with time and $\langle \ln[R_{12}(t)/R_{12}(0)]\rangle = \lambda t$ where λ is called a senior Lyapunov exponent of the flow (remark that for the description of the scalar we need the flow taken backwards in time which is different from that taken forward because turbulence is irreversible). Dimensionally, $\lambda = \Lambda f(Re)$ where the limit of the function f at $Re \to \infty$ is unknown. We thus obtain:

$$F_2(r) = \Phi(0)\lambda^{-1}\ln(L/r) = 2P\lambda^{-1}\ln(L/r) \ . \tag{17}$$

In a similar way, one shows that for $n \ll \ln(L/r)$ all F_n are expressed via F_2 and the structure functions $S_{2n} = \langle[\theta(\mathbf{r},t) - \theta(0,t)]^{2n}\rangle \simeq (P/\lambda)^n \ln^n(r/r_d)$ for $n \ll \ln(r/r_d)$. That can be generalized for an arbitrary statistics of pumping as long as it is finite-correlated in time (Balkovsky and Fouxon 1999, Falkovich et al 2001). Note that those F_{2n} anf S_{2n} are completely determined by $\Phi(0)$ which is the flux of θ^2, only sub-leading corrections depend on the fluxes of the high-order integrals.

2d Enstrophy cascade. Now, one can use the analogy between passive scalar and vorticity in 2d (Kraichnan 1967,Falkovich and Lebedev 1994). For the enstrophy cascade, one derives the flux relation analogous to (13):

$$\langle(\mathbf{v}_1\cdot\nabla_1 + \mathbf{v}_2\cdot\nabla_2)\omega_1\omega_2\rangle = 2D \ , \tag{18}$$

where $D = \langle \nu(\nabla\omega)^2 \rangle$. The flux relation along with $\omega = curl\,\mathbf{v}$ suggests the scaling $\delta v(r) \propto r$ that is velocity being close to spatially smooth (of course, it cannot be perfectly smooth to provide for a nonzero vorticity dissipation in the inviscid limit, but the possible singularities are indeed shown to be no stronger than logarithmic). That makes the vorticity cascade similar to the Batchelor regime of passive scalar cascade with a notable change in that the rate of stretching λ acting on a given scale is not a constant but is logarithmically growing when the scale decreases. Physically, for smaller blobs of vorticity there are more large-scale velocity gradients that are able to stretch them. Since λ scales as vorticity, the law of renormalization can be established from dimensional reasoning and one gets $\langle \omega(\mathbf{r},t)\omega(0,t) \rangle \sim [D \ln(L/r)]^{2/3}$ which corresponds to the energy spectrum $E_k \propto D^{2/3}k^{-3} \ln^{-1/3}(kL)$. High-order correlation functions of vorticity are also logarithmic, for instance, $\langle \omega^n(\mathbf{r},t)\omega^n(0,t) \rangle \sim [D \ln(L/r)]^{2n/3}$. Note that both passive scalar in the Batchelor regime and vorticity cascade in 2d are universal that is determined by the single flux (P and D respectively) despite the existence of high-order conserved quantities. Experimental data and numeric simulations support those conclusions (Falkovich et al 2001, Tabeling 2002).

5. Zero modes and anomalous scaling

How one builds the Lagrangian description when the velocity is not spatially smooth, for example, that of the energy cascades in the inertial interval? Again, the only exact relation one can derive for two fluid particles separated by a distance in the inertial interval is for the Lagrangian time derivative of the squared velocity difference (Falkovich et al 2001):

$$\left\langle \frac{d|\delta\mathbf{v}|^2}{dt} \right\rangle = 2\epsilon$$

— this is the Lagrangian counterpart to (6,9,24). One can *assume* that the statistics of the distances between particles is also determined by the energy flux. That assumption leads, in particular, to the Richardson law for the asymptotic growth of the inter-particle distance:

$$\langle R_{12}^2(t) \rangle \sim \epsilon t^3 \ , \tag{19}$$

first inferred from atmospheric observations (in 1926) and later from experimental data on the energy cascades both in 3d and in 2d. There is no consistent theoretical derivation of (19) and it is unclear whether it is exact (likely to be in 2d) or just approximate (possible in 3d). Semi-heuristic argument usually presented in textbooks is based on the mean-field estimate: $\dot{\mathbf{R}}_{12} = \delta\mathbf{v}(\mathbf{R}_{12},t) \sim (\epsilon R_{12})^{1/3}$ which upon integration gives: $R_{12}^{2/3}(t) - R_{12}^{2/3}(0) \sim \epsilon^{1/3}t$. While this argument is at best a crude estimate in 3d (where there is no definite velocity scaling since every moment has its own exponent ζ_n) we use it to discuss implications for the passive scalar‡.

‡ What matters here and below is that in a non-smooth flow $R_{12}^a(t) - R_{12}^a(0) \sim t$ with $a < 1$, not the precise value of a

For two trajectories, the Richardson law gives the separation time from r to L: $T_2(r,L) \sim \epsilon^{-1/3}[L^{2/3} - r^{2/3}]$. Note that $T_2(r,L)$ has a finite limit at $r \to 0$ — infinitesimally close trajectories separate in a finite time. That leads to non-uniqueness of Lagrangian trajectories (non-smoothness of the velocity field means that the equation $\dot{\mathbf{R}} = \mathbf{v}(\mathbf{R})$ is non-Lipschitz). As discussed in much details elsewhere [30], that leads to a finite dissipation of a transported passive scalar even without any molecular diffusion (which corresponds to a dissipative anomaly and time irreversibility). Indeed, substituting $T_2(r,L)$ into (15), one gets the steady-state pair correlation function of the passive scalar: $F_2(r) \sim \Phi(0)\epsilon^{-1/3}[L^{2/3} - r^{2/3}]$ as suggested by Oboukhov (1949) and Corrsin (1952). The structure function is then $S_2(r) \sim \Phi(0)\epsilon^{-1/3}r^{2/3}$. Experiments measuring the scaling exponents $\sigma_n = d\ln S_n(r)/d\ln r$ generally give σ_2 close to 2/3 but higher exponents deviating from the straight line even stronger than the exponents of the velocity in 3d as seen in Fig. 2. Moreover, the scalar exponents σ_n are anomalous even when advecting velocity has a normal scaling like in 2d energy cascade (described in Sec. 7 below).

To explain the dependence $\sigma(n)$ and describe multi-point correlation functions or high-order structure functions one needs to study multi-particle statistics. Here an important question is what memory of the initial configuration remains when final distances far exceed initial ones. To answer this question one must analyze the conservation laws of turbulent diffusion. We now describe a general concept of conservation laws which, while conserved only on average, still determine the statistical properties of strongly fluctuating systems. In a random system, it is always possible to find some fluctuating quantities which ensemble averages do not change. We now ask a more subtle question: is it possible to find quantities that are expected to change on the dimensional grounds but they stay constant (Falkovich et al 2001, Falkovich and Sreenivasan 2006). Let us characterize n fluid particles in a random flow by inter-particle distances R_{ij} (between particles i and j) as in Figure 3. Consider homogeneous functions f of inter-particle distances with a nonzero degree ζ, i.e. $f(\lambda R_{ij}) = \lambda^\zeta f(R_{ij})$. When all the distances grow on the average, say according to $< R_{ij}^2 > \propto t^a$, then one expects that a generic function grows as $f \propto t^{a\zeta/2}$. How to build (specific) functions that are conserved on the average, and which ζ-s they have? As the particles move in a random flow, the n-particle cloud grows in size and the fluctuations in the shape of the cloud decrease in magnitude. Therefore, one may look for suitable functions of size and shape that are conserved because the growth of distances is compensated by the decrease of shape fluctuations.

