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Some discrete models of growth and interfaces, and their continuum counterpart (A short introduction to critical interfaces in 2d)

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A short introduction to critical interfaces in 2d

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Foreword

The purpose of these notes is very modest. They are meant to introduce readers gently to the concepts of Loewner chains, local growth and stochastic Loewner evolutions (SLEs). These concepts have played an important role in physics and mathematics during the recent years.

The first chapter describes two discrete examples, the exploration process and loop-erased random walks. It can be read almost without any prerequisites. The aim is to show that even for curves defined purely in geometrical terms, it is useful to have a statistical mechanics viewpoint where the measure on curves is derived from a measure on local degrees of freedom of some model. A third model, DLA is also introduced.

The second chapter introduces Loewner chains and their relevance for the description of growth processes. A prerequisite is a minimal knowledge of complex analysis.

The third (and last) chapter contains the derivation of the relevance of SLE in the description of interfaces when two properites, conformal invariance and the domain Markov property, are assumed/proved. The prerequisites are some knowledge of complex analysis and probability theory.

The discussion is informal. There is little or no claim at originality. We try to give some intuition based on explicit examples. Physical applications are sometimes mentioned but never explained in detail.

There is no reference but it is easy to find more detailed and/or rigorous and/or applied and/or ... presentations with references in the available

literature on the web by typing keywords.

Chapter 1

Discrete models

Random curves have focussed the interest of physicists and mathematicians for decades. The simplest and perhaps oldest example is the random walk or its continuous counterpart, Brownian motion. For dimensions ≤ 4 it is not a simple curve. On the other hand, polymers have a strong tendency to be self avoiding, and they can be modeled crudely as random simple walks with a statistical weight giving fugacity μ to each monomer. But there is a wealth of interesting models of random simple walks. Among them are interfaces in 2d systems. Under certain circumstances, such systems are expected to have a continuum limit.

Recently a lot of progress has been achieved. First, a classification of random curves in the continuum with certain special properties has been obtained. It has received the name "Stochastic Loewner Evolutions", and it is the subject of chapter 3. It is hard to overestimate the impact of SLE : it has given tools to solve formidable problems by routine computations, but moreover it has made it possible to prove that families of random simple walks and interfaces have a continuum limit.

The purpose of the examples that follow is to illustrate the connexion between geometrical random curves and statistical mechanics. It turns out that partition functions under various disguises play a huge role in the study of SLE

1.1 Discrete domains

In what follows, a domain $\mathbb{D}$ is an open simply connected (i.e. no holes) strict subset in the euclidean plane.

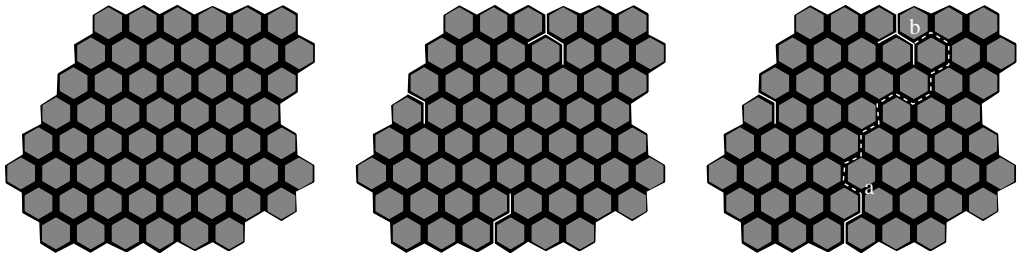

Figure 1.1: Left: a smooth hexagonal domain. Middle: a non-smooth hexagonal domain. Right: an admissible boundary condition.

The euclidean plane admits regular tilings by hexagons, by triangles or by squares. The following definitions are given for hexagonal tilings, but they can easily be adapted for tilings by triangles and squares.

All hexagonal tilings can be obtained from one of them by similarities (in complex notation $z \mapsto \lambda z + \rho$. Fix such a tiling $\mathcal{T}$, for instance one whose hexagons have unit area . The plane is the disjoint union of vertices , open edges and open faces of $\mathcal{T}$: every point in the plane is either a vertex, or an interior point of an edge, or an interior point of a face.

An hexagonal domain $\mathbb{D}$ with reference $\mathcal{T}$ is a domain in the usual sense which is the union of vertices, open edges and open faces of $\mathcal{T}$.

This definition accomodates "smooth" domains like the left one in Fig.1.1 whose boundary is a simple curve but also more irregular shapes like the middle one in Fig.1.1 whose boundary is not a simple curve. If $\varepsilon > 0$ is much smaller than the size of an edge of $\mathcal{T}$, the points in the hexagonal domain $\mathbb{D}$ whose distance to the complement of $\mathbb{D}$ is ε form a simple curve, but the limit $\varepsilon \to 0^+$ is singular. The knowledge of the side from which a boundary point is approached is naively lost in the limit, but one can decide to keep track of it and this is the most useful definition of boundary in this context. For hexagonal domains we have thus a notion of boundary wich makes it a curve even for a non smooth domain[1].

An admissible boundary condition is a couple of distinct vertices (a, b) of $\mathcal{T}$, $a, b \notin \mathbb{D}$ such that there is a path (or simple walk) from a to b in $\mathbb{D}$ i.e. a number $n \geq 1$ and a sequence $s_1, \cdots, s_{2n+1}$ where $a = s_1, b = s_{2n+1}$,

[1]That such a boundary can also be defined for general domains is a nontrivial matter. This is related to the theory of "prime ends" and of the so-caled Poisson/Martin boundaries that parametrize solutions of the Laplace equation.

the s_{2m+1}, $1 \leq m < n$, (if any) are distinct vertices of $\mathcal{T}$ in $\mathbb{D}$ and the s_{2m}, $1 \leq m < n$, are distinct edges of of $\mathcal{T}$ in $\mathbb{D}$ with boundary $\{s_{2m-1}, s_{2m+1}\}$. This is illustrated on the left of Fig.1.1. Any such path splits $\mathbb{D}$ into a left and a right piece.

If $s_1, \cdots, s_{2n+1}$ is a path from a to b in $\mathbb{D}$ and $0 \leq m < n$, the set $\mathbb{D}'$ obtained by removing from $\mathbb{D}$ the sets s_l, $1 < l \leq s_{2m+1}$ is still a domain, and (s_{2m+1}, b) is an admissible boundary condition for $\mathbb{D}'$.

Our main interest in the next subsections will be in measures on paths from a to b in $\mathbb{D}$ when $\mathbb{D}$ is an domain and (a, b) an admissible boundary condition.

Hexagonal domains have a special property which is crucial for what follows. Suppose $(\mathbb{D}, a, b)$ is an hexagonal domain with admissible boundary condition. The right (resp. left) hexagons are by definition those which are on the right (resp. left) of every path from a to b in $\mathbb{D}$. Left and right hexagons are called boundary hexagons. The other hexagons of $\mathbb{D}$ are called inner hexagons[2]. Color the left hexagons in black (say) and the right hexagons in white as in Fig.1.2 on the left. If one colors the inner hexagons arbitrarily in black or white, then there is a single path from a to b in $\mathbb{D}$ such that the hexagon on the left (resp. right) of any of its edges is black (resp. white). This is illustrated in Fig.1.2 on the right. This path can be defined recursively because a is on the boundary of at least one left and at least one right hexagon: as a is not in $\mathbb{D}$, in any coloring there is exactly one edge in $\mathbb{D}$ with a on its boundary and bounding two hexagons of different colors. Start the path with this edge and go on.

If $\mathbb{D}$ is an arbitrary domain and $\lambda \neq 0, \rho$ are complex numbers, one can approximate $\mathbb{D}$ by an hexagonal domain with reference tiling $\lambda\mathcal{T} + \rho$. It is useful for the general theory to have a quantititive notion of how close such an approximation is to the original domain and to have quantitiative notion of convergence of approximations (when λ gets smaller and smaller) that garanties that some phenomena on discrete domains (for instance some properties of certain statistical mechanics models) have an interpretation in the limit. We shall not give a formal definition of convergence, but simply mention that it exists.

