The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

1952-6

School on Stochastic Geometry, the Stochastic Lowener Evolution, and Non-Equilibrium Growth Processes

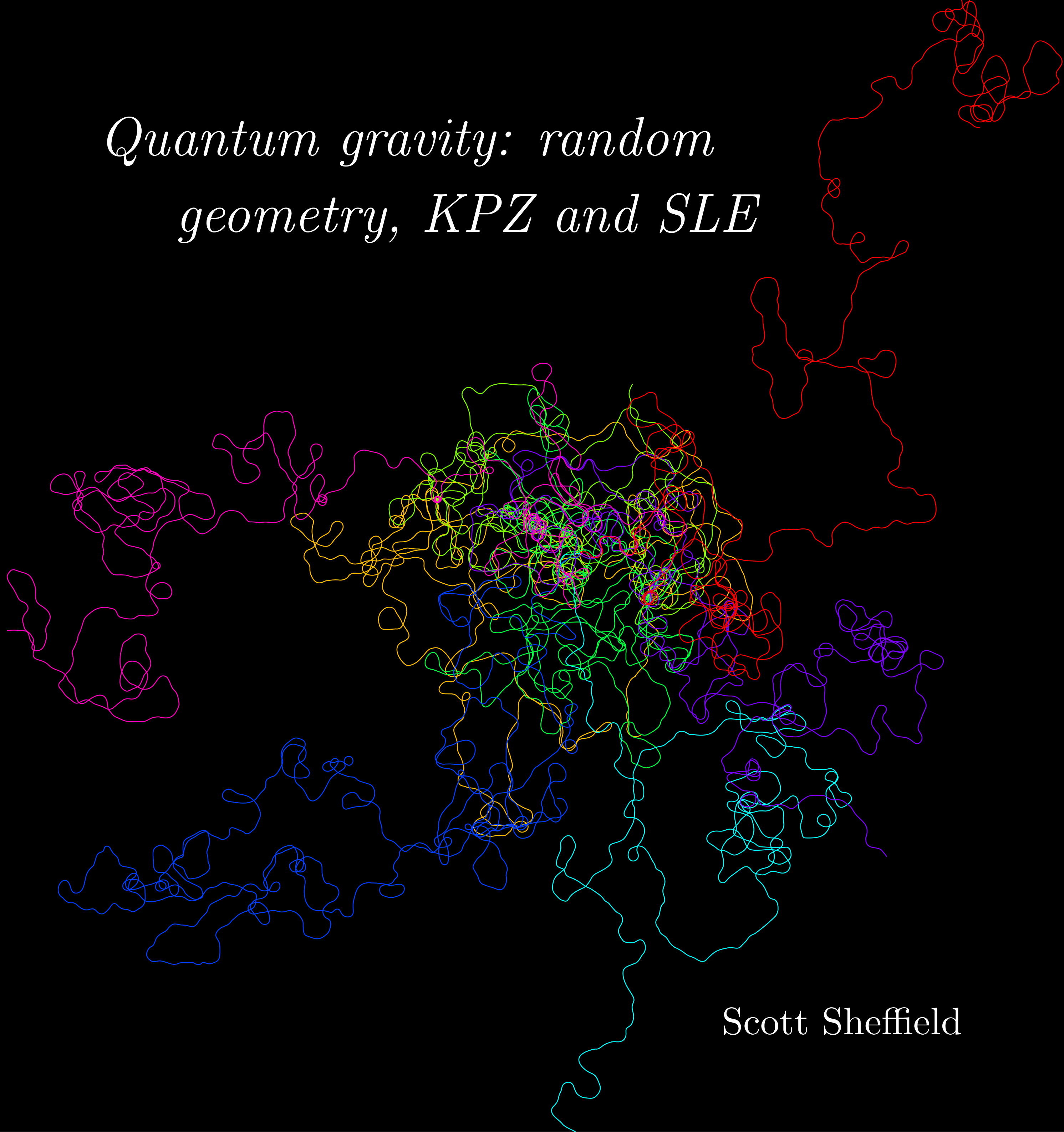
7 - 18 July 2008

Quantum gravity: KPZ, SLE, conformal welding and scaling limits (Quantum gravity: random geometry, KPZ and SLE)

Scott SHEFFIELD

Courant Institute of Mathematical Sciences, Dept. of Mathematics, New York University, NY, U.S.A.

Quantum gravity: random geometry, KPZ and SLE



Scott Sheffield

Riemann uniformization theorem

Uniformization Every smooth simply connected Riemannian manifold $\mathcal{M}$ can be conformally mapped to either the unit disc $\mathbb{D}$, the complex plane $\mathbb{C}$, or the complex sphere $\mathbb{C} \cup \{\infty\}$.

Isothermal coordinates: $\mathcal{M}$ can be parameterized by points $z = x + iy$ in one of these spaces in such a way that the metric takes the form $e^{\lambda(z)}(dx^2 + dy^2)$ for some real-valued function λ. The (x, y) are called *isothermal coordinates* or *isothermal parameters* for $\mathcal{M}$.

Write D for the parameter space and suppose D is a simply connected bounded subdomain of $\mathbb{C}$ (which is conformally equivalent to $\mathbb{D}$ by the Riemann mapping theorem).

Isothermal coordinates

LENGTH of path in $\mathcal{M}$ parameterized by a smooth path P in D is $\int_P e^{\lambda(s)/2} ds$, where ds is the Euclidean length measure on D.

AREA of subset of $\mathcal{M}$ parameterized by a measurable subset A of D is $\int_A e^{\lambda(z)} dz$, where dz is Lebesgue measure on D.

GAUSSIAN CURVATURE DENSITY in D is $-\Delta\lambda$, i.e., if A is a measurable subset of the D, then the integral of the Gaussian curvature with respect to the portion of $\mathcal{M}$ parameterized by A is $\int_A -\Delta\lambda(z) dz$.

"There are methods and formulae in science, which serve as master-keys to many apparently different problems. The resources of such things have to be refilled from time to time. In my opinion at the present time we have to develop an art of handling sums over random surfaces. These sums replace the old-fashioned (and extremely useful) sums over random paths. The replacement is necessary, because today gauge invariance plays the central role in physics. Elementary excitations in gauge theories are formed by the flux lines (closed in the absence of charges) and the time development of these lines forms the world surfaces. All transition amplitude are given by the sums over all possible surfaces with fixed boundary."