For the simplest case of Brownian random walk, inter-particle distances grow by the diffusion law: $\langle R_{ij}^2(t)\rangle = R_{ij}^2(0) + \kappa t$, $\langle R_{ij}^4(t)\rangle = R_{ij}^4(0) + 2(d+2)[R_{ij}^2(0)\kappa t + \kappa^2 t^2]/d$, etc. Here d is the space dimensionality. Two particles are characterized by a single distance. Any positive power of this distance grows on the average. For many particles, one can build conserved quantities by taking the differences where all powers of t cancel out: $f_2 = \langle R_{12}^2 - R_{34}^2\rangle$, $f_4 = \langle 2(d+2)R_{12}^2R_{34}^2 - d(R_{12}^4 + R_{34}^4)\rangle$, etc. These polynomials are called harmonic since they are zero modes of the Laplacian in the $2d$-dimensional

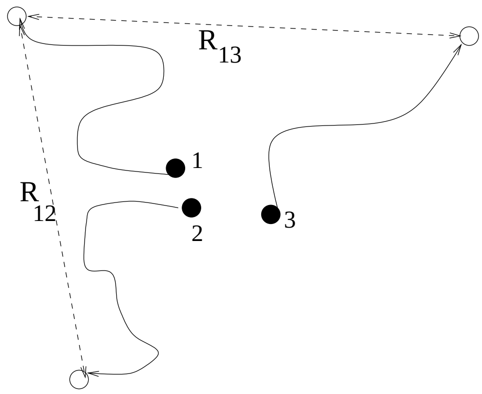

Figure 3. Three fluid particles in a flow.

space of $\mathbf{R}_{12}$, $\mathbf{R}_{13}$. One can write the Laplacian as $\Delta = R^{1-2d}\partial_R R^{2d-1}\partial_R + \Delta_\theta$, where $R^2 = R_{12}^2 + R_{13}^2$ and Δ_θ is the angular Laplacian on $2d-1$-dimensional unit sphere. Introducing the angle, $\theta = \arcsin(R_{12}/R)$, which characterizes the shape of the triangle, we see that the conservation of both $f_2 = \langle R^2 \cos 2\theta\rangle$ and $f_4 = \langle R^4[(d+1)\cos^2 2\theta - 1]\rangle$ can be also described as due to cancellation between the growth of the radial part (as powers of t) and the decay of the angular part (as inverse powers of t). For n particles, the polynomial that involves all distances is proportional to R^{2n} (i.e. $\zeta_n = n$) and the respective shape fluctuations decay as t^{-n}.

The scaling exponents of the zero modes are thus determined by the laws that govern decrease of shape fluctuations. The zero modes, which are conserved statistically, exist for turbulent macroscopic diffusion as well. However, there is a major difference since the velocities of different particles are correlated in turbulence. Those mutual correlations make shape fluctuations decaying slower than t^{-n} so that the exponents of the zero modes, ζ_n, grow with n slower than linearly. This is very much like the total energy of the cloud of attracting particles does not grow linearly with the number of particles. Indeed, power-law correlations of the velocity field lead to super-diffusive behavior of inter-particle separations: the farther particles are, the faster they tend to move away from each other, as in Richardson's law of diffusion. That is the system behaves as if there was an attraction between particles that weakens with the distance, though, of course, there is no physical interaction among particles (but only mutual correlations because they are inside the correlation radius of the velocity field). Let us stress that while zero modes of multi-particle evolution exist for all velocity fields—from those that are smooth to those that are extremely rough as in Brownian motion—only those non-smooth velocity fields with power-law correlations provide for an anomalous scaling. Zero modes were discovered in Gawedzki and Kupiainen 1995, Shraiman and Siggia 1995, Chertkov et al 1995 and then described in Chertkov and Falkovich 1996, Bernard et al 1996, Balkovsky and Lebedev 1998.

The existence of multi-particle conservation laws indicates the presence of a long-time memory and is a reflection of the coupling among the particles due to the simple fact that they are all in the same velocity field.

We now ask: How does the existence of these statistical conservation laws (called martingales in the probability theory) lead to anomalous scaling of fields advected by turbulence? According to (14), the correlation functions of θ are proportional to the times spent by the particles within the correlation scales of the pumping. The structure functions of θ are differences of correlation functions with different initial particle configurations as, for instance, $S_3(r_{12}) \equiv \langle[\theta(\mathbf{r}_1)-\theta(\mathbf{r}_2)]^3\rangle = 3\langle\theta^2(\mathbf{r}_1)\theta(\mathbf{r}_2)-\theta(\mathbf{r}_1)\theta^2(\mathbf{r}_2)\rangle$. In calculating S_3, we are thus comparing two histories: the first one with two particles initially close to the position $\mathbf{r}_1$ and one particle at $\mathbf{r}_2$, and the second one with one particle at $\mathbf{r}_1$ and two particles at $\mathbf{r}_2$— see Fig 4. That is, S_3 is proportional to the time

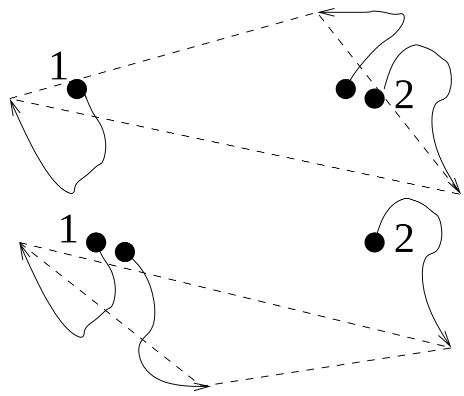

Figure 4. Two configurations (upper and lower) whose difference determines the third structure function.

during which one can distinguish one history from another, or to the time needed for an elongated triangle to relax to the equilateral shape. That time grows with r_{12} (as it takes longer to forget more elongated triangle) by the law that can be inferred from the law of the decrease of the shape fluctuations of a triangle.