All the examples of interfaces we shall deal with in the sequel can be defined on arbitrary hexagonal domains with admissible boundary condi-

[2]Note that being a boundary or an inner hexagon depends on (a, b).

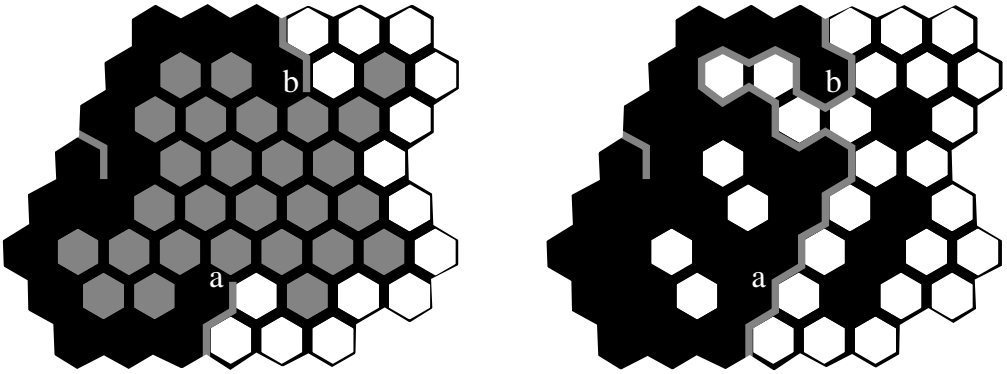

Figure 1.2: Left: the boundary of an hexagonal lattice domain with boundary conditions. Right: the interface associated to a configuration.

tion, though sometimes we shall use square domains. Certain geometrical examples will define directly a law for the interface or a probabilistic algorithm to construct samples. Examples from statistical mechanics will give a weight for each coloring of the inner hexagons, and the law for the interface can be derived (at least in principle) from this more fundamental weight. The model of interface can depend on some parameters, called collectively p (for instance, temperature can be one of those).

Consider an interface model with parameter family p on discrete domains. Fix an arbitrary domain with two marked boundary points, $(\mathbb{D}, a, b)$ and approximate it by a "convergent" sequence of discrete domains with boundary conditions $(\mathbb{D}_n, a_n, b_n)$ whose reference tiling $\lambda_n\mathcal{T}$ has scale $\lambda_n \to 0^+$. A continuum limit exists when there is a (domain independent) function $p(\lambda)$ such that the distribution of interfaces in $(\mathbb{D}_n, a_n, b_n)$ with parameters $p(\lambda_n)$ converges to some limit. The limiting value $p(0)$ is called the critical value p_c. The choice $p(\lambda) = p_c$ leads to a scale invariant theory.

A map $f : \mathbb{D} \to \mathbb{D}'$ between two domains sending marked boundary points to marked boundary points (i.e. $f(a) = a'$ and $f(b) = b'$) is said to be conformal if it preserves angles. Riemann's theorem, to be explained in more detail in section 2.1 asserts the existence of such maps. One can then ask, for a given interface model, whether the distribution of interfaces in $(\mathbb{D}, a, b)$ and in $(\mathbb{D}', a', b')$ are conformally equivalent. This can be checked numerically on good lattice approximations of these domains.

In two dimensions scale invariance plus locality is often enough to ensure conformal invariance. Thus the limiting theory of a discrete model at $p = p_c$ is a good candidate for confromal invariance. More generaly, there is often a threshold function $p_s(\lambda)$ such that if $p(\lambda) - p_c = o(p_s(\lambda) - p_c)$ the limiting continuum theory is the same as the critical theory, if $p(\lambda) - p_c \propto p_s(\lambda) - p_c$ a limiting continuum theory exist but is not scale invariant, and if $p_s(\lambda) - p_c = o(p(\lambda) - p_c)$ the limiting theory either does not exist or is trivial in some sense (concentrated on a single curve for instance). It is clear that only the small λ behavior of p_s matters and commonly $p_s(\lambda) - p_c$ can be taken to be simply a power of λ. The exponent is one of the critical exponents of the model.

1.2 Exploration process

1.2.1 Definition

Let $(\mathbb{D}, a, b)$ be an hexagonal domain with admissible boundary condition. Color the left hexagons in black (say) and the right hexagons in white. If a is incident to no inner hexagon of $\mathbb{D}$, all paths from a to b in $\mathbb{D}$ start with the same edge. Else, a is incident to exactly one inner hexagon of $\mathbb{D}$. Color it black or white using a biased coin (say black has probability p and white $1 - p$), and make a step along the edge of $\mathbb{D}$ adjacent at a whose adjacent faces have different colors. Then remove from $\mathbb{D}$ the edge corresponding to the first step and its other end point, call it $\dot{a}$ to get a new domain $\dot{\mathbb{D}}$. If $\dot{a} = b$ stop. Else $(\dot{\mathbb{D}}, \dot{a}, b)$ is a new hexagonal domain with admissible boundary condition and one can iterate as shown on the Fig.1.3. Each choice of color is made independently of the preceeding ones but with the same bias. This random process is called the exploration process, and by construction it results in a simple path from a to b.

The fact that at some times the next step can be decided without tossing (for example, in Fig.1.3, for the transition from the second picture to the third one, the choice of color for one hexagon is enough to fix two steps of the exploration path) results in a subtle interaction between the abstract independant coin tossings and their intricate effect on the geometry of the path.

There is exactly one coin toss for each inner face of $\mathbb{D}$ touching an edge of the path : this toss takes place the first time the inner face is touched by

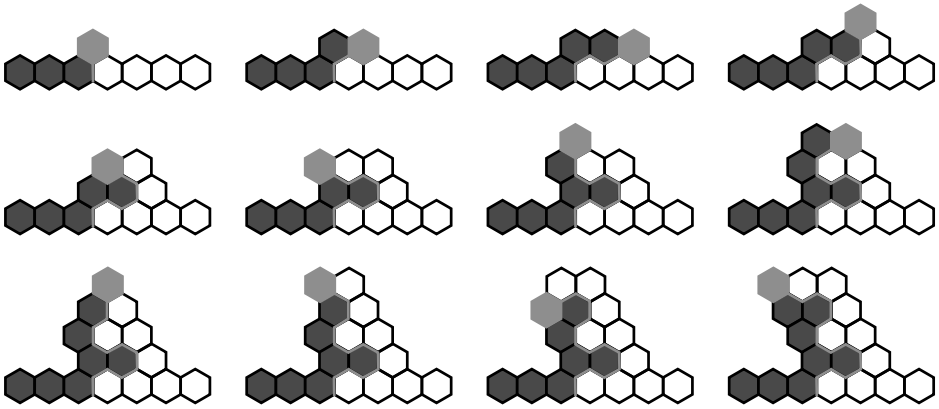

Figure 1.3: The exploration process.

the tip of the path. In the rest of the process, this face becomes a boundary hexagon. But the path can have more than one egde along it.

The exploration process has a very important property: locality. It means that if $(\mathbb{D}, a, b)$ and $(\mathbb{D}', a, b)$, $\mathbb{D}' \subset \mathbb{D}$ are two hexagonal domains with the same admissible boundary condition (a, b), the distributions of the exploration path in $\mathbb{D}$ and $\mathbb{D}'$ coincide up to the first time it touches a boundary hexgagon of $\mathbb{D}'$ wich is an inner hexagon of $\mathbb{D}$. This notion of locality should not be confused with the notion of locality in quantum field theory.

By symmetry, if there is a single value of p for which the theory is critical, it has to be $p_c = 1/2$ and the numerics confirms this intuition. Fig.1.4 shows a few samples of the symmetric exploration process. They join the middle horizontal sides of similar rectangles of increasing size. The pseudo random sequence is the same for the four samples.

Even for small samples, the exploration process makes many twists and turns. By construction, the interface is a simple curve, but with the resolution of the figure, the exploration proacess for large samples does not look like a simple curve at all!