A.M. Polyakov, Moscow 1981

The *standard Gaussian* on n-dimensional Hilbert space

has density function $e^{-(v,v)/2}$ (times an appropriate constant). We can write a sample from this distribution as

$$\sum_{i=1}^{n} \alpha_i v_i$$

where the v_i are an orthonormal basis for $\mathbb{R}^n$ under the given inner product, and the α_i are mean zero, unit variance Gaussians.

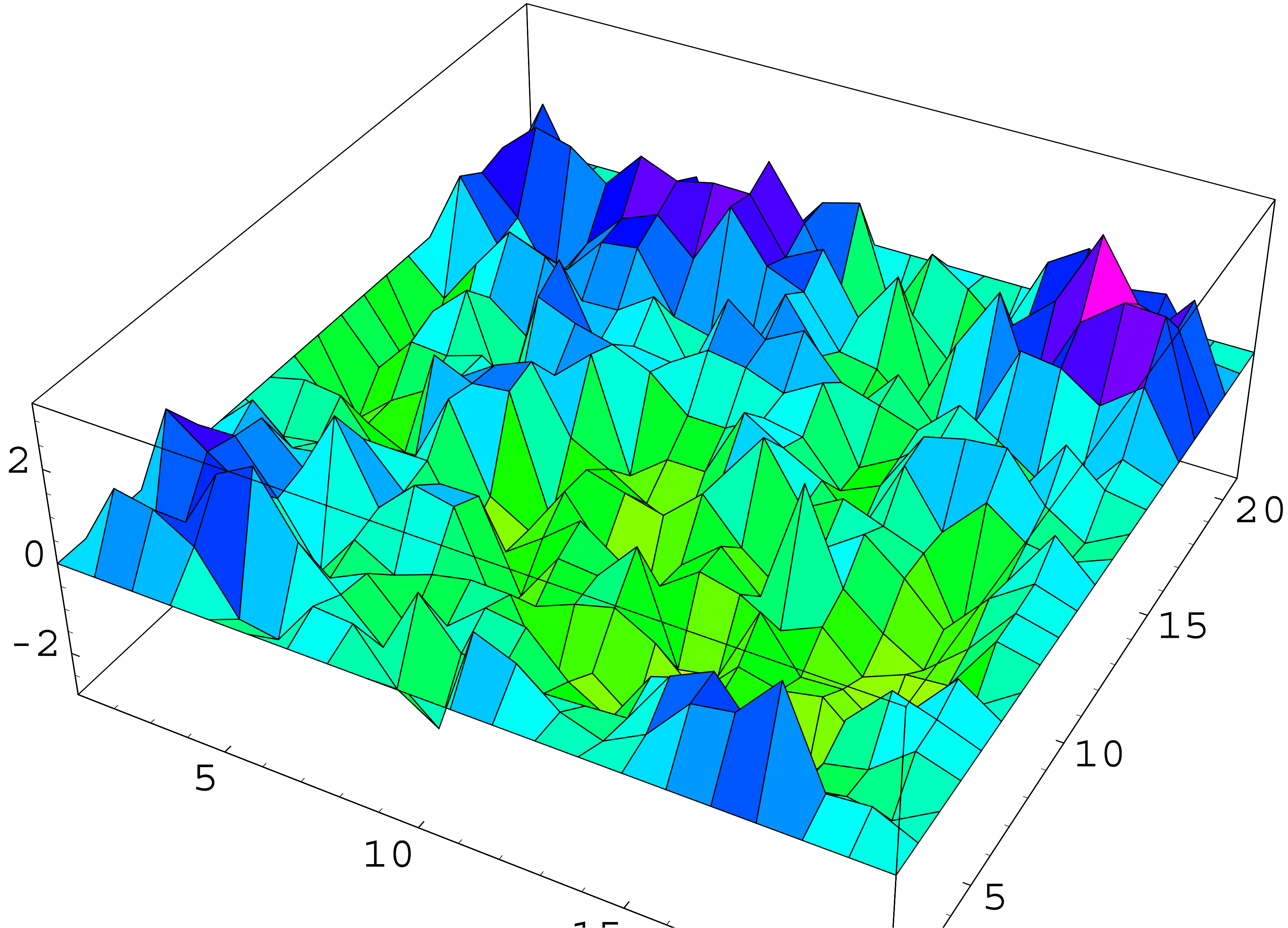
The discrete Gaussian free field

Let f and g be real functions defined on the vertices of a planar graph Λ. The **Dirichlet inner product** of f and g is given by

$$(f,g)_\nabla = \sum_{x\sim y} (f(x)-f(y))\,(g(x)-g(y))\,.$$

The value $H(f)=(f,f)_\nabla$ is called the **Dirichlet energy of f.** Fix a function f_0 on boundary vertices of Λ. The set of functions f that agree with f_0 is isomorphic to $\mathbb{R}^n$, where n is the number of interior vertices. The **discrete Gaussian free field** is a random element of this space with probability density proportional to $e^{-H(f)/2}$.