Quantitative details can be worked out for the white in time velocity (Kraichnan 1968). Profound insight of Kraichnan was that it is spatial rather than temporal non-smoothness of the velocity that is crucial for an anomalous scaling. The Kraichnan model is described in much detail in the course by Gawędzki, here we mention few salient points. The velocity ensemble is defined by the second moment:

$$\langle v^i(\mathbf{r},t)v^j(0,0)\rangle = \delta(t)\Big[D_0\delta_{ij} - d_{ij}(\mathbf{r})\Big]\,,$$
$$d_{ij} = D_1\, r^{\xi}\Big[(d-1+\xi)\,\delta^{ij} - \xi r^i r^j r^{-2}\Big]\,. \tag{20}$$

Here the exponent $\xi \in [0,2]$ is a measure of the velocity non-smoothness with $\xi = 2$ corresponding to a smooth velocity while $\xi = 0$ to a velocity very rough in space (distributional). Richardson-Kolmogorov scaling of the energy cascade corresponds to $\xi = 4/3$. Lagrangian flow is a Markov random process for the Kraichnan ensemble (20). Every fluid particle undergoes a Brownian random walk with the so-called eddy diffusivity D_0. The PDF $P(r,t)$ for two particles to be separated by r after time t

satisfies the diffusion equation (see e.g. Falkovich et al 2001)

$$\partial_t P = L_2 P\,, \quad L_2 = d_{ij}(\mathbf{r})\nabla^i\nabla^j = D_1(d-1)r^{1-d}\partial_r r^{d+\xi-1}\partial_r\,, \tag{21}$$

with the scale-dependent diffusivity $D_1(d-1)r^{\xi}$. The asymptotic solution of (21) is $P(r,t) = r^{d-1}t^{d/(2-\xi)}\exp(-\mathrm{const}\, r^{2-\xi}/t)$, log-normal for $\xi = 2$. For $\xi = 4/3$, it reproduces, in particular, the Richardson law. Multi-particle probability distributions also satisfy diffusion equations in the Kraichnan model as well as all the correlation functions of θ. Multiplying (12) by $\theta_2 \ldots \theta_{2n}$ and averaging over the Gaussian statistics of $\mathbf{v}$ and φ one derives

$$\partial_t F_{2n} = L_{2n}F_{2n} + \sum_{l,m} F_{2n-2}\Phi(\mathbf{r}_{lm})\,, \quad L_{2n} = \sum d_{ij}(\mathbf{r}_{lm})\nabla_l^i\nabla_m^j\,. \tag{22}$$

This equation enables one, in principle, to derive inductively all steady-state F_{2n} starting from F_2. The equation $\partial_t F_2(r,t) = L_2F_2(r,t) + \Phi(r)$ has a steady solution $F_2(r) = 2[\Phi(0)/(2-\xi)d(d-1)D_1][dL^{2-\xi}/(d-2+\xi) - r^{2-\xi}]$, which has the Corrsin-Oboukhov form for $\xi = 4/3$. Further, F_4 contains the so-called forced solution having the normal scaling $2(2-\xi)$ but also, remarkably, a zero mode Z_4 of the operator L_4: $L_4Z_4 = 0$. Such zero modes necessarily appear (to satisfy the boundary conditions at $r \simeq L$) for all $n > 1$ and the scaling exponents of Z_{2n} are generally different from $n\gamma$ that is anomalous. In calculating the scalar structure functions, all terms cancel out except a single zero mode (called irreducible because it involves all distances between $2n$ points). Analytically and numerical calculations of Z_n and their scaling exponents σ_n (described in detail in the review Falkovich et al 2001) give σ_n lying on a convex curve (see Fig. 2) which saturates (Balkovsky and Lebedev 1998) to a constant at large n. Such saturation is a signature that most singular structures in a scalar field are shocks like in Burgers turbulence, the value σ_n at $n \to \infty$ is the fractal codimension of fronts in space (Celani et al 2001).

The existence of statistical conserved quantities breaks the scale invariance of scalar statistics in the inertial interval and explains why scalar turbulence knows about pumping “more” than just the value of the flux. Here again the statistics in the inertial interval, apart from the flux of θ^2, depends on the infinity of pumping-related parameters. However, those parameters neither are fluxes of θ^n, nor we can interpret them as any other fluxes. At the present level of understanding, we thus describe an anomalous scaling in Burgers and in passive scalar in quite different terms. Of course, the qualitative appeal to structures (shocks) is similar but the nature of the conservation laws is different. The anomalies produced by dynamically conserved quantities (like anomalous scaling in Burgers and time irreversibility in all cases of turbulence) are qualitatively different from the anomalies produced by statistically conserved quantities (like breakdown of scale invariance in passive scalar turbulence). Indeed, dissipation is a singular perturbation which breaks conservation of dynamical integrals of motion and imposes (one or many) flux-constancy conditions, very much similar to quantum anomalies. On the contrary, there are no cascades of conserved quantity related to zero modes, nor their conservation is broken by dissipation. Anomalous scaling of zero modes is due to correlations between different fluid trajectories. On the other hand, the two types of anomalies are related intimately: the flux constancy requires a certain degree of velocity non-smoothness, which generally leads to an anomalous scaling of zero modes.

Both symmetries, one broken by pumping (scale invariance) and another by damping (time reversibility) are not restored even when $r/L \to 0$ and $r_d/r \to 0$.

For the vector field (like velocity or magnetic field in magnetohydrodynamics) the Lagrangian statistical integrals of motion may involve both the coordinate of the fluid particle and the vector it carries. Such integrals of motion were built explicitly and related to the anomalous scaling for the passively advected magnetic field in the Kraichnan ensemble of velocities (Falkovich et al 2001). Doing that for velocity that satisfies the 3d Navier-Stokes equation remains a task for the future.

6. Inverse cascades

Here we consider inverse cascades and discover that, while time reversibility remains broken, the scale invariance is restored in the inertial interval. Moreover, even wider symmetry of conformal invariance may appear there.

Passive scalar in a compressible flow. Similar to (15) one can derive from (14)

$$\langle \theta(t,\mathbf{r}_1)\dots\theta(t,\mathbf{r}_{2n})\rangle = \int_0^t dt_1\dots dt_n \\ \times \langle \Phi(R(t_1|T,\mathbf{r}_{12}))\dots\Phi(R(t_n|T,\mathbf{r}_{2n-1,2n}))\rangle + \dots\ , \tag{23}$$

The functions Φ in (23) restrict integration to the time intervals where $R_{ij} < L$. If the Lagrangian trajectories separate, the correlation functions reach at long times the stationary form for all r_{ij}. Such steady states correspond to a direct cascade of the tracer (i.e. from large to small scales) considered above. That generally takes place in incompressible and weakly compressible flows.

It is intuitively clear that in compressible flows the regions of compressions can trap fluid particles counteracting their tendency to separate. Indeed, one can show that particles cluster in flows with high enough compressibility [30, 20, 35]. In particular, all the Lyapunov exponents are negative when the compressibility degree of a short-correlated flow exceeds $d/4$ [20]. Even in the non-smooth flow with high enough compressibility, the trajectories are unique, particles that start from the same point will remain together throughout the evolution [35]. That means that advection preserves all the single-point moments $\langle\theta^N\rangle(t)$. Note that the conservation laws are statistical: the moments are not dynamically conserved in every realization, but their average over the velocity ensemble are. In the presence of pumping, the moments are the same as for the equation $\partial_t\theta = \varphi$ in the limit $\kappa \to 0$ (nonsingular now). It follows that the single-point statistics is Gaussian, with $\langle\theta^2\rangle$ coinciding with the total injection $\Phi(0)t$ by the forcing. That growth is produced by the flux of scalar variance toward the large scales. In other words, the correlation functions acquire parts which are independent of r and grow proportional to time: when Lagrangian particles cluster rather than separate, tracer fluctuations grow at larger and larger scales — phenomenon that can be loosely called an inverse cascade of a passive tracer [30, 20, 35]. As is clear from (23), correlation functions at very large scales are related to the probability for initially distant particles to come close. In a strongly compressible flow, the trajectories are typically contracting, the particles tend to approach and the distances will reduce to the forcing correlation length L (and smaller) for long enough times. On a particle language, the larger the time the large the distance starting from which particle come within L. The correlations of the field θ at larger and larger scales are therefore established as time increases, signaling the inverse cascade process.