To estimate the (Hausdorff, fractal) dimension of the symmetric exploration process, one can generate samples in similar rectangular domains of different sizes and made the statistics of the number of steps S of the interface as a function of the size L of the rectangle domain. One observes that $S \propto L^{\delta}$ and a modest numerical effort (a few hour of CPU) leads to

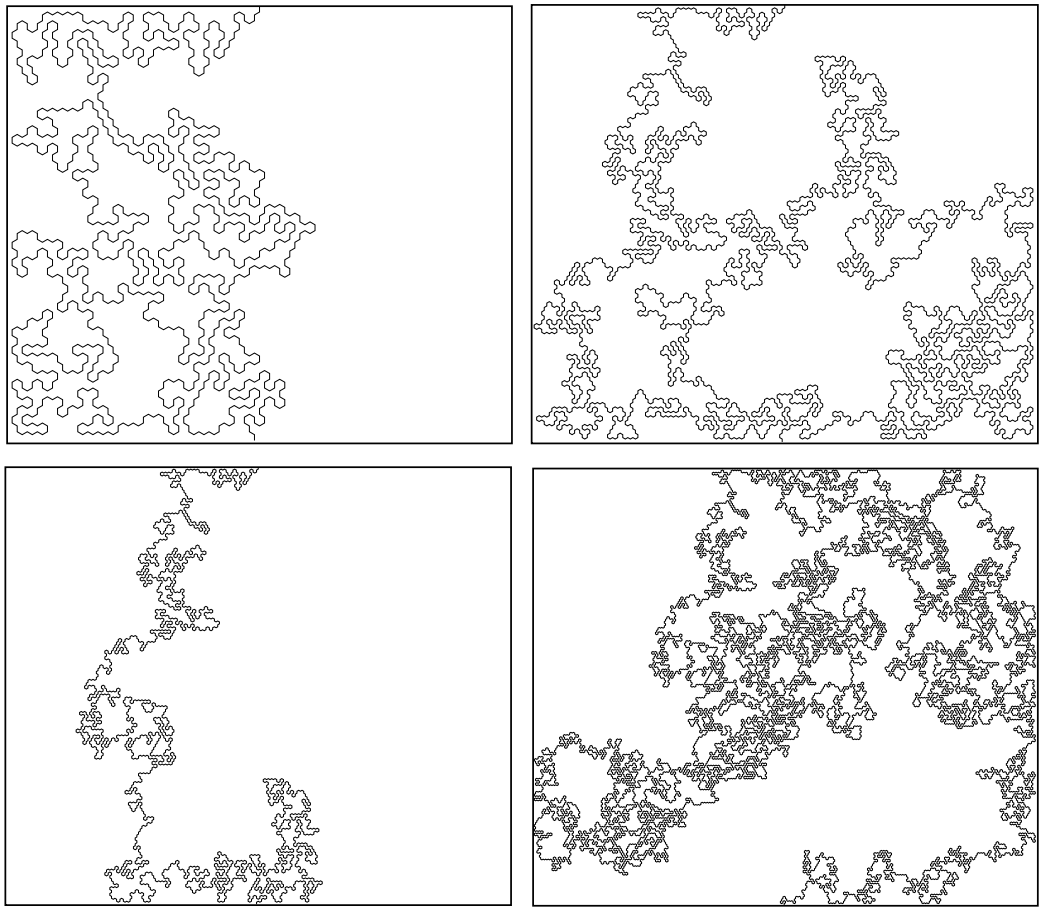

Figure 1.4: Samples of the exploration process for increasing sizes.

$\delta = 1.75 \pm .01$. To get an idea of how small the finite size corrections are, observe Fig.1.5.

The exploration process is build by applying local rules involving only a few nearby sites, and we could wave our hands to argue that its scale invariance (for $p = p_c = 1/2$) should imply its conformal invariance in the continuum limit. But the exploration process is one which has been rigorously proved to have a conformally invariant distribution in the continuum limit, the fractal dimension being exactly $7/4$. As suggested by numerical simulations, the continuum limit does not describe simple curves but curves with a dense set of double points, and in fact the –to be defined later– SLE_6 process describes not only the exploration path but also the exploration hull, which is the complement of the set of points that can be

joined to infinity by a continuous path that does not intersect the exploration path. As we shall mention later, among SLE_κ's, SLE_6 is the only one that satisfies locality, so what is really to prove in this case is conformal invariance in the continuum limit (a nontrivial task), and the value of κ is for free.

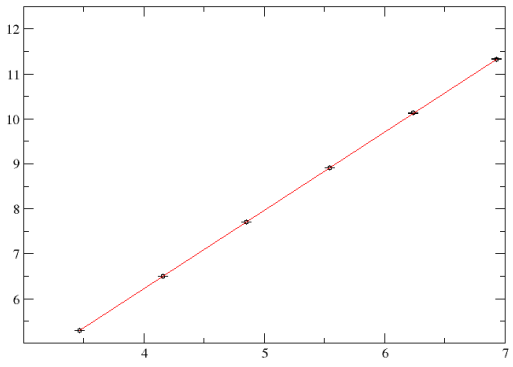

Figure 1.5: The logarithm of the average length of the exploration path versus the logarithm of the domain size. The numerical results are the circles, the straight line is the linear regression, the error bars are shown.

1.2.2 Relation to percolation and coupling

The exploration process has been presented as a growth process, but in fact it is related to statistical mechanics in a simple way. Indeed, suppose that once the exploration sample has been constructed one tosses repeatedly (independently) the same coin to color also the hexagons that have not been colored during the construction of the path. One gets a configuration in which all hexagons have been colored independently, and from which the exploration path can be reconstructed has the sole curve joining the marked points with bording hexagons all black one the left side and white on the right side. So one could also construct exploration samples by coloring all the hexagons independently at once and then drawing the separating curve. So the exploration path is the interface for the statistical mechanics of percolation.

Of course this appproach is a poor idea for numerical simulations of the exploration process for a fixed p, because many hexagons are colored for no use. But it has several advantages. First, it shows plainly that the law for the exploration process is reversible (i.e. the choice of which of the two marked is used to start the exploration path is irrelevant). Second, percolation can be studied with other boundary conditions. Third, it makes it possible to use the powerful probabilistic tool of coupling.

It happens frequently that on some measure space $(A, \mathcal{F})$ one has to deal with a family of probability laws P_u where u is some parameter. In some favorable circumstances, one can find another probability space $(E, \mathcal{G}, \mu)$ and a family of measurable maps $f_u : E \to A$ such that the image measure of μ by f_u is P_u: if B is a measurable subset of A (i.e. $B \in \mathcal{F}$) then $f_u^{-1}(B)$ is a measurable subset of E (i.e. $f_u^{-1}(B) \in \mathcal{G}$) and $P_u(B) = \mu(f_u^{-1}(B))$. Thus we can fix a configuration in E and by changing u see a "movie" of configurations in A. This is known as *coupling*.

In the case of percolation, it is easy to find a coupling. Let the parameter u vary in $[0, 1]$. If H is the set of inner hexagons (the ones whose colors are not fixed by boundary conditions) of some finite hexagonal domain, set $A = \{b, w\}^H$ with $\mathcal{F} = \mathcal{P}(A)$ (all subsets of A are measurable), and set $E = [0, 1]^H$ with μ the product Lebesgue measure. So A is the set of assignments of a color, b(lack) or w(hite), to each inner hexagon, and E is the set of assignments of a real number $\in [0, 1]^H$ to each inner hexagon. A configuration in A can be seen equivalently as a map from H to $\{b, w\}$, or as a partition of H in black and white. If $x \in [0, 1]$, set $f_u(x) = b$ if $u < x$ and $f_u(x) = w$ if $u \geq x$. Use the product structure of A and E to define $f_u : E \to A$ so that an hexagon h is white if and only if its assigned value is $\geq u$. Obviously the image measure of μ by f_u colors the inner hexagons independently, each being black with probability $1 - u$ and white with probability u.