Discrete GFF on 20×20 grid, zero boundary



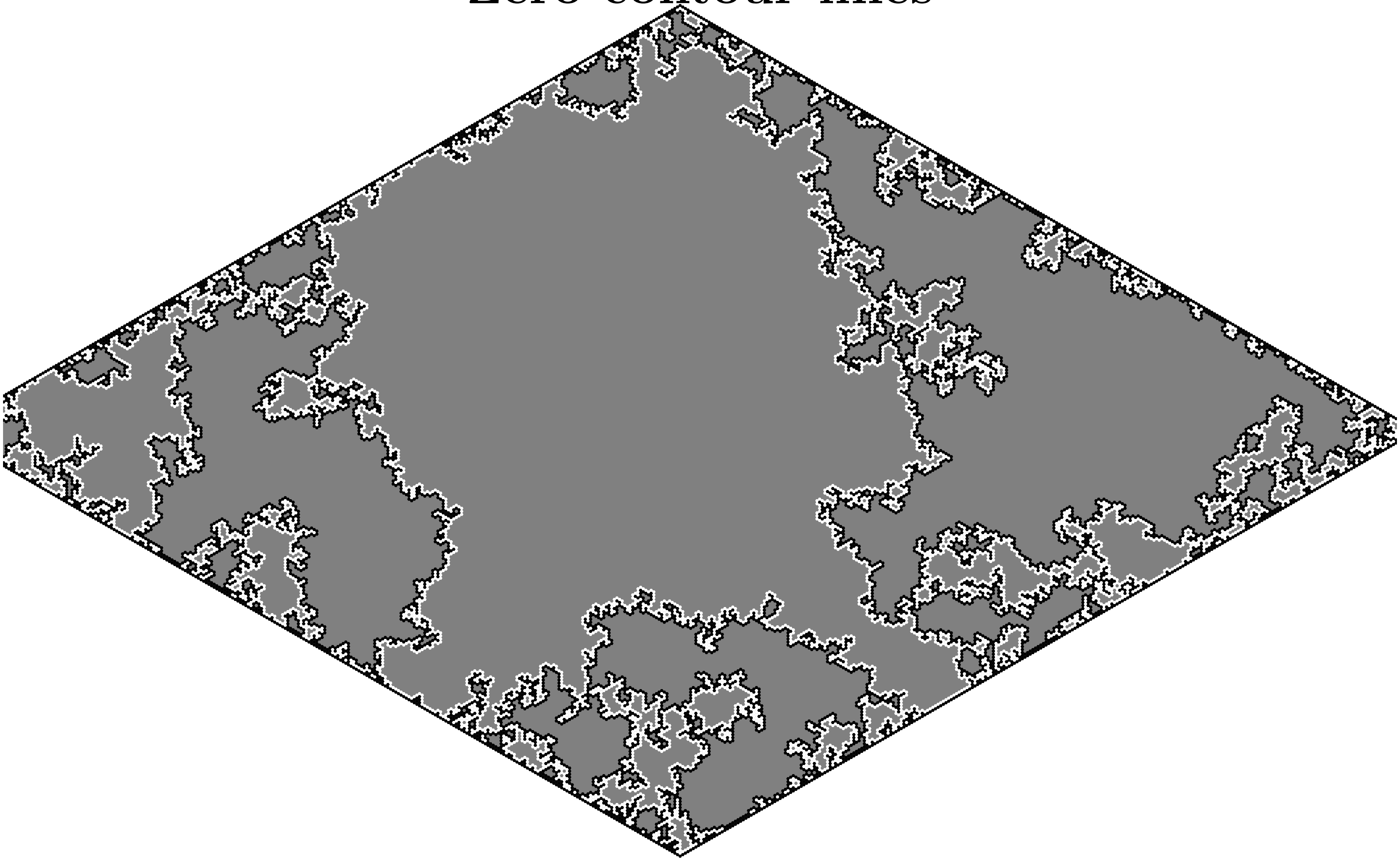
The continuum Gaussian free field

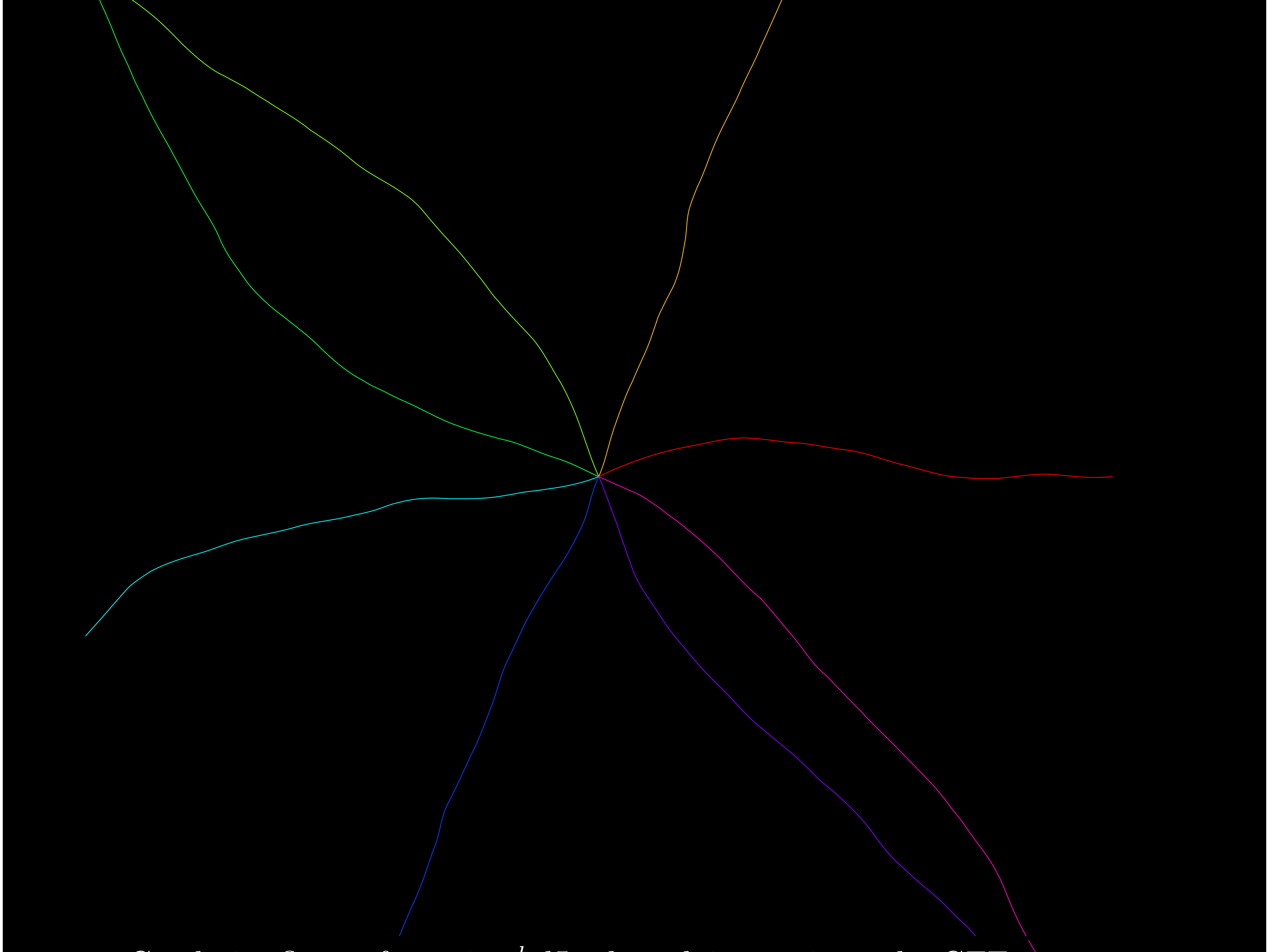
is a "standard Gaussian" on an *infinite* dimensional Hilbert space. Given a planar domain D, let $H(D)$ be the Hilbert space closure of the set of smooth, compactly supported functions on D under the conformally invariant *Dirichlet inner product*