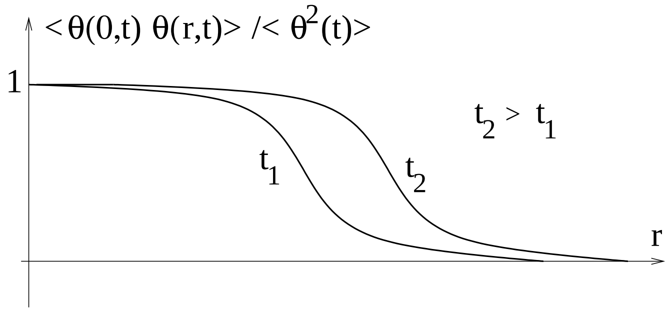

Figure 5. Growth of large-scale correlations with time.

The uniqueness of the trajectories greatly simplifies the analysis of the PDF $\mathcal{P}(\delta\theta, r)$. Indeed, the structure functions involve initial configurations with just two groups of particles separated by a distance r. The particles explosively separate in the incompressible case and we are immediately back to the full N-particle problem. Conversely, the particles that are initially in the same group remain together if the trajectories are unique. The only relevant degrees of freedom are then given by the intergroup separation and we are reduced to a two-particle dynamics. It is therefore not surprising that the statistics of the passive tracer is scale invariant in the inverse cascade regime [35].

An example of strongly compressible flow is given by Burgers turbulence (1) where there is clustering (in shocks) for the majority of trajectories (full measure in the inviscid limit). Considering passive scalar in such a flow, $\theta_t + u\theta_x - \kappa\Delta\theta = \phi$, we conclude that it undergoes an inverse cascade. The statistics of θ is scale invariant at the scales exceeding the correlation scale of the pumping ϕ. While the limit $\kappa \to 0$ is regular (i.e. no dissipative anomaly), the statistics is time irreversible because of the flux towards large scales. It is instructive to compare u and θ which are both Lagrangian invariants (tracers) in the unforced undamped limit. Yet passive quantity θ (and all its powers) go to large scales under pumping while all powers of u cascade towards small scales and are absorbed by viscosity. Physically, the difference is evidently due to the fact that the trajectory depends on the value of u it carries, the larger the velocity the faster it ends in a shock and dissipates the energy and other integrals. Formally, for active tracers like u^n one cannot write a formula like (23) obtained by two independent averages over the force and over the trajectories.

7. Inverse energy cascades in hydrodynamics

For the inverse energy cascade in 2d Navier-Stokes equation, there is no consistent theory except for the flux relation that can be derived similarly to (9):

$$S_3(r) = 4\epsilon r/3 \ . \tag{24}$$

This scaling one can also get from phenomenological dimensional arguments, though in two seemingly unrelated ways. Consider the velocity difference v_r at the distance r. On the one hand, one may require that the kinetic energy v_r^2 divided by the typical time r/v_r must be constant and equal to the energy flux, ϵ: $v_r^3 \sim \epsilon r$. On the other hand, it can be argued that vorticity, which cascades to small scales, must be in equipartition in the inverse cascade range. If this is the case, the enstrophy $r^d\omega_r^2$ accumulated in a volume of size r is proportional to the typical time r/v_r at such scale, i.e. $r^d\omega_r^2 \sim r/v_r$. Using $\omega_r \sim v_r/r$ we derive $v_r^3 \sim r^{3-d}$ which for $d = 2$ is exactly the requirement of constant energy flux. Amazingly, the requirements of vorticity equipartition (i.e. equilibrium) and energy flux (i.e. turbulence) give the same Kolmogorov-Kraichnan scaling in 2d. Let us stress that (24) means that time reversibility is broken in the inverse cascade. Experiments (Tabeling 2002, Kellay and Goldburg 2002, Chen et al 2006) and numerical simulations (Boffetta et al 2000), however, demonstrate a scale-invariant statistics with the vorticity having scaling dimension 2/3: $\omega_r \propto r^{-2/3}$. In particular, $S_2 \propto r^{2/3}$ which corresponds to $E_k \propto k^{-5/3}$. It is ironic that probably the most widely known statement on turbulence, the 5/3 spectrum suggested by Kolmogorov for 3d, is not correct in this case (even though the true scaling is close) while it is probably exact in the Kraichnan's inverse 2d cascade. Qualitatively, it is likely that the absence of anomalous scaling in the inverse cascade is associated with the growth of the typical turnover time (estimated, say, as $r/\sqrt{S_2}$) with the scale. As the inverse cascade proceeds, the fluctuations have enough time to get smoothed out as opposite to the direct cascade in 3d, where the turnover time decreases in the direction of the cascade. Note in passing that passive scalar undergoes direct cascade in the flow of the 2d inverse energy cascade, scalar statistics is not scale invariant since the velocity is non-smooth (compare with the relation between the Lagrangian invariants u and θ for Burgers turbulence).

Two-dimensional Navier-Stokes equation belongs to a family of models that describe a transport of a scalar quantity by an incompressible velocity related to a scalar by an instantaneous linear scale-invariant relation. Consider a real function of time and coordinates, $a(\mathbf{r}, t)$, which evolves according to the equation

$$\partial a/\partial t + (\mathbf{v} \cdot \nabla)a = f + \nu\Delta a - \alpha a \ . \tag{25}$$

Here $\mathbf{r} = (x, y)$ belongs to a two-dimensional manifold (plane, disc or torus) where one defines a solenoidal vector field of velocity: $\mathbf{v} = (\partial\Psi/\partial y, -\partial\Psi/\partial x)$. The stream function Ψ is related to the quantity a by a linear scale-invariant relation $\Psi(\mathbf{r}, t) \simeq \int d\mathbf{r}'\, |\mathbf{r}-\mathbf{r}'|^{m-2} a(\mathbf{r}', t)$. That is a is carried by the velocity $\mathbf{v}$, is pumped by the force f and is dissipated by the viscous and uniform (bottom) friction with the friction coefficients respectively ν and α.

For the models of physical interest, m is integer. 2d Navier-Stokes equation corresponds to $m=2$ when the pseudo-scalar $a=\nabla\times\mathbf{v}=\Delta\Psi$ is called vorticity and $\Psi(\mathbf{r},t)=-(2\pi)^{-1}\int d\mathbf{r}'\ \ln|\mathbf{r}-\mathbf{r}'|a(\mathbf{r}',t)$. We also consider $m=1$, which corresponds to surface quasi-geostrophic (SQG) model that describes rotating buoyancy-driven flows near solid surface, a is the temperature in this case [37, 53]. The case $m=-2$, which describes large-scale flows of a rotating shallow fluid, is not considered here [44].