In such a setting, a useful tool is Russo's formula. Let us derive it abstractly and then interpret it. Suppose we partition A in two subsets $A = B \cup W$ in such a way that being in W is a so-called *increasing property*: if $\gamma \in W$ and if $\gamma' \in A$ is such that all hexagons which are white in the configuration γ are also white in the configuration γ' then $\gamma' \in W$. If we order $\{b, w\}$ by saying that $w > b$ and view γ and γ' as maps from H to $\{b, w\}$, this says that if $\gamma \in W$ and $\gamma' \geq \gamma$ then $\gamma' \in W$. Then it is intuitively clear, and coupling makes it obvious, that $P_u(W)$ is an increasing

function of u. If $\gamma \in A$ is a configuration, call an hexagon h pivotal for $A = B \cup W$ in the configuration γ either if $\gamma \in B$, h is colored in black and changing it into white yields a configuration in W or if $\gamma \in W$, h is colored in white and changing it into black yields a configuration in B. In the first case, we say that h is pivotal in γ to enter W and in the second case that h is pivotal in γ to enter B. In each configuration $\gamma \in A$ there is a certain number (possibly 0) of pivotal hexagons $\Pi(\gamma)$, and Π is thus a random variable on A. Russo's formula states that

$$\frac{d}{du} P_u(W) = E_u(\Pi),$$

i.e. that the derivative of $P_u(W)$ is the expected number of pivotal points for the probability P_u.

The proof is easy. We shall prove a slightly more refined identity, namely that if $\Pi_W(\gamma)$ is the number of hexagons in γ pivotal to enter W then $(1-u)\frac{d}{du}P_u(W) = E_u(\Pi_W)$. By symmetry if $\Pi_B(\gamma)$ is the number of hexagons in γ pivotal to enter B then $u\frac{d}{du}P_u(W) = E_u(\Pi_B)$. As $\Pi = \Pi_W + \Pi_B$, the sum of these two equalities gives Russo's formula. Suppose $0 \leq u < v \leq 1$. By definition $P_v(W) - P_u(W) = \mu(f_v(X) \in W) - \mu(f_u(X) \in W)$ and by the increasing property of W this is $\mu(f_v(X) \in W \text{ and } f_u(X) \notin W)$. We can split this as a double sum to get

$$P_v(W) - P_u(W) = \sum_{\beta \in B} \sum_{\omega \in W} \mu(f_v(X) = \omega \text{ and } f_u(X) = \beta).$$

Note that the summand can be nonzero only if $\beta < \omega$ i.e. if one can go from β to ω by turning some black hexagons to white ones because this is what happens to $f_\cdot(X)$ for a fixed X by tuning the parameter from u to v. For a given X the hexagons h that change color are those for which $X(h) \in]u, v]$, so in the above double sum only the pairs (β, ω) which disagree on a single hexagon can contribute to first order in $v - u$.

For instance we can sum first over β's to get

$$P_v(W) - P_u(W) = \sum_{\beta \in B} \sum_{h \text{ pivotal in } \beta} \mu(f_u(X) = \beta \text{ and } X(h) = \in]u, v]) + O((v-u)^2).$$

But by the definition of μ and f_u, $\mu(f_u(X) = \beta \text{ and } X(h) = \in]u, v]) = \mu(f_u(X) = \omega)\frac{v-u}{1-u} = P_u(\beta)\frac{v-u}{1-u}$. So

$$P_v(W) - P_u(W) = \frac{v-u}{1-u} \sum_{\beta \in B} P_u(\beta) \#\{\text{pivotal points in } \beta\} + O((v-u)^2).$$

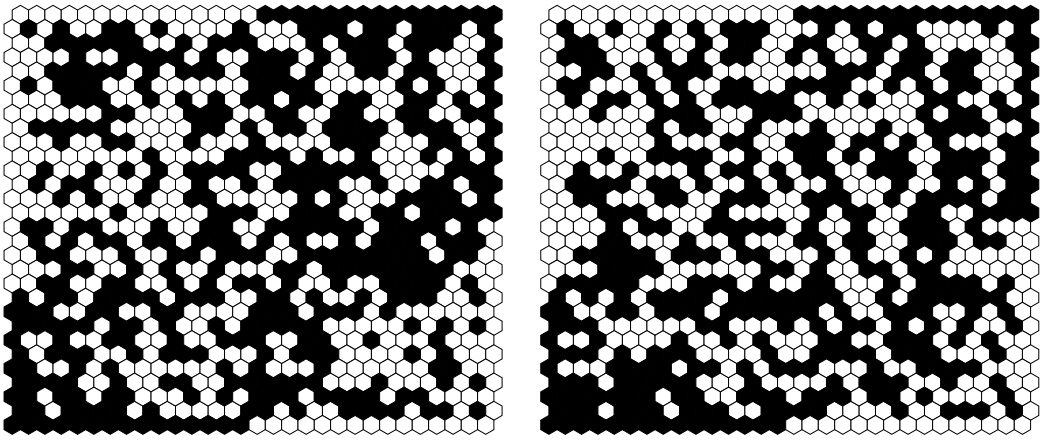

Figure 1.6: Left: a sample of critical percolation with a black crossing and no pivotal point. Right: a sample of critical percolation with a white crossing and several pivotal points. Find these!

The sum is just the expected number of pivotal points to enter W for P_u, and taking the limit leads to the annouced result :

$$(1-u)\frac{d}{du}P_u(W) = E_u(\Pi_W).$$

Had we decided to sum first over ω's, we would have obtained

$$u\frac{d}{du}P_u(W) = E_u(\Pi_B).$$

Now that we have proved Russo's formula abstractly, let us apply it to a concrete decomposition of A relevant for percolation. Take a domain and split its boundary into four segments, such that the colors of the hexagons are fixed on each segment but alternate from one segment to the next as on Fig.1.6. Then a simple topological argument shows that in any configuration either there is a black cluster connecting the two black boundary componants, or there is a white cluster connecting the two white boundary componants. In the first case put the configuration in B and in the second case put it in W. That being in W is an increasing property is clear. Pivotal hexagons are the ones which change the color of the connecting cluster, so they have a impact on the long range properties of a configuration. Fig.1.6 shows two samples.

Such pivotal points could be called "global pivotal points" because they are defined with respect to global boundary conditions. However, in an arbitrary configuration of percolation one can look at windows of a certain size and define pivotal points with respect to that window. Anyway, at $u = p_c = 1/2$, the number of pivotal points can be shown to behave like $(L/\lambda)^{3/4}$ where L is the linear size of the system and λ is the scale of the tiling. So $p_s(\lambda) - 1/2 = \lambda^{3/4}$ is a good candidate for the threshold function: by coupling, changing u from the critical value $1/2$ to $1/2 + g\lambda^{3/4}$ just flips of order 1 pivotal points, and Russo's formula indicates that $P_{1/2+g\lambda^{3/4}}(W) - P_{1/2}(W)$ is a finite function of g in the continuum limit. The validity of this threshold function is in fact rigorously proved (only in some weak sense at the moment, but progress is rapid).

So even if the exploration process can be defined purely as a geometric random curve model, it proves very useful to introduce a (seemingly trivial) two dimensional statistical mechanics model from whom the law of the the exploration process is obtained as a "marginal" of the full distribution.

1.3 Loop erased random walks

This example still keeps some aspects of a growth process, in that new pieces of the process can be added recursively. Loop-erased random walk were invented by Lawler as an example of random paths more tractable than the canonical self avoiding walks. A loop-erased random walk is a random walk with loops erased along as they appear.

1.3.1 Definition

More formally, if $X_0, X_1, \cdots, X_n$ is a finite sequence of abstract objects, we define the associated loop-erased sequence by the following recursive algorithm.

Initialize counters $l = 0$ and $m = 1$ and set $Y_0 = X_0$
Iterate while $m \leq n$
{
-If there is a k with $0 \leq k \leq l$ such that $Y_k = X_m$ set $l = k$
-Else increment l by 1 and set $Y_l = X_m$.
}

The loop-erased sequence is $Y_0, \cdots, Y_l$.