$$(f_1, f_2)_\nabla = \int_D (\nabla f_1 \cdot \nabla f_2) dx dy.$$

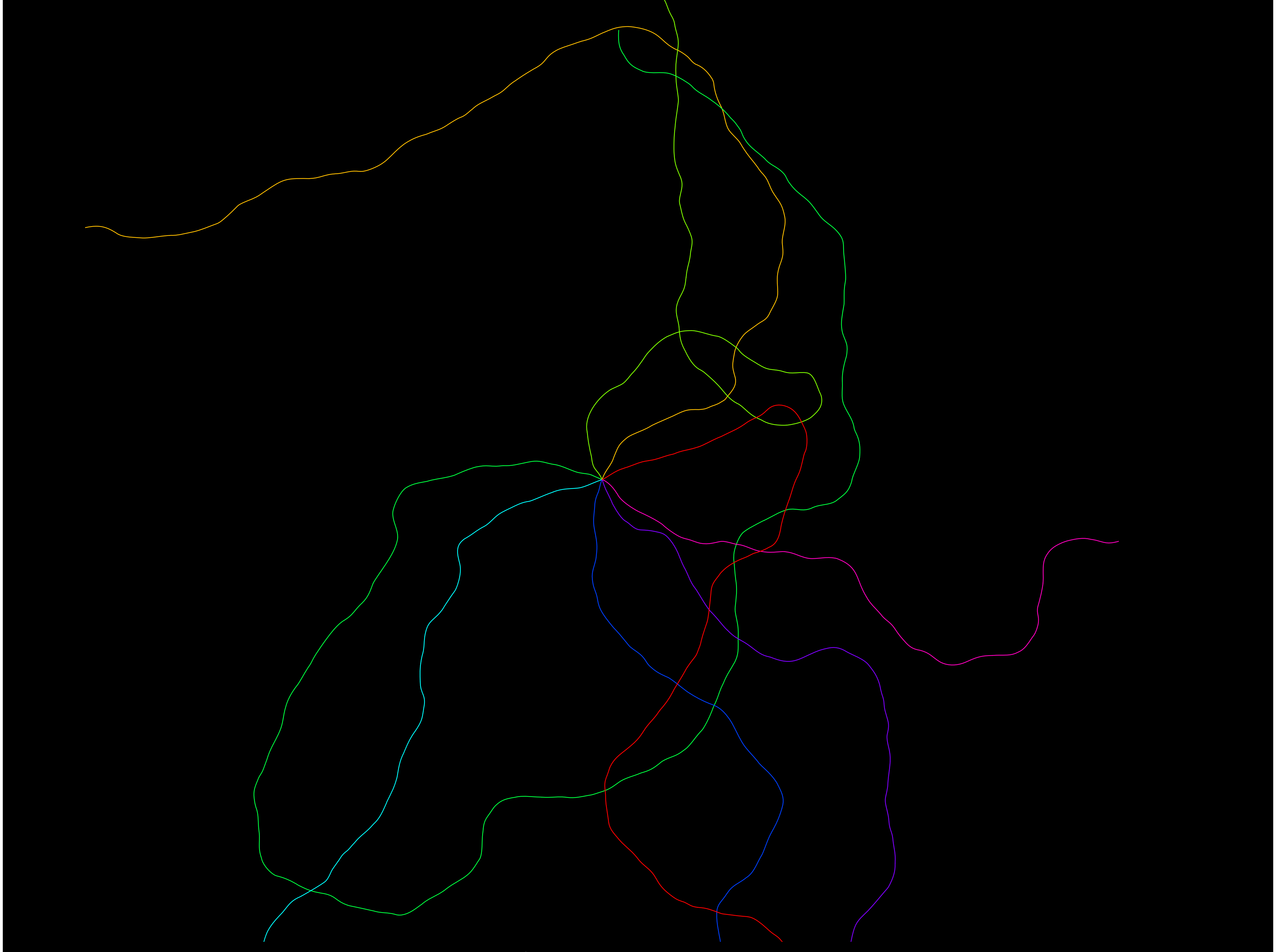
The GFF is the formal sum $h = \sum \alpha_i f_i$, where the f_i are an orthonormal basis for H and the α_i are i.i.d. Gaussians. The sum does not converge point-wise, but h can be defined as a *random distribution*—inner products (h, ϕ) are well defined whenever ϕ is sufficiently smooth.