In the Fourier representation, $\mathbf{v}_\mathbf{k}=-\imath(k_2,-k_1)\Psi_\mathbf{k}=-\imath(k_2,-k_1)k^{-m}a_\mathbf{k}$. For example, for the torus $2\pi\times 2\pi$ the Fourier coefficients $a_\mathbf{k}(t)=\int a(\mathbf{x},t)e^{\imath(\mathbf{k}\cdot\mathbf{x})}d\mathbf{x}$ evolve according to the equation

$$\frac{\partial a_\mathbf{k}}{\partial t}-\sum_\mathbf{j}[\mathbf{k},\mathbf{j}]j^{-m}a_\mathbf{j}a_{\mathbf{k}-\mathbf{j}}=f_\mathbf{k}-(\alpha+\nu k^2)a_\mathbf{k}\,, \tag{26}$$
$$\mathbf{k},\mathbf{j}\in Z^2\backslash(0,0)\,,\quad [\mathbf{k},\mathbf{j}]=k_1j_2-k_2j_1\,,\quad k=|\mathbf{k}|\,,j=|\mathbf{j}|\,.$$

The random force is usually taken Gaussian: $df(\mathbf{r},t)=\sum_\mathbf{k}\sqrt{D(\mathbf{k})}\exp[\imath(\mathbf{k}\cdot\mathbf{r})]dB_{\mathbf{k}t}$, where $B_{\mathbf{k}t}$ are standard Brownian random walks independent for different $\mathbf{k}$. The spectral density $D(\mathbf{k})$ is nonzero in the ring $k_f<k<Ak_f$, here A - is a order-unity factor. The random fields $a,\mathbf{v},\Psi$ are generally non-Gaussian since they are related to the force by a nonlinear equation.

The left part of the equation (26) conserves L^2-norm $\sum|a_\mathbf{k}|^2$ and the "energy" $E=\sum|a_\mathbf{k}|^2k^{-m}$, the right side describes generation and dissipation. We assume the existence of a steady state where the expectations $E[*]$ are time-independent so that the following relations must hold:

$$\sum_\mathbf{k}(\alpha+\nu k^2)E[|a_\mathbf{k}|^2]=\sum D(\mathbf{k})\equiv P\,, \tag{27}$$
$$\sum_\mathbf{k}(\alpha+\nu k^2)k^{-m}E[|a_\mathbf{k}|^2]=\sum D(\mathbf{k})k^{-m}\equiv Q\,. \tag{28}$$

Let us fix P,Q and consider the limit $k_f\to\infty,\nu\to 0$. For $m>0$ it is natural to assume that the terms with $k\simeq k_\nu\gg k_f$ give the main contribution into the left sum (27) and that with $k\simeq k_\alpha\ll k_f$ into the left sum (28). Such arguments form the basis of the Kraichnan's double-cascade picture which postulates the existence of two inertial intervals where the nonlinear (inertial) term of (25) dominates and provides for the spectral transfer of P and Q respectively in the direct and inverse cascades [40]. Here we consider the inverse cascade determined by the flux Q. Power-law dependencies can be guessed from dimensional reasoning. Comparing centimeters and seconds one ought to remember that the dimensionality of a is $\mathrm{sec}^{-1}{\cdot}\mathrm{cm}^{m-2}$ while that of Q is $\mathrm{sec}^{-3}{\cdot}\mathrm{cm}^{4-m}$. Then $k_\alpha\simeq(\alpha^3/Q)^{1/(4-m)}$, where $\simeq$ means equality up to a dimensionless factor (generally dependent on the force details). assuming that the statistics in the inertial interval of the inverse cascade (that is at $k_\alpha\ll k\ll k_f$)is determined by the energy flux Q and wavenumber k one obtains from power counting:

$$E[|a_\mathbf{k}|^2]\simeq Q^{2/3}k^{(4m-4)/3}\,. \tag{29}$$

In the r-representation

$$E[a_r^2]=\sum_\mathbf{k}E[|a_\mathbf{k}|^2](1-e^{\imath(\mathbf{k}\cdot\mathbf{r})})\simeq Q^{2/3}r^{(4-4m)/3}\propto r^{2h}\,. \tag{30}$$

For $m = 1$, $E[a_r^2] \propto \ln(k_f r)$. Somewhat more rigorous way to get the exponent h is to require the constancy of the triple correlation function that describes the energy flux. For instance, in that way one derives the identity $\langle(\delta\mathbf{v}\cdot\mathbf{r})^3\rangle = 3Qr^4/2$ that gives Kolmogorov-Kraichnan scaling $h = -2/3$ for the Euler equation ($m = 2$)[40, 32, 30]. The best thing one can say about such arguments is that they are confirmed by the data of experiments and numerics that is within the precision determined by the finiteness of the inertial interval and experimental errors, the probability distribution is invariant with respect to global (uniform) scale transformations: $\mathcal{P}(a_r, r) \sim a_r^{-1} f(a_r r^{2/3})$ for $m = 2$ [61, 38, 17, 11, 15] and $\mathcal{P}(a_r, r) \sim a_r^{-1} f(a_r/\ln(k_f r))$ for $m = 1$ [56, 62, 15]. Conformal transformations realize non-uniform change of scale (preserving the angles) so that conformal invariance can be thought of as local scale invariance. Note that nonlocal relation between the velocity $\mathbf{v}$ and the field a it carries makes our systems dynamically nonlocal. However, we excite the systems by a noise with the short radius of correlation k_f^{-1} and hope to find locality in statistics. Specifically, consider turbulence in some connected domain $\mathcal{D} \subset C$ and the family of measures $\mu_{\mathcal{D}}(z_1, \ldots, z_n)$, depending on the points $z_i \in \mathcal{D}$ (for instance, probabilities of the velocity differences in different points). Turbulence excited by the force with the same k_f in another domain $\mathcal{D}'$ produces another family $\mu_{\mathcal{D}'}$. We call the measure conformal invariant if it is invariant with respect to the conformal map $f : \mathcal{D} \to \mathcal{D}'$, that is $\mu_{\mathcal{D}}(z_1, \ldots, z_n) = \mu_{\mathcal{D}'}(f(z_1), \ldots, f(z_n))$. That property takes place for some remarkable class of random curves which we describe now.

8. Schramm-Loewner evolution (SLE)

Non-self-intersecting curve growing from the domain boundary can be described by a conformal map of the domain with the curve inside into a domain without the curve. For example, in the simplest case the curve $\gamma(t)$ starts at the real axis of the half-plane H. Here t parameterizes the curve, it should not be confused with the time from (25). The map $g_t : H\backslash\gamma(t) \to H$ is fixed by the asymptotics $g_t(z) \sim z + 2t/z + \mathrm{O}(1/z^2)$ at infinity. If the curve touches itself, one must define the domain $K(t)$ as the union of the curve and all points that cannot be reached from infinity and consider $g_t : H\backslash K(t) \to H$. The growing tip of the curve is mapped into a real point $\xi(t)$. Loewner found in 1923 that the conformal map $g_t(z)$ and the curve $\gamma(t)$ are fully parameterized by tip image $\xi(t)$ called the driving function [49]. For that one needs to solve the remarkably simple Loewner equation $dg_t(z)/dt = 2[g_t(z) - \xi(t)]^{-1}$. Almost eighty years later, Schramm considered random curves in planar domains and showed (first, in a particular case) that the measure on the curves is conformal invariant if and only if $\xi(t) = \sqrt{\kappa}B_t$, where B_t is a standard one-dimensional Brownian walk [57]. In addition, the measure $\mu_H(\gamma; z_1, z_2)$ on the curves γ connecting z_1 and z_2 is Markovian: if to divide γ into two pieces γ_1 from the boundary z_1 to z γ_2 from z to z_2, then the conditional measure is as follows: $\mu_H(\gamma_2|\gamma_1; z_1, z_2) = \mu_{H\backslash\gamma_1}(\gamma_2; z, z_2)$. Diffusivity κ allows one to classify the classes of conformal invariance random curves called SLE_κ. Such curves have been encountered in physics before as the boundaries of clusters of 2d critical phenomena described by conformal field theories. The language and formalism of SLE is a new natural communication tool for physicists and mathematicians, they lead to an explosive growth of new results in mathematics, field theory and the theory of critical phenomena [45, 46, 47, 48, 12, 5]. We shall see in the next Section that SLE is encountered in hydrodynamics as well.