Let us look at two examples.
For the "month" sequence *j,f,m,a,m,j,j,a,s,o,n,d*, the first loop is *m,a,m*, whose removal leads to *j,f,m,j,j,a,s,o,n,d*, then *j,f,m,j*, leading to *j,j,a,s,o,n,d*, then *j,j* leading to *j,a,s,o,n,d* where all terms are distinct.
For the "reverse" month sequence *d,n,o,s,a,j,j,m,a,m,f,j*, the first loop is *j,j*, leading to *d,n,o,s,a,j,m,a,m,f,j*, then *a,j,m,a* leading to *d,n,o,s,a,m,f,j*.

This shows that the procedure is not "time-reversal" invariant. Moreover, terms that are within a loop can survive: in the second example *m,f*, which stands in the *j,m,a,m,f,j* loop, survives because the first *j* is inside the loop *a,j,m,a* which is removed first.

The above algorithm is most useful if the sequence $X_0, X_1, \cdots, X_n$ is viewed as a stream of data that is treated "on the fly". If $X_0, X_1, \cdots, X_n$ is known at once, another algorithm erases the loop in possibly fewer steps. It goes as follows:

Initialize counters $l = 0$ and $m = n$
Until $l = m$, iterate
{
- Find the largest $k \leq m$ such that $X_k = X_l$
- If $k > l$ remove the terms with indices from $l + 1$ to k, and shift the indices larger than k by $l - k$ to get a new sequence.
- Decrement m by $k - l$ and increment l by 1.
}

For the month sequence, this leads at once from *j,f,m,a,m,j,j,a,s,o,n,d* to *j,a,s,o,n,d*, and then the counter l is incremented from 0 to 5 without further removals. For the "reverse" month sequence the counter l is incremented from 0 to 4, a loop is removed leading from *d,n,o,s,a,j,j,m,a,m,f,j* to *d,n,o,s,a,m,f,j*, then the counter l is incremented from 5 to 7 without further removals.

A loop-erased random walk is when this procedure is applied to a (two dimensional for our main interest) random walk. In the full plane this is very easy to do. Fig.1.7 represents a loop-erased walk of 200 steps obtained by removing the loops of a 4006 steps random walk on the square lattice. The thin grey lines build the shadow of the random walk (where shadow means that we do not keep track of the order and multiplicity of the visits)

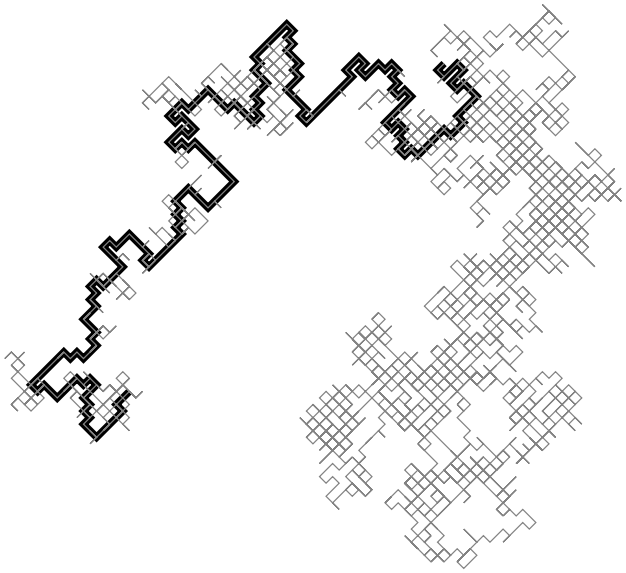

Figure 1.7: A loop-erased random walk with its shadow.

and the thick line is the corresponding loop-erased walk. The time asymmetry is clearly visible and allows to assert with little uncertainty that the walk starts on the top right corner.

In this setting, it is trivial to get samples but the measure remains in the background. One possible approach is the following. Consider a symmetric random walk on the square lattice and view the successive positions as a stream of data. Remove the loops as they show up, and stop the random walk at the first time n for which the associated loop-erased walk has reached length N. The probability of the random walk is 4^{-n}. Note that the set of random walks for which the loop-erasure never reaches size N has probability 0, for instance as a subset of the set of random walks that remain in the ball of radius N centered at the origin forever. So the total probability for the set of random walks stopped when their loop-erasure reaches length N is 1. This procedure leads to a finite family of loop-erased walks, each of them can be obtained via the loop erasure of an infinite number of random walks. The probability of a given loop-erased walk is taken to be sum of the individual probabilities of its random walk ancestors.

This can be adapted to the setting of discrete domains with admissible boundary condition. Let $(\mathbb{D}, a, b)$ be such a domain, and let ν be the valency of the associated tiling, i.e. the number of egdes adjacent to a

vertex: ν is 6 or the triangular tilings, 4 for the square tilings, and 3 for the hexagonal tilings. Consider the set of walks from a to b in $\mathbb{D} \cup \{a, b\}$ that visit a and b only once, and give each step weight ν^{-1}, so that the weight of a walk is the usual random walk probability on the tiling. However, the total weight of walks from a to b in $\mathbb{D} \cup \{a, b\}$ is < 1. As before, these walks can be loop-erased, and the weight of a simple path γ from a to b in $\mathbb{D} \cup \{a, b\}$ is taken to be sum of the weights of all random walks from a to b in $\mathbb{D} \cup \{a, b\}$ whose loop erasure leads to γ. To get a probability measure, one needs to divide the weight by the total weight of all random walks from a to b in $\mathbb{D} \cup \{a, b\}$. This is easy in principle, and is closely related to the solution of the discrete Laplace equation with appropriate boundary conditions; in practice this normalization can be computed explicitely for only a handful of examples.

In the same spirit, if we have an arbitrary weight assignment for walks from a to b in $\mathbb{D} \cup \{a, b\}$, we can use it to induce a weight on simple paths from a to b in $\mathbb{D} \cup \{a, b\}$ again by taking the weight of a simple path γ to be sum of the weights of all random walks from a to b in $\mathbb{D} \cup \{a, b\}$ whose loop erasure leads to γ. What is special about the standard random walk weight is that, as is well known, the random walk has a scale invariant limit (Brownian motion of course), so the corresponding loop-erased random walk can be expected to have a scale invariant limit. The loop-erased random walk is one of the first systems that has been proved to have a (not only scale but even) conformally invariant continuum limit, the fractal dimension being $5/4$. A naive idea to get directly a continuum limit representation of loop-erased walks would be to remove the loops from a Brownian motion. This turns out to be impossible due to the proliferation of overlapping loops of small scale. However, the SLE_2 process, to be defined later, gives a direct definition. In fact, it is the consideration of loop-erased random walks that led Schramm to propose SLE as a description of interfaces.

1.3.2 Simulation

We have seen that is is very simple to generate loop-erased random walks of a fixed length N in the plane. We could use this technique to get a probability measure on the first N steps of loop-erased random walks of length M. However, it is unclear whether this probability measure stabilizes

if we fix N and let M go to infinity. One of the problems is that in two dimensions, random walks are recurrent: with probability one they visit every site (and then they have to do it infinitely many times). So if we erase the loops of a random walk, the resulting loop-erased walk never stabilizes; if we wait long enough, the random walk comes back to the origin and at that instant the loop-erased walk starts anew from scratch.

The numerical simulation of a loop-erased random walk in domains $(\mathbb{D}, a, b)$ is not easy either, because the random walks have a tendency to leave $\mathbb{D}$. Note that it would bias the sampling if we would forbid them to leave by simply dispatching the weight of steps leaving $\mathbb{D}$ to the ones staying in $\mathbb{D}$. What one has to do is to *condition* on random walks staying in $\mathbb{D}$. So most samples would have simply to be rejected and only from time to time would a sample be a walk from a to b in $\mathbb{D} \cup \{a, b\}$.

There is one exceptional domain in which at the same time an infinite loop-erased random walk can easily be defined and simulated. It is when $\mathbb{D}$ is the square tiling of an half-space, conventionally taken to be $\mathbb{H}_{\text{int}}$, the tiling of the upper-half plane with vertices at the points $(n, m) \in \mathbb{Z} \times \mathbb{N}$, a is $O \equiv (0, 0)$ (by translation invariance along the real axis, any boundary vertex would do) and b is infinity. Let us explain why random walks on the square lattice conditioned to go from O to infinity while staying in $\mathbb{H}_{\text{int}}$ have a simple description.