Zero contour lines

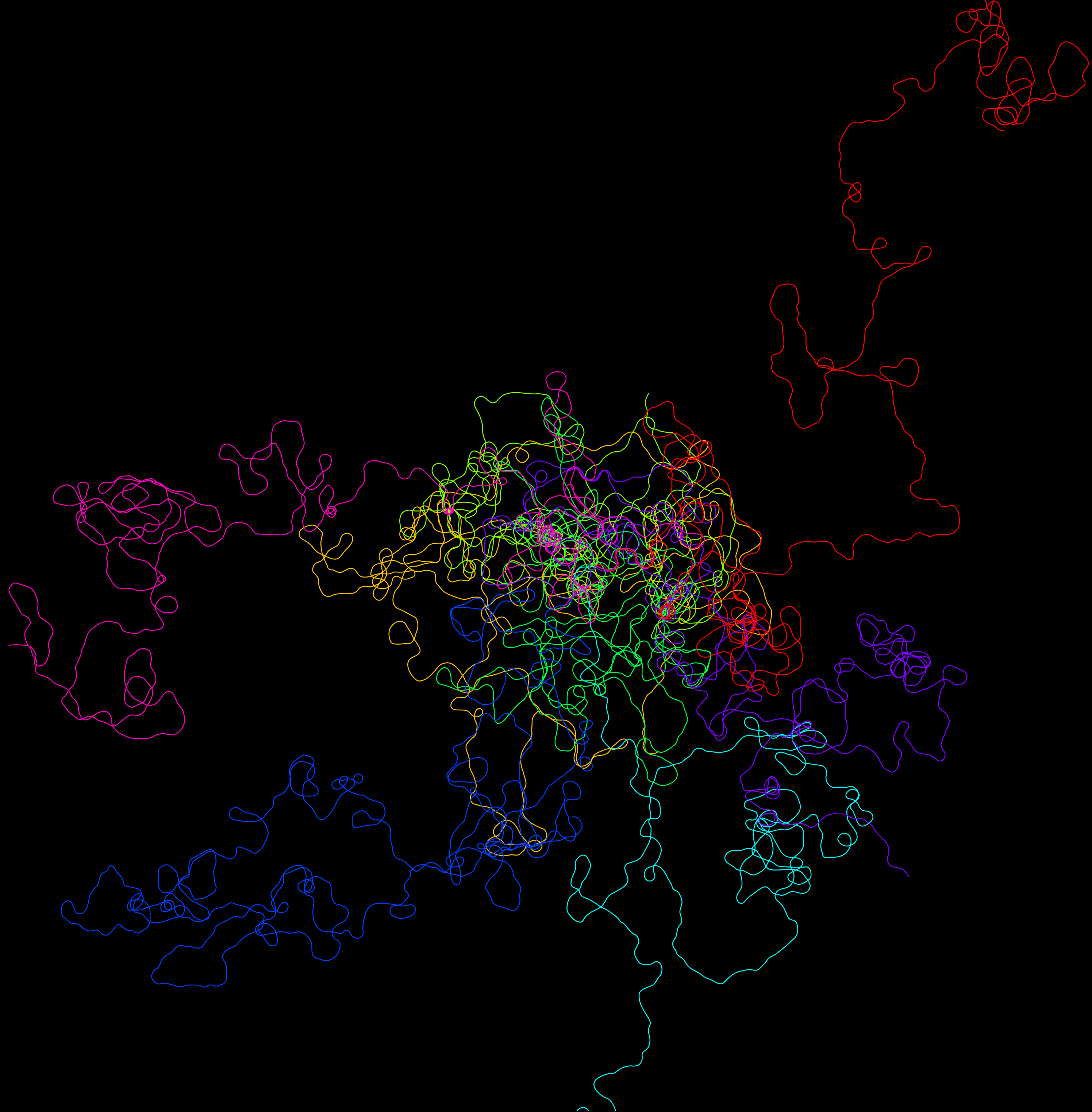




Geodesics flows of metric $e^h dL$ where h is .05 times the GFF.



Geodesics flows of metric $e^h dL$ where h is .2 times the GFF.



Geodesics flows of metric $e^h dL$ where h is 1 times the GFF.

"The gravitational action we are going to discuss has the form

$$S = \frac{d}{96\pi} \int_M (R \frac{1}{\Delta} R).$$

Here d^{-1} will play the role of coupling constant, Δ is a Laplacian in the metric g_{ab}, R is a scalar curvature and M is a manifold in consideration. This action is naturally induced by massless particles and appears in the string functional integral.
The most simple form this formula takes is in the conformal gauge, where $g_{ab} = e^{\phi}\delta_{ab}$ where it becomes a free field action. Unfortunately this simplicity is an illusion. We have to set a cut-off in quantizing this theory, such that it is compatible with general covariance. Generally, it is not clear how to do this. For that reason, we take a different approach..."