Let us list here few basic facts about SLE curves. When $\kappa = 0$, γ is a vertical straight line. The larger the κ, the more curve wiggles. The curve is simple (i.e. with probability 1 does not touch nether itself nor real axis) when $0 \le \kappa < 4$. For SLE$_\kappa$ with $4 \le \kappa < 8$, the curve touches itself but does not fill the space. In this case, one can define an external perimeter (as a part one can reach from infinity) which belongs to a dual class SLE$_{\kappa_*}$ with $\kappa_* = 16/\kappa$ [55, 6, 24]. The fractal dimension of SLE$_\kappa$ curves is $D_\kappa = 1 + \kappa/8$ for $\kappa < 8$.

Among the dual pairs, κ κ_*, one is special from the viewpoint of locality. The curves from SLE$_6$ do not feel the boundary until they touch it (a rigorous definition of that property called SLE locality can be found in [48]).The dual curve SLE$_{8/3}$ have the "restriction property": the statistics of the curves conditioned not to visit some region is the same as in the domain without this region. Intuitively, one can appreciate these properties by considering lattice (discrete) models which turn into the respective SLE in the continuous limit [48, 5]. For example, consider a honeycomb lattice. A random walk along the bonds starts from the boundary point that has all black hexagons to the left and white to the right and keeps that property as it moves turning right/left as it meets black/white hexagon. SLE$_6$ is obtained from the classical model of critical percolation when hexagons get their colors independently with the probability 1/2. SLE$_{8/3}$ corresponds to a self-avoiding random walk when every bond is visited only once. Also the value $\kappa = 4$ is special because it is self-dual it corresponds to the so-called harmonic navigator. In this case, the probability of the color for the hexagon encountered is determined by the harmonic function defined in the domain with the boundary that includes the hexagons colored before; in other words, a new random walk starts from the hexagon and colors it by the color of the boundary the walk hits [48, 5, 58]. Both SLE$_6$ and SLE$_4$ appear as isolines of Gaussian random fields. If one considers the surface of a random function of two variables, $a(x, y)$, as a landscape during a great flood then at some water level the probability to sail across is equal to probability to walk. At this level, the shoreline belongs to SLE$_6$ (critical percolation) if the correlation functions of $a(x, y)$ decay sufficiently fast. In particular, non-rigorous but plausible Harris criterium claims that if $\langle a(\mathbf{r})a(0)\rangle \sim r^{-2h}$ and $h \ge 3/4$, then isolines of the Gaussian field a are equivalent to critical percolation [63]. That follows from the fact that when a is non-zero, percolation is non-critical even for a short-correlated field, and a finite correlation length appears which scales as $l_c \propto a^{-4/3}$; that means that the non-zero isoline cannot be distinguished from the zero isoline at the scales shorter than l_c. In other words, on a scale r one is allowed fluctuation of the field less that $r^{-4/3}$. Therefore, if on the scale r the fluctuations are of the size r^{-h} with $h \ge 3/4$ then the fluctuations of the field $a(x, y)$ are small and its nodal line belong to SLE$_6$. On the contrary, isolines of the Gaussian field a with $h < 3/4$ are not equivalent to critical percolation i.e. do not belong to SLE_6. As far as SLE_4 is concerned, this class contain isolines of Gaussian (free) fields with $\langle a(\mathbf{r})a(0)\rangle \sim \ln r$ [58, 14, 13]. How all that is related to turbulence where the only thing we are sure about its being non-Gaussian (because the flux makes the third moment nonzero)?

9. Isolines in turbulence

The fractal dimension of SLE_κ curves is known to be $D_\kappa = 1 + \kappa/8$ for $\kappa < 8$. To establish possible link between turbulence and critical phenomena, let us try to relate the Kolmogorov-Kraichnan phenomenology to the fractal dimension of the boundaries of vorticity clusters. Note that one ought to distinguish between the dimensionality 2 of the full vorticity level set (which is space-filling) and a single zero-vorticity line that encloses a large-scale cluster. Consider the vorticity cluster of gyration radius L

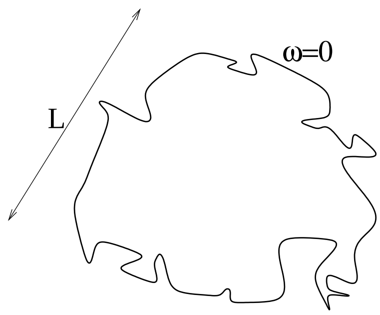

Figure 6. Vorticity nodal line with the gyration radius L.

which has the "outer boundary" of perimeter P (that boundary is the part of the zero-vorticity line accessible from outside, see Fig. 6 for an illustration). The vorticity flux through the cluster, $\int \omega dS \sim \omega_L L^2$, must be equal to the velocity circulation along the boundary, $\Gamma = \oint \mathbf{v} \cdot d\ell$. The Kolmogorov-Kraichnan scaling is $\omega_L \sim \epsilon^{1/3} L^{-2/3}$ (coarse-grained vorticity decreases with scale because contributions with opposite signs partially cancel) so that the flux is $\propto L^{4/3}$. As for circulation, since the boundary turns every time it meets a vortex, such a contour is irregular on scales larger than the pumping scale. Therefore, only the velocity at the pumping scale L_f is expected to contribute to the circulation, such velocity can be estimated as $(\epsilon L_f)^{1/3}$ and it is independent of L. Hence, circulation should be proportional to the perimeter, $\Gamma \propto P$, which gives $P \propto L^{4/3}$, i.e. the fractal dimension of the exterior of the vorticity cluster is expected to be $4/3$. This remarkable dimension correspond to a self-avoiding random walk (SLE curve) which is also known to be an exterior boundary (without self-intersections) of percolation cluster (yet another SLE curve).

Figure 7 shows a nodal line of vorticity obtained by a numerical solution of (25) with $m = 2$ on a torus (that is 2d Navier-Stokes equation with periodic boundary conditions and added external force and uniform friction), the details can be found in [11, 9]. Force scale is $l_f = 2\pi/k_f = 0.05$. The curve looks fractal at the scales exceeding l_f, i.e. in the interval of an inverse cascade. Indeed, the length P grows nonlinearly with the end-to-end distance L [9]. Power-law exponents of this grows for the curve and its external perimeter are found to be close within the resolution to the dimensionalities $7/4, 4/3$ of the dual pair SLE_6 $\text{SLE}_{8/3}$ (historically, dimensionality 4/3 of the external perimeter has been guessed from the Kolmogorov-Kraichnan scaling $a_r \sim r^{-2/3}$, which stimulated the search for SLE in turbulence [9]). Let us briefly describe how we identified possible