The horizontal steps are not an issue, and we can concentrate on vertical steps. For a simple random walk in one dimension it is well known that a walk started at $m \in [0, l]$ touches the boundary for the first time at the endpoint 0 with probability $1 - m/l$ and at the endpoint l wiht probability m/l. Indeed, if $p(m)$ is the probability to touch the boundary for the first time at the endpoint 0, then $p(0) = 1$, $p(l) = 0$ and if $m \in]0, l[$, $p(m) = \frac{1}{2}(p(m-1) + p(m+1))$ as can be seen by conditioning on the first step of the walk. So by the usual rules of conditional probabilities, if the random walk is conditioned to exit at l and is at $m \in]0, l[$ at time t, it has probability $\frac{m+1}{2m}$ to go to $m+1$ and $\frac{m-1}{2m}$ to go to $m-1$. This has three striking consequences. First, the process remains Markov and time time homogeneous. Second, the transition probabities do not depend on l, so they can be used even if l is infinite. Third, taking l infinite, the probability, starting at m, never to visit $m' < m$ is $1 - m'/m > 0$ as can be seen by conditioning on the first step of the walk.

These three properties imply that each site is visited only a finite num-

ber of times, i.e. the walk escapes to infinity. Let us explain this in more detail. Suppose the walk starts from 0, goes to 1 at the first step, and then the above transition probabilities are used. Then at the second step the walk goes to 2. With probability $1/2$ it *never* goes back to 1 again. With probability $1/2$ it comes back to 1 at some point, and then the walk starts anew. Thus the number of visits to 1 follows a geometric law : 1 is visited $k \geq 1$ times with probability $1/2^k$. In particular the probability to visit 1 at least k times is $1/2^{k-1}$ which goes to 0 (in fact exponentially). Hence with probability 1 the number of visits of point 1 is finite. The same argument generalizes. First, let s be the probability that point $m \geq 1$ is never visited. Suppose the walk is at point m. With probability $\frac{m+1}{2m}$ it goes to $m+1$ and then with probability $1 - \frac{m}{m+1} = \frac{1}{m+1}$ it *never* goes back to m again. So the total probability that the walk starting from m never visits m again is $r \geq \frac{1}{2m} > 0$. It follows that that the number of visits to m follows essentially a geometric law : m is visited 0 times wiht probability s and $k \geq 1$ times with probability $(1-s)r(1-r)^{k-1}$. Again, the probability to visit m at least $k \geq 1$ times is $(1-s)(1-r)^{k-1}$ which goes to 0 (in fact exponentially). Hence with probability 1 the number of visits of point m is finite. This is true for any m and any starting point for the walk. Hence, in particular we see recursively that if the walk is at $m' < m$, m will be visited later with probability one, because with probability 1 all points in $[0, m-1]$ are visited only finitely many times. This means that in fact $r = \frac{1}{2m}$. To summarize, if the walk starts from 0, the number of visits to $m \geq 1$ is $k \geq 1$ with probability $\frac{1}{2m}\left(1 - \frac{1}{2m}\right)^{k-1}$. In particular, the walk is transient, i.e. it escapes to ∞ with probability 1.

So we use the usual random walk in the horizontal direction but the conditioned random walk in the vertical direction. Explicitely, at the first step the walk goes from $(0,0)$ to $(-1,1)$ or $(1,1)$ with probability $1/2$, and later, if at (n,m), the walk makes a step in the NE or NW directions with probability $\frac{m+1}{4m}$ and in the SE or SW directions with probability $\frac{m-1}{4m}$. Call the vertical coordinate "altitude" for convenience. As explained before, the altitude of the walk goes to ∞ with probability one, and the associated loop-erased walk converges. More precisely, for $m < M$ stop the random walk the first time it reaches altitude M and stop the corresponding loop-erased random walk at altitude m. Then with probability $> 1 - m/M$ the loop-erased random walk up to altitude m will not be modified by the subsequent evolution of the random walk. This is because to close a loop,

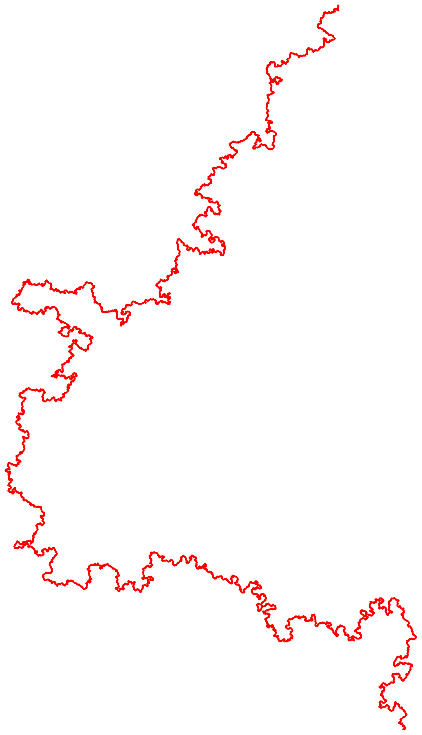

Figure 1.8: A sample of the loop-erased random walk.

the walk has to come back to the same point, which is more stringent than to come back to the same altitude. Hence, letting M go to infinity, we get a well defined limiting distribution for loop-erased random walks from O to altitude m for any m, hence for loop-erased random walks from O to ∞. Accurate numerical simulations are made by taking $M \gg m$. However, the process for which $m = M$ is interesting as well. It has a continuum limit which can be studied with the so-called dipolar variant of stochastic Loewner evolutions.

Fig.1.8 shows a sample of lerw of about 10^5 steps. At first glance, one observes a simple (no multiple points) irregular curve with a fractal structure. The intuitive explanation why a loop-erased random walk has a tendency not to come back too close to itself is that if it would do so, then with large probability a few more steps of the random walk would close a loop.

1.3.3 Relation with statistical mechanics

Again, it is useful to make the connexion between the purely geometric description of loop-erased random walks and more conventional statistical mechanics.

The starting point of the correspondance is a formula for the expansion of the determinant of $\det(\mathbf{1}-\mathbf{A})$ when $\mathbf{A}=(A_{\nu\nu'})_{\nu,\nu'\in V}$ is a matrix whith index set V. For later convenience, we call the elements of V vertices. A cycle of length $k\geq 1$ in V is sequence $(\nu_1,\cdots,\nu_k)$ of distinct vertices of V modulo cyclic permutation; so that $(\nu_1,\cdots,\nu_k)$, $(\nu_2,\cdots,\nu_k,\nu_1)$, $\cdots$ represent one and the same cycle. Cycles are said to be disjoint if no vertex appears in more than one of them. The subsets $\{C_1,\cdots,C_n\}$ of $\mathcal{P}(V)$ made of n disjoint cycles of V form a set that we denote $\mathcal{C}_n$. The weight of a cycle C represented by $(\nu_1,\cdots,\nu_k)$ is by definition $w(C)\equiv A_{\nu_1\nu_2}\cdots A_{\nu_{k-1}\nu_k}A_{\nu_k\nu_1}$ (for $k=1$, this reduces to $A_{\nu_1\nu_1}$) which indeed is invariant under cyclic permutations.

An elementary reorganization of Cramer's formula yields

$$\det(\mathbf{1}-\mathbf{A})=\sum_{n\geq 0}(-1)^n\sum_{\{C_1,\cdots,C_n\}\in\mathcal{C}_n}w(C_1)\cdots w(C_n). \tag{1.1}$$

Similarly, for $\nu,\nu'\in V$ we define a walk of k steps from ν to ν' in V to be any sequence of vertices $(\nu_0,\cdots,\nu_k)$ with $\nu_0=\nu$ and $\nu_k=\nu'$ but with $\nu_1,\cdots,\nu_{k-1}$ distinct from ν and ν'. Hence with this definition a walk visits is starting and end point only once. This restriction is a bit unusual, but it is not really crucial. The weight of a walk $W=(\nu_0,\cdots,\nu_k)$ is taken to be $w(W)\equiv A_{\nu_0\nu_1}\cdots A_{\nu_{k-1}\nu_k}$.