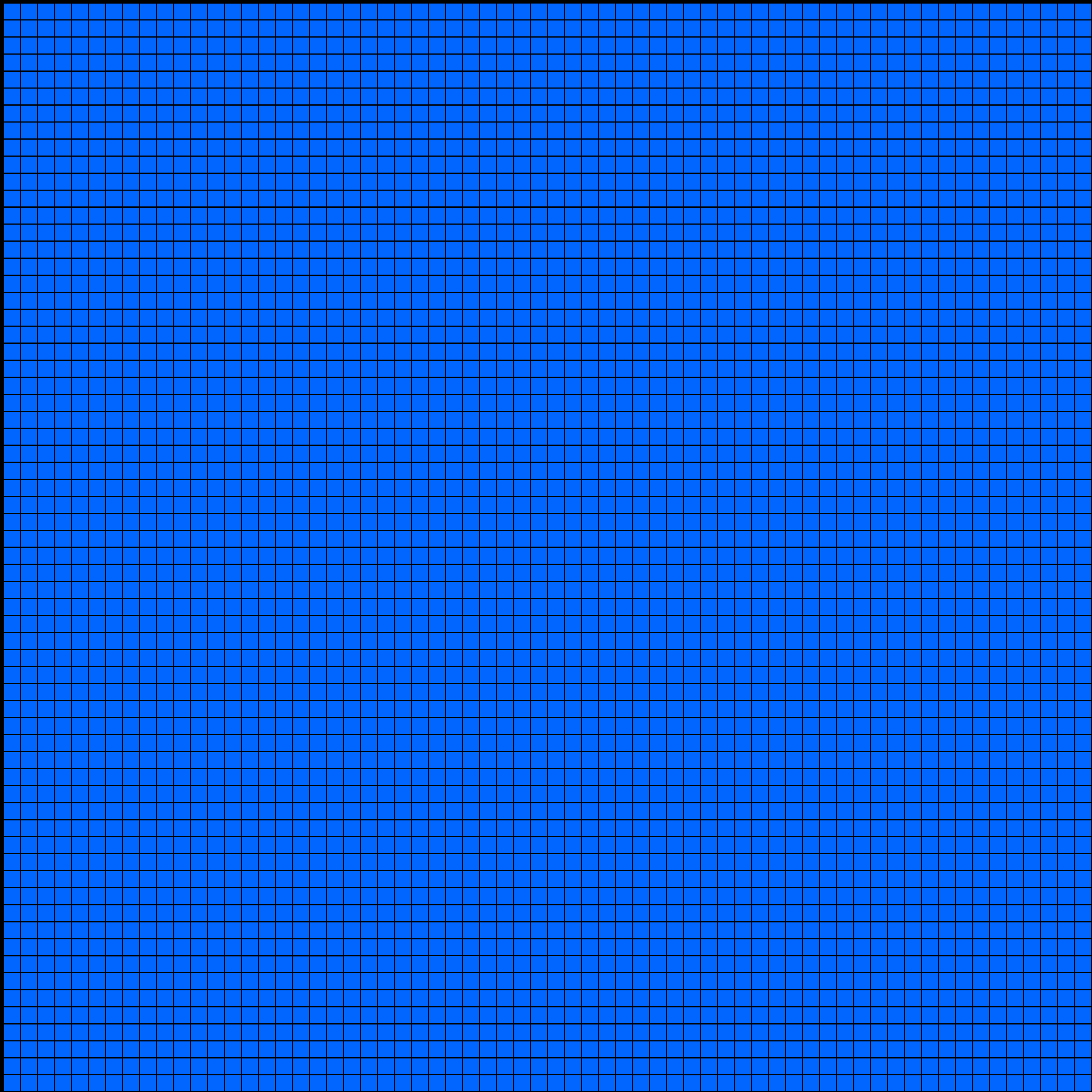
A.M. Polyakov, Moscow 1987

Constructing the random metric

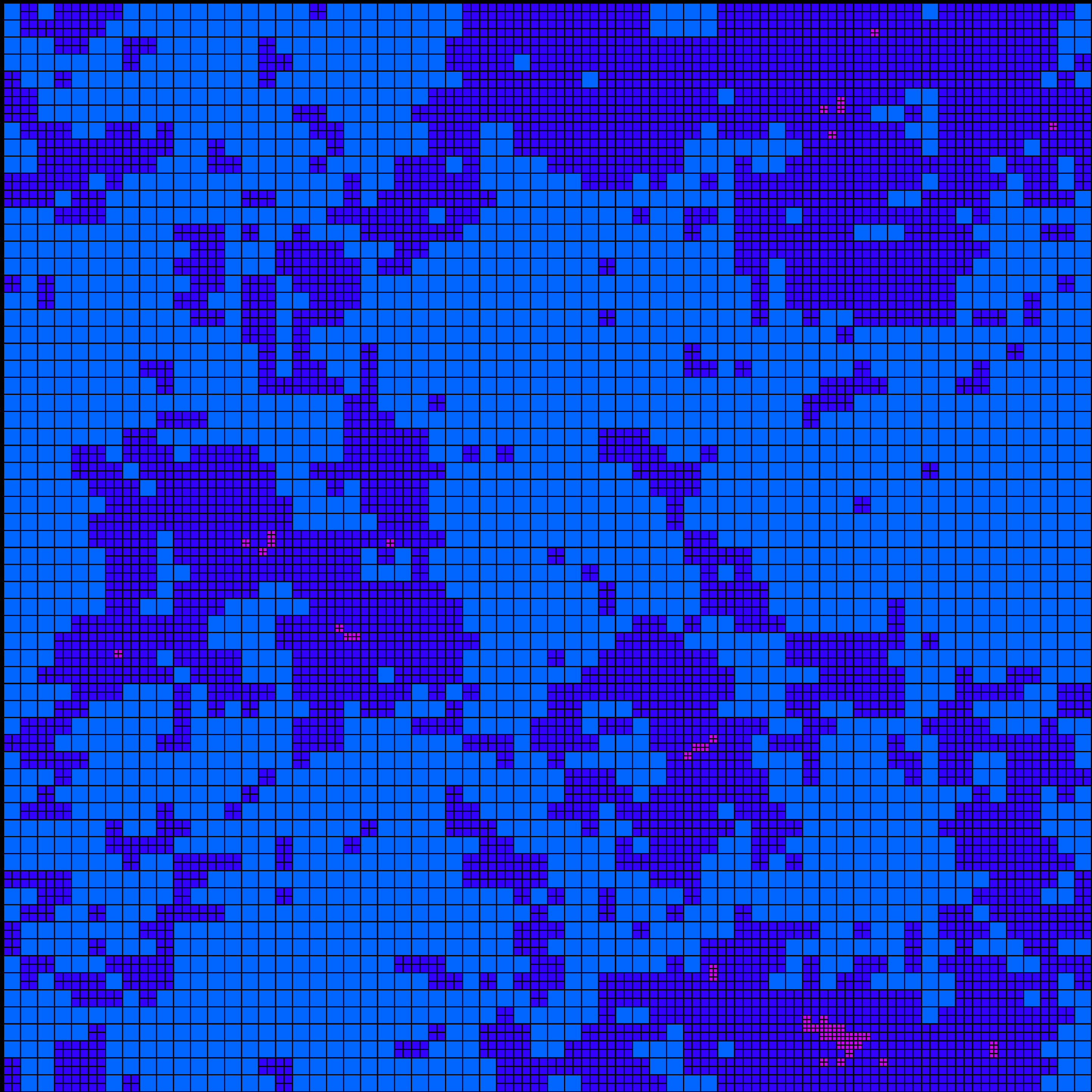
Let $h_\epsilon(z)$ denote the mean value of h on the circle of radius ϵ centered at z. This is almost surely a locally Hölder continuous function of (ϵ, z) on $(0, \infty) \times D$. For each fixed ϵ, consider the surface $\mathcal{M}_\epsilon$ parameterized by D with metric $e^{\gamma h_\epsilon(z)}(dx^2 + dy^2)$.

We define $\mathcal{M} = \lim_{\epsilon \to 0} \mathcal{M}_\epsilon$, but what does that mean?