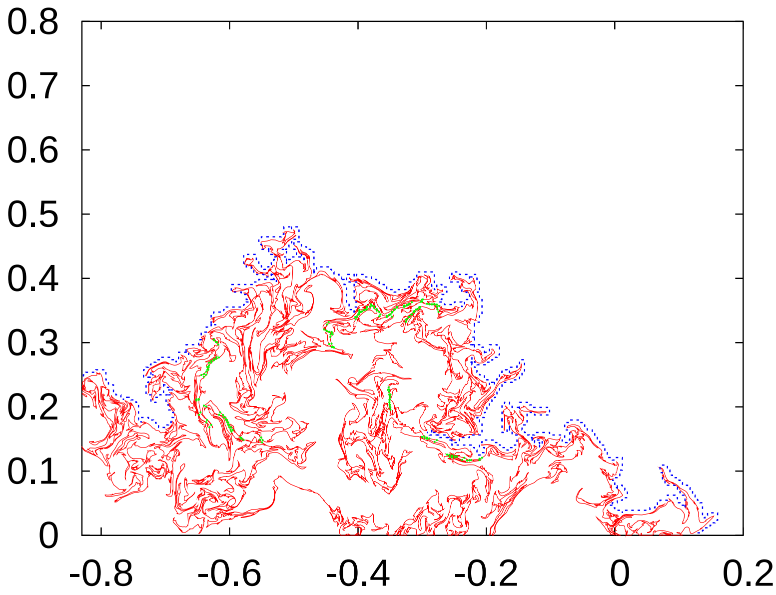

Figure 7. A portion of a candidate SLE trace obtained from the vorticity field. The figure has been adapted from [9]

curves from an SLE class and determined the driving function $\xi(t)$. We drew quite arbitrarily a straight line to be a real axis and at the end checked that translations and rotations of the axis did not change the results. We then start from the intersection of a zero isoline and the axis and move along the curve or along the axis (when return to it) preserving orientation i.e. keeping positive vorticity always to the right. Such a procedure faithfully reproduces the statistics only in the local case, indeed we expected (and found!) $\kappa \approx 6$. We then divided our curve into small straight segments and approximated the family of conformal maps $g_t(z)$ by a discrete set of standard conformal maps absorbing one segment one by one [50, 9]). The resulting set of "times" t_i and values ξ_i defines the driving function $\xi(t)$. The only thing left is to run the Schramm test i.e. to check how well this function corresponds to a Brownian walk. The data presented by upward oriented triangles in Figure 8 show that the ensemble average $\langle \xi(t)^2 \rangle$ indeed grows linearly in time: the diffusion coefficient κ is very close to the value 6, with an accuracy of 5% (lower inset). The probability distribution functions of $\xi(t)/\sqrt{\kappa t}$ collapse onto a standard Gaussian distribution at all times t (upper inset). Therefore, we expect that in the limit of vanishingly small L_f the driving $\xi(t)$ tends to a true Brownian motion and zero-vorticity lines become SLE_κ traces with κ very close to 6. Note that the vorticity field has $h = 2/3 < 3/4$, that is the Harris criterium is violated. However, our field is non-Gaussian - while the probability distribution looks like Gaussian, the deviations are measurable including the third moment [61, 38, 17, 11, 15]). Triangles pointing down on the lower are obtained for the isolines of a Gaussian field having the same Fourier spectrum as vorticity but randomized phases. Apparently, our accuracy is sufficient to make sure that it does not correspond to any SLE including SLE_6. Indeed, $E[\xi^2]/t \equiv \langle \xi^2 \rangle /t$ is not constant and approaches the limiting value $\kappa = 6$ only at the scales exceeding $2\pi/k_\alpha$ where the power-law correlation is already cut-off by friction and the field becomes truly uncorrelated. Something remarkable happens here: non-Gaussianity of the vorticity field, i.e. multi-point correlations, somehow conspire to make zero-vorticity line statistically equivalent to the isoline of a short-correlated field even though the pair correlation function decays slower that Harris criterium requires.

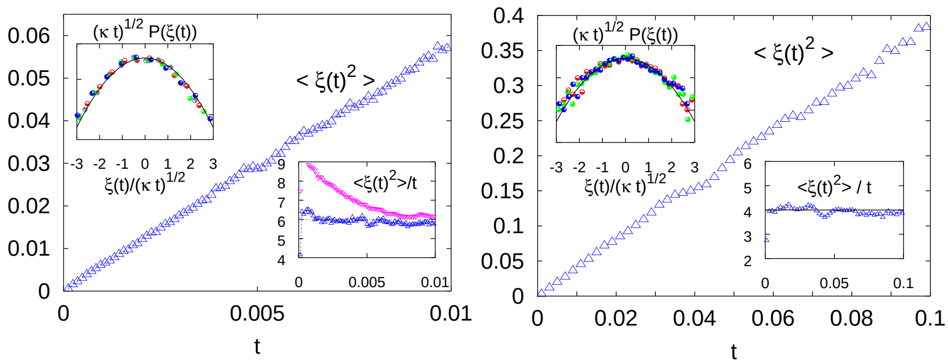

Figure 8. Demonstration of conformal invariance of the isolines of vorticity in the Euler equation (left) and of the temperature in the surface geostrophic model (right). The driving function is an effective diffusion process with diffusion coefficient $\kappa = 6 \pm 0.3$ (left [9]) and with $\kappa = 4 \pm 0.2$ (right [10]). Right (lower) inset: triangles pointing up correspond to the vorticity, triangles pointing down to the Gaussian field with the same second moment. Left (upper) insets: the probability density function of the re-scaled driving function $\xi(t)/\sqrt{\kappa t}$ at four different times $t = 0.0012, 0.003, 0.006, 0.009$ (left) and $t = 0.02, 0.04, 0.08$ (right); the solid lines are the Gaussian distribution $g(x) = (2\pi)^{-1/2} \exp(-x^2/2)$.

The identification of isolines as SLE_κ curves allows to apply powerful techniques borrowed from the theory of stochastic differential equations and conformal mapping theory and to obtain analytic predictions for some nontrivial statistical properties of lines, vortices and clusters in turbulence. For example, vorticity nodal lines are boundaries of the vorticity clusters. The statements from the percolation theory that the probability of a cluster (island) decays with area as $s^{-96/91}$, and with the perimeter as $P^{-8/7}$, can be directly confirmed for turbulence [9]. Moreover, SLE allows exact analytic formulas for the probabilities that a nodal line crosses different figures (triangles, rectangles etc). Such probabilities are determined by the second-order ordinary differential equation and are expressed via the hypergeometric functions, which miraculously describe turbulence data. It is worth to stress that maximum one aspired in turbulence theory before was to predict a single number (usually a scaling exponent and often from dimensional reasoning), now we are able to predict non-trivial functions. An ability to make exact predictions rather than order-of-magnitude estimates is heartening too. Most important though is that the inverse vorticity cascade , for instance, is described by the Euler equation so that the correspondence between the nodal lines in turbulence and SLE hints at some fundamental properties of this equation which we do not yet grasp.