The sequence W can be loop-erased to yield a path from ν to ν' (remember that paths are walks in which a given vertex appears at most once). If γ is a path, we define

$$\tilde{w}(\gamma)\equiv\sum_{W\mapsto\gamma}w(W),$$

where the sum is over all walks whose associated loop-erased walk is γ. We aim at a general formula for $\tilde{w}$.

Let $\gamma=(\nu_0,\cdots,\nu_k)$ be a path from $\nu,\nu'\in V$. Let $V^{(0)}\equiv V\backslash\{\nu_k,\nu_0\}$, $V^{(1)}\equiv V\backslash\{\nu_k,\nu_0,\nu_1\},\cdots$. For $l=0,\cdots,k-1$ let $\mathbf{A}^{(l)}$ be the matrix $\mathbf{A}$ restricted to the vertex set $V^{(l)}$

A walk W which yields γ after loops have been erased can be decomposed as follows (see the second loop-erasing algorithm): the walk (ν_0,ν_1),

followed by an arbitrary number of walks from v_1 to v_1 in $V^{(0)}$, followed by the walk (v_1, v_2), followed by an arbitrary number of walks from v_2 to v_2 in $V^{(1)}$ and so on. Take $1 \leq l \leq k-1$. Note that if one expands $\left(\frac{1}{1-\mathbf{A}^{(l-1)}}\right)_{v_l v_l}$ in formal formal power series in the coefficents of $\mathbf{A}$, one gets exactly the sum of the weights for the concatenation of an arbitrary number of walks from v_l to v_l in $V^{(l-1)}$. Hence we infer that

$$\begin{aligned}\tilde{w}(\gamma) \;=\; A_{v_0,v_1}\left(\frac{\mathbf{1}}{\mathbf{1}-\mathbf{A}^{(0)}}\right)_{v_1 v_1} A_{v_1,v_2}\left(\frac{\mathbf{1}}{\mathbf{1}-\mathbf{A}^{(1)}}\right)_{v_2 v_2} A_{v_2,v_3}\cdots \\ \cdots A_{v_{k-2},v_{k-1}}\left(\frac{\mathbf{1}}{\mathbf{1}-\mathbf{A}^{(k-2)}}\right)_{v_{k-1} v_{k-1}} A_{v_{k-1},v_k}\end{aligned}$$

But by Cramer's formula for the inverse of a matrix,

$$\left(\frac{\mathbf{1}}{\mathbf{1}-\mathbf{A}^{(l-1)}}\right)_{v_l v_l} = \frac{\det(\mathbf{1}-\mathbf{A}^{(l)})}{\det(\mathbf{1}-\mathbf{A}^{(l-1)})} \text{ for } l = 0, \cdots, k-1.$$

Hence the product in the above formula for $\tilde{w}(\gamma)$ is telescopic, and we get the representation we were aiming at :

$$\tilde{w}(\gamma) = w(\gamma)\frac{\det(\mathbf{1}-\mathbf{A}^{(k-1)})}{\det(\mathbf{1}-\mathbf{A}^{(0)})}. \tag{1.2}$$

A first use of this formlual is that is shows clearly that if the matrix $\mathbf{A}$ is symmetric, the loop-erased random walk weight is reversible i.e. the same for a path and its opposite or time reversal. In all cases the asymmetry comes solely from the weight of γ.

It is time to interpret the formulæ obtained so far in connection with statistical mechanics.

We start with Eq.(1.1) but read from right to left. The right-hand side can be seen as a partition function for a gas of oriented loops on a graph. Indeed, if E is an arbitrary subset of $V \times V$, we can consider the corresponding oriented graph $G = (V, E)$ i.e. view E as the set of edges if G. We give each edge in $(v, v') \in E$ the weight $A_{vv'}$ and impose that $A_{vv'} = 0$ if $(v, v') \notin E$. An oriented loop on G is a sequence $(v_1, \cdots, v_k)$ of distinct vertices of V modulo cyclic permutation, with the condition that $(v_1, v_2), (v_2, v_3), \cdots, (v_{k-1}, v_k), (v_k, v_1)$ are in E. Except for the last condition, this is what we called a cycle before: note that "cycle" reminds of the permutation context whereas "loop" reminds of geometric context.

A configuration is a family of disjoint oriented loops, each oriented loop counts for a weight which is the product of the weight of the traversed edges and an overall factor (-1). Then the partition function, i.e. the sum of the weights of all possible configuration is by definition the right-hand side of Eq.(1.1), and this reconstructs the determinant on the left-hand side. We can specialize more by assuming further that E is a symmetric subset of $V \times V$ that does not meet the diagonal, and that $\mathbf{A}$ is symmetric. Then there is no loop of length 1, and the loop $(v_1, \cdots, v_k)$ has the same weight as the loop traversed in the opposite order $(v_k, \cdots, v_1)$. If $k = 2$ a loop and its opposite are the same, but not if $k \leq 3$. So we get the same partition function if instead of summing over oriented loops, we sum over unoriented loops counting each unoriented loop of of length ≥ 3 twice, i.e. giving unoriented loop of length ≥ 3 a overall factor (-2) instead of (-1). Finally, we could also give each edge in E the same weight μ so that the weight of a loop configuration would be

$$(-1)^{\#\text{ loops of length }2}(-2)^{\#\text{ loops of length }\geq 3}\mu^{\#\text{traversed edges}}$$

where of course loops of length 2 count for 2 traversed edges.

This statistical weight could be used as a definition of the so-called $O(-2)$ model, where -2 reminds of the overall weight of each loop (of length ≥ 3). This model has several avatars, which are supposed to be in the same universality class, i.e. to describe the same macroscopic physics in the continuum limit. In certain versions, loops of length 2 are completely forbidden. Replacing the factor (-2) by a factor n yields the general $O(n)$ model, which describes other systems of interest. For instance $n = 2$ is related to the XY model, the Kosterlitz-Thouless transition and the gaussian free field. Also $n = 1$ is related to the Ising model, $n = 0$ to self-avoiding walks and so on.

Note that the partition function, i.e. $\det(\mathbf{1} - \mathbf{A})$ has a simple "field theory" interpretation : if χ_v and $\bar{\chi}_v$, $v \in V$ are a collection of independant Grasman variables, the fundamental result of Grasman integration is

$$\det(\mathbf{1} - \mathbf{A}) = \int \prod_{v \in V} d\chi_v \, d\bar{\chi}_v e^{\sum_{v,v'} \chi_v (\delta_{vv'} - A_{vv'}) \bar{\chi}_{v'}}.$$

This is the clue to the quantum field theory approach to loop-erased random walks.

Before we interpret Eq.(1.2), let us start with a general observation. Suppose $\mathcal{C}$ is a configuration space, assumed to be finite for simplicity and consider a model of statistical mechanics on $\mathcal{C}$. Each $c \in \mathcal{C}$ has a weight $w(c)$. The partition function is $Z \equiv \sum_{c \in \mathcal{C}} w(c)$. Suppose $\mathcal{C}$ can be partitioned as $\mathcal{C} = \cup_{\gamma \in \Gamma} \mathcal{C}_\gamma$. Then we can define $Z_\gamma \equiv \sum_{c \in \mathcal{C}_\gamma} w(c)$ for $\gamma \in \Gamma$, and Z_γ can be interpreted as the marginal weight of $\mathcal{C}_\gamma$. The probability of $\mathcal{C}_\gamma$ is simply Z_γ/Z. In concrete situations, the splitting $\mathcal{C} = \cup_{\gamma \in \Gamma} \mathcal{C}_\gamma$ will usually have some interpretation. For instance, in the cases we are interested in in these notes, we shall look at configuration spaces $\mathcal{C}$ that describe a statistical mechanics model on domains $(\mathbb{D}, a, b)$ with boundary conditions, in such a way that in each $c \in \mathcal{C}$ we can identify unambiguously a path γ joining a to b. Of course γ depends on c, and we can use this γ to split c. Then Z_γ/Z is simply the probability to observe the path γ. The reader should have another glance at Sec.1.2.2 to look at the relationship between the exploration path and percolation from this viewpoint.