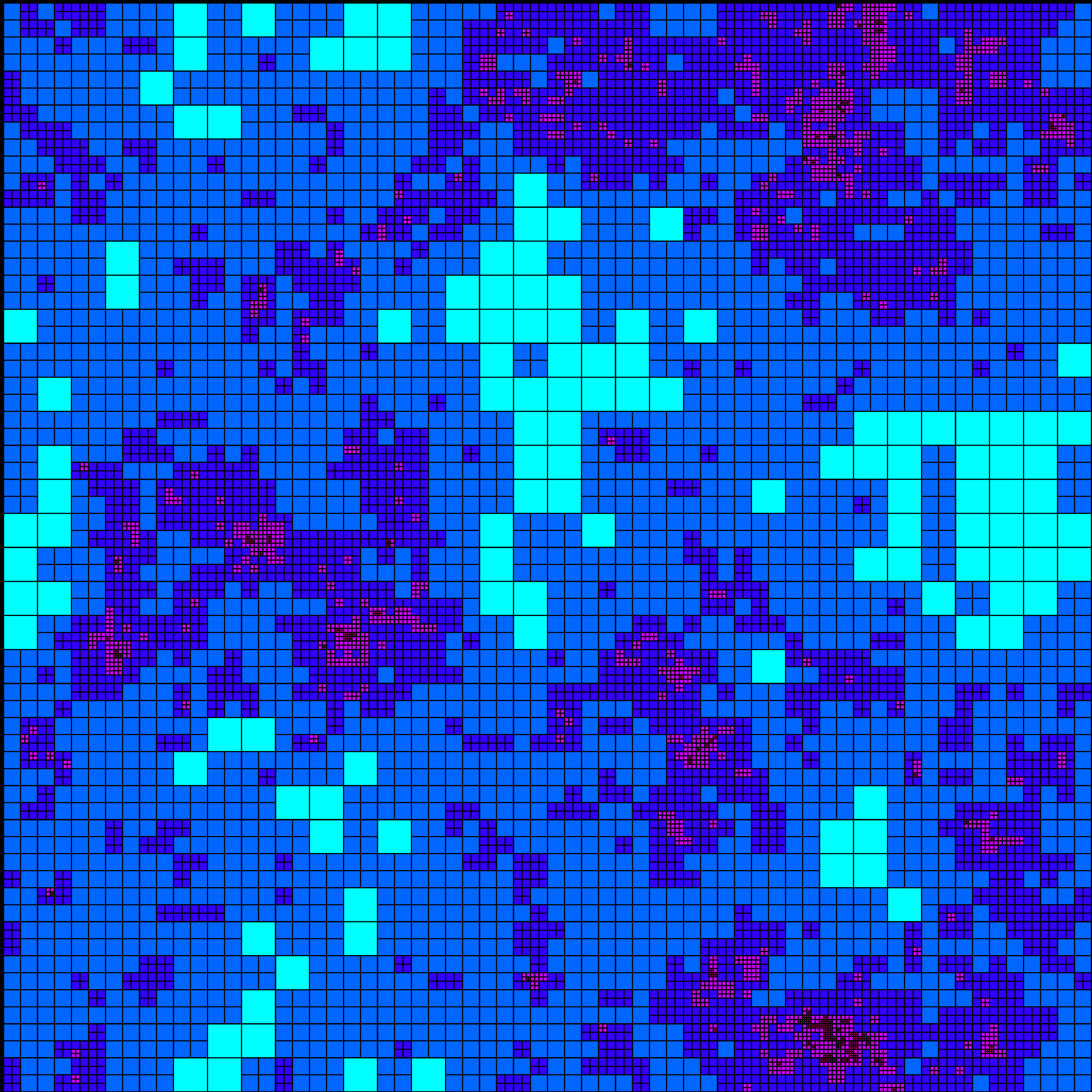
PROPOSITION: Fix $\gamma \in [0, 2)$ and define h, D, and μ_ϵ as above. Then it is almost surely the case that as $\epsilon \to 0$ along powers of two, the measures $\mu_\epsilon := \epsilon^{\gamma^2/2} e^{\gamma h_\epsilon(z)} dz$ converge weakly to a non-trivial limiting measure, which we denote by $\mu = \mu_h = e^{\gamma h(z)} dz$.



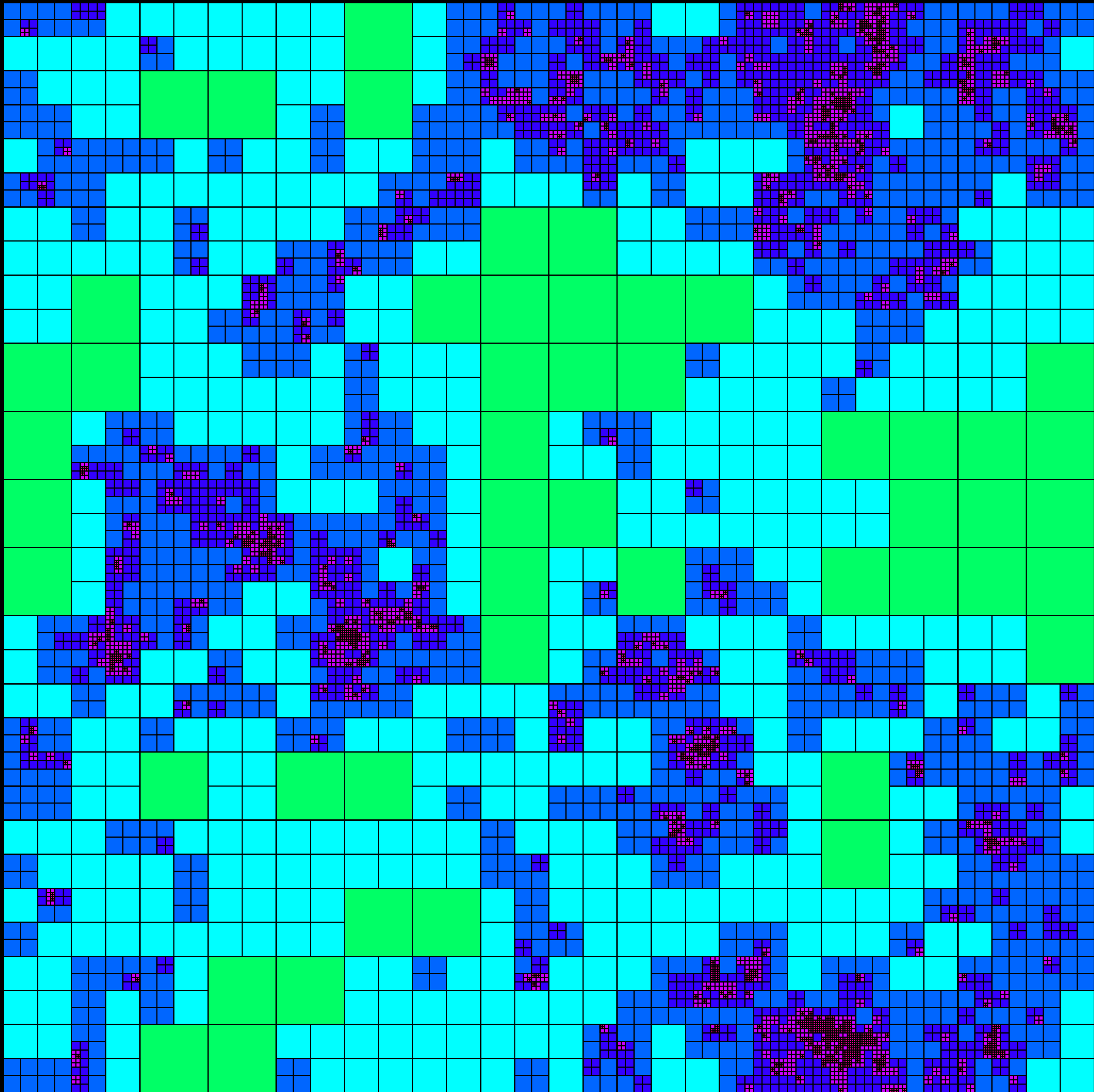
Area/4096 square decomposition of $e^{\gamma h} d^2 z$ for $\gamma = 0$