Let us now describe briefly the results for the surface quasi-geostrophic model, $m = 1$. In this case, the zero-temperature isoline crosses the straight line rarely so that we simply choose as the candidates for SLE the pieces of the curve returning to the line at the distance far exceeding $2\pi/k_\alpha$. An example is shown in Figure 9 (a). Such a procedure is self-consistent for $\kappa \le 4$,which is indeed what we obtain. To recover the driving function for the curve going from 0 to x_∞ in the upper half-plane, one needs to solve the equation $\partial_t g_t = 2/\{\varphi'(g_t)[\varphi(g_t) - \xi_t]\}$, where $\varphi(z) = x_\infty z/(x_\infty - z)$. This equation on g_t can be solved for a constant ξ: $G_{t,\xi}(z) = x_\infty\{\eta x_\infty(x_\infty - z) + [x_\infty^4(z-\eta)^2 + 4t(x_\infty - z)^2(x_\infty-\eta)^2]^{1/2}\}/\{x_\infty^2(x_\infty - z) + [x_\infty^4(z-\eta)^2 + 4t(x_\infty-z)^2(x_\infty-\eta)^2]^{1/2}\}$, where $\eta = \varphi^{-1}(\xi)$. In this case, the curve is the semi-circle connecting η and x_∞. We divide the interval $[0, T]$ into small intervals $[t_n, t_{n+1})$ $t_0 = 0$, $t_{N+1} = T$, where driving function can be considered piecewise constant, $\xi_n = \xi(t_n)$. The map g_t is found as a composition $G_{t_N - t_{N-1}, \xi_{N-1}} \circ \cdots \circ G_{t_1,\xi_0}$. Approximating the curve by a finite number of points, $\{z_0, z_1, \ldots, z_{N+1}\}$, where $z_0 = 0$ and $z_{N+1} = x_\infty$, such a discrete procedure defines the set $\xi_n = \xi(t_n)$. The first step is to find the semi-circle passing through x_∞ and z_1 [see Fig. 9 (b)]. That gives the values $\eta_0 = \varphi^{-1}(\xi_0) = [\mathrm{Re} z_1\, x_\infty - (\mathrm{Re} z_1)^2 - (\mathrm{Im} z_1)^2]/(x_\infty - \mathrm{Re} z_1)$ $t_1 = (\mathrm{Im} z_1)^2 x_\infty^4/\{4[(\mathrm{Re} z_1 - x_\infty)^2 + (\mathrm{Im} z_1)^2]^2\}$. The map G_{t_1,ξ_0} then maps z_k into a new sequence which is one element shorter: $z'_k = G_{t_1,\xi_0}(z_{k+1})$ $k = 1 \ldots N$. Iterating such procedure one defines two sets t_k and ξ_k, which approximate the driving function. The procedure was checked first in applying to a self-avoiding random walk where it gave the right value $\kappa = 8/3$ with the accuracy better than 5%. Applying the procedure to the surface quasi-geostrophic model we obtain $\xi(t)$, whose statistics converges at

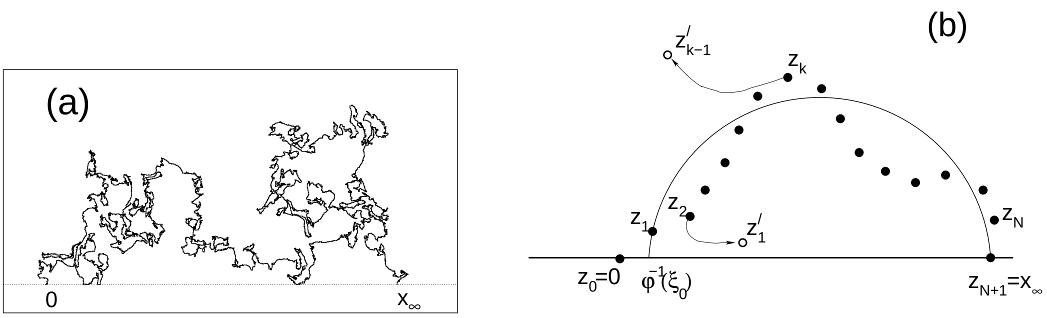

Figure 9. (a) A part of the temperature isoline that is a candidate for SLE. (b) The algorithm to extract the driving function (see text). The figure is adapted from [10]

$l_f^2 < \kappa t < 2\pi/k_\alpha$ to the Gaussian statistics with $\langle \xi^2(t) \rangle = \kappa t \quad \kappa = 4 \pm 0.2$, as shown at the right part of the Fig. 8. We conclude that within our accuracy the temperature isolines behave locally as curves from SLE_4. Worth stressing that the temperature field is substantially non-Gaussian [15, 10] so that it is completely unclear how it can have isolines with the same statistics as that of the isolines of the free Gaussian field with the same second moment.

Remark that if the contour $z(l)$ belongs to the class SLE_κ, then the unit vector z_l has a Gaussian statistics with the second moment proportional to the logarithm of the contour length. That property has also been found for the isolines of temperature and vorticity for both our models [9, 10].

Let us briefly discuss weak wave turbulence from the viewpoint of conformal invariance. Such turbulence has statistics close to Gaussian. Gaussian scalar field in 2d is conformal invariant if its correlation function is logarithmic i.e. the spectral density decays as k^{-2}. Such is the case, for instance, for the fluid height in gravitational-capillary weak wave turbulence on a shallow water (see Zakharov et al 1992, Sect. 5.1.2). It is interesting if deviations from Gaussianity due to wave interaction destroy conformal invariance. Another interesting example is the inverse cascade of 2d strong optical turbulence described by the Nonlinear Schrödinger Equation,

$$i\Psi_t + \Delta\Psi + T|\Psi|^2\Psi = 0 \ . \tag{31}$$

This equation also describes Bose-Einstein condensation (then it is usually called Gross-Pitaevsky equation). Weak turbulence is determined by $|T|^2$ and is the same both for $T < 0$ (wave repulsion) and $T > 0$ (wave attraction). Inverse cascade tends to produce a uniform condensate $\Psi(k = 0) = A$. At high levels of nonlinearity, different signs of T correspond to dramatically different physics. At $T < 0$ the condensate is stable, it renormalizes the linear dispersion relation from $\omega_k = k^2$ to the Bogolyubov form $\omega_k^2 = k^4 - 2TA^2k^2$. That dispersion relation is close to acoustic at small k, it allows for three-wave interactions. The resulting over-condensate turbulence is a mixture of phonons, solitons, kinks and vortices. On the contrary, the condensate and sufficiently long waves are unstable at $T > 0$; that instability leads to wave collapse at $d = 2, 3$ with the energy being fast transferred from large to small scales where it dissipates (Dyachenko et al 1992). No analytic theory is yet available for such strong turbulence. Numerics hint that in the case of a stable growing condensate, the statistics of the finite-scale fluctuations approach Gaussian with a logarithmic correlation function [25].

10. Conclusion

We reiterate the conclusions on the status of symmetries in turbulence.
Turbulence statistics is always time-irreversible.
Weak turbulence is scale invariant and universal (determined solely by the flux value). It is generally not conformal invariant.
Strong turbulence: Direct cascades often have symmetries broken by pumping (scale invariance, isotropy) non-restored in the inertial interval. In other words, statistics at however small scales is sensitive to other characteristics of pumping besides the flux. That can be alternatively explained in terms of either structures or statistical conservation laws (zero modes). Anomalous scaling in a direct cascade may well be a general rule apart from some degenerate cases like passive scalar in the Batchelor case (where all the zero modes have the same scaling exponent, zero, as the pair correlation function). Inverse cascades in systems with strong interaction may be not only scale invariant but also conformal invariant. It is an example of emerging or restored symmetry.
For Lagrangian invariants, we explain the difference between direct and inverse cascades in terms of separation or clustering of fluid particles. Generally, it seems natural that the statistics within the pumping correlation scale (direct cascade) is more sensitive to the details of the pumping statistics than the statistics at much larger scales (inverse cascade).

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