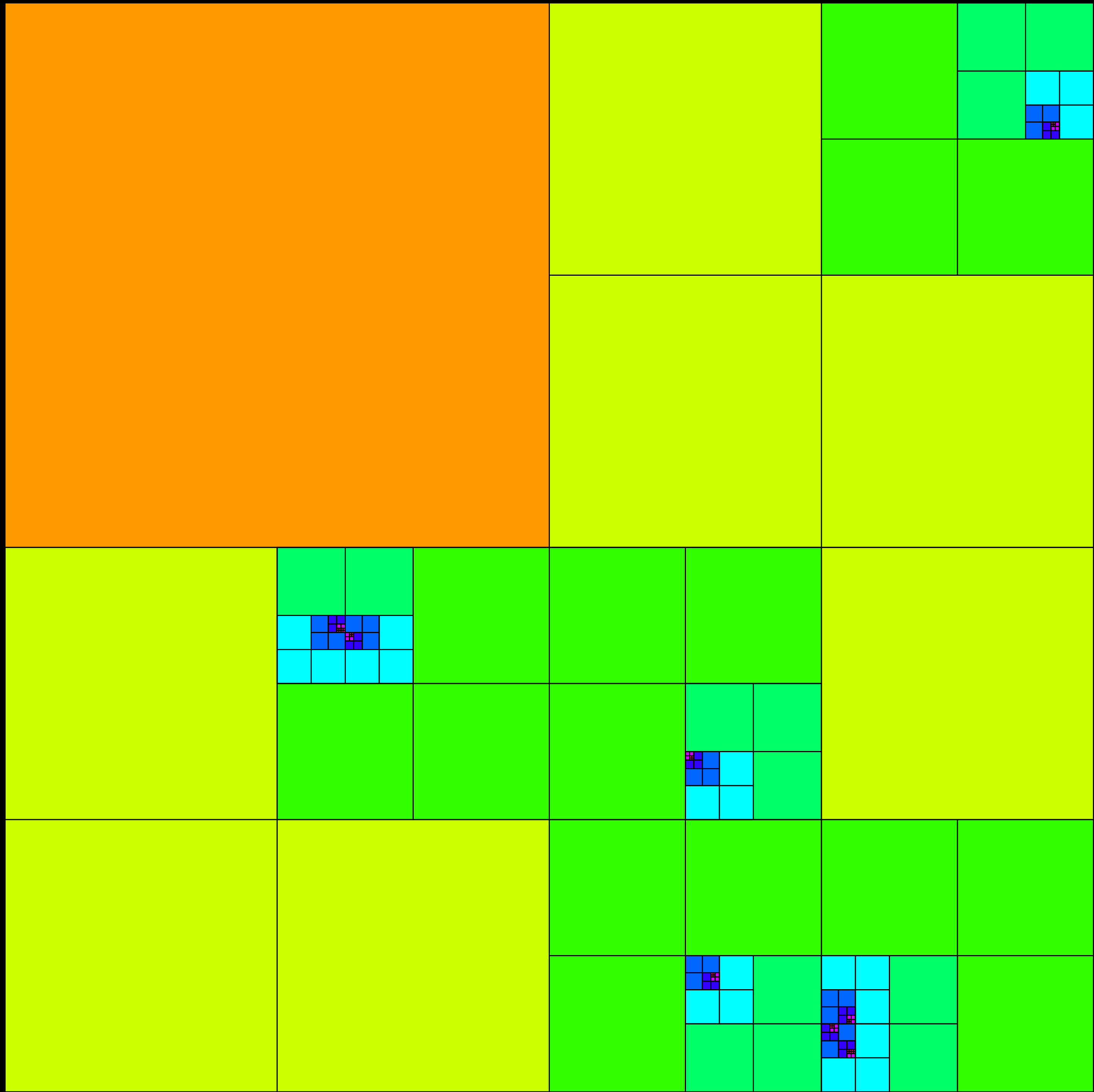
Area/4096 square decomposition of $e^{\gamma h} d^2 z$ for $\gamma = 1/2$



Area/4096 square decomposition of $e^{\gamma h} d^2 z$ for $\gamma = 1$



Area/4096 square decomposition of $e^{\gamma h}d^2z$ for $\gamma = 2$



Area/4096 square decomposition of $e^{\gamma h} d^2 z$ for $\gamma = 10$

Knizhnik-Polyakov-Zamolodchikov (KPZ) Formula

THEOREM [Duplantier, S.]: Fix $\gamma \in [0,2)$ and let X be a random subset of a deterministic compact subset of D. Let $N(\mu,\delta,X)$ be the number of (μ,δ) boxes intersected by X and $N(\epsilon,X)$ the number of diadic squares intersecting X that have edge length ϵ (a power of 2). Then if

$$\lim_{\epsilon\to 0} \frac{\log \mathbb{E}[N(\epsilon,X)]}{\log \epsilon^2} = x - 1.$$

for some $x > 0$ then

$$\lim_{\delta\to 0} \frac{\log \mathbb{E}[N(\mu,\delta,X)]}{\log \delta} = \Delta - 1,$$

where Δ is the non-negative solution to

$$x = \frac{\gamma^2}{4}\Delta^2 + \left(1 - \frac{\gamma^2}{4}\right)\Delta.$$