Eq.(1.2) can then be interpreted straightforwardly. We consider now configurations made not simply of (mutually avoiding) loops, but of (mutually avoiding) loops avoiding a path from ν to ν'. The total weight of configurations for a fixed path from ν to ν' is simply the numerator of the right-hand side of Eq.(1.2). The denominator depends on ν and ν' but not on the simple path between them. So from the point of view of statistical mechanics explained before, the weight the loop-erased random walk model assigns to a path γ, , i.e. the left-hand side of Eq.(1.2), is proportional to the marginal weight of configurations of "loops plus that path" in the loop gas model.

Hence we have succeeded in giving an interpetation of the loop erased random walks as interfaces in a statistical mechanics model. We are cheating a bit here because even if we take a positive edge weight μ, because of the $(-)$ sign associated to each loop, individual configurations may well have a negative weight, so that a straightforward probabilistic interpretation is not available.

Our interest is of course the case when the graph G is the one associated with a discrete domain $(\mathbb{D}, a, b)$ with admissible boundary conditions. If we take for the edge weight μ the inverse of the valency ν of the tiling, $\mathbf{1} - \mathbf{A}$ if essentially the discrete laplacian with Dirichlet boundary conditions. This suggests again that a continuum limit exists, for which (continuum) loop-erased random walks in a (continuum) domain D from a to b are related

to the field theory of so-called symplectic fermions with measure

$$\mathcal{D}\chi\,\mathcal{D}\bar{\chi}\,\exp\int_{\mathbb{D}}\nabla\chi\cdot\nabla\bar{\chi}$$

with Dirichlet boundary conditions. This field theory is well-known to be conformally invariant. But $\mu - 1/\nu = -\lambda^2$ is a scaling function which leads to the addition of a mass term to the action.

We conclude this section by noting without justification that the way to impose the existence of a path from a to b is to insert in correlation functions the obervable $J(a)\bar{J}(b)$ where $J(a)$ (resp.$\bar{J}(b)$) is the normal derivative of χ (resp. $\bar{\chi}$) at a (resp. at b).

1.4 Another example of growth: DLA

Up to now, the two growth processes we have defined shared some common features. The next one, DLA, is of a rather different nature. It is believed to have a scale invariant but not conformally invariant limiting distribution. Another reason to introduce DLA is that it can also be modelled via Loewner chains, a subject we touch in the next chapter. Finally, DLA seems to be a relevant model for a variety of phenomena in physics, for instance aggregation or deposition phenomena, but also in biology, for instance growth of bacterial colonies under certain circumstances.

DLA stands for diffusion limited aggregation. It refers to processes in which the domains grow by aggregating diffusing particles. Namely, one imagines building up a domain by clustering particles one by one. These particles are released from the point at infinity, or uniformly from a large circle around the growing sample, and diffuse as random walkers. They will eventually hit the sample and the first time this happens they stick to it. Then the procedure goes on. By convention, time is incremented by unity each time a particle is added to the domain. Thus at each time step the area of the domain is increased by the physical size of the particle. The position at which the particle is added depends on the probability for a random walker to visit the boundary for the first time at this position, which is essentially what is called the harmonic measure at that position. During this process the clustering domain gets ramified and develops branches and fjords of various scales. The probability for a particle to stick on the cluster is much higher on the tip of the branches than deep inside the fjords. This

property, relevant at all scales, is responsible for the fractal structure of the DLA clusters.

In a discrete approach one may imagine that the particles are tiny squares whose centers move on a square lattice whose edge lengths equal that of the particles, so that particles fill the lattice when they are glued together. The center of a particle moves as a random walker on the square tiling. The probability $Q(x)$ that a particle visits a given tile x satisfies the discrete version of the Laplace equation $\nabla^2 Q = 0$. It vanishes on the boundary of the domain, i.e. $Q = 0$ on the boundary, because the probability for a particle to visit a tile already occupied, i.e. a point of the growing cluster, is zero. The local speed at which the domain is growing is proportional to the probability for a tile next to the interface but on the outer domain to be visited. This probability is proportional to the discrete normal gradient of Q, since the visiting probability vanishes on the interface. So the local speed is $v_n = (\nabla Q)_n$. To add a new particle to the growing domain, a random walk has to wander around and the position at which it finally sticks is influenced by the whole domain. To rephrase this, for each new particle one has to solve the outer Laplace equation, a non-local problem, to know the sticking probability distribution. This is a typical example when scale invariance is not expected to imply conformal invariance.

It is not so easy to make an unbiased simulation of DLA on the lattice. One of the reasons is that on the lattice there is no such simple boundary as a circle, for which the hitting distribution from infinity is uniform. The hitting distribution on the boundary of a square is not such a simple function. Another reason is that despite the fact that the symmetric random walk is recurrent is 2d, each walk takes many steps to glue to the growing domain. The typical time to generate a single sample of reasonable size with an acceptable bias is comparable to the time it takes to make enough statistics on loop-erased random walks or percolation to get the scaling exponent with two significant digits. Still this is a modest time, but it is enough to reveal the intricacy of the patterns that are formed. Fig.1.9 is such a sample. The similarity with the sample in Fig.2.1, obtained by iteration of conformal maps, is striking. But a quantitative comparison of the two models is well out of analytic control and belongs to the realm of extensive simulations. There is now a consensus that the fractal dimension of 2d DLA clusters is $D_{dla} \simeq 1.71$. It has been long debated whether or

not discrete simulations done right nevertheless do keep a remnant of the lattice at large distance. There is some consensus now that for instance the orientation of the lattice can be seen even in the large, and rotation invariant algorithms should be preferred.

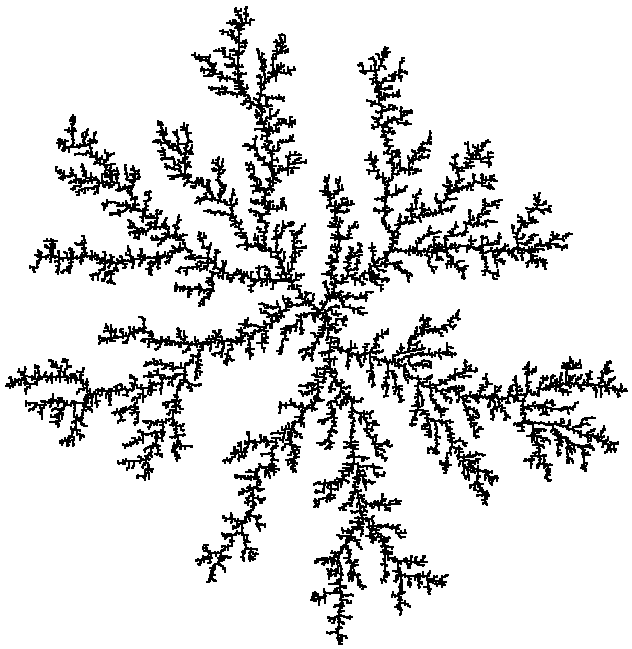

Figure 1.9: A DLA sample.

Chapter 2

Loewner chains

There are many possible descriptions of subsets of a set. Some may look more natural than others but it is the problem at hand that decides which one is the most efficient. Growth processes in two dimensions involve time dependent subsets of the complex plane $\mathbb{C}$. Loewner chains have proved to be an invaluable tool in this context. The simplest situation is when they are used to describe families of domains. These notes deal (almost) exclusively with that case.

Loewner chains were introduced (by Loewner!) in the context of the Bieberbach conjecture, now a theorem proved by de Branges in 1985. It states that if $f(z) = z + \sum_{n\geq 2} a_n z^n$ is a holomorphic function injective in the unit disc $\mathbb{U} = \{z \in \mathbb{C}, |z| < 1\}$ then $|a_n| \leq n$ for $n \geq 2$. Bieberbach proved that $|a_2| \leq 2$ in 1912, and Loewner proved in 1923 that $|a_3| \leq 3$ using a dynamical picture of the changes of $f(\mathbb{U})$ when the a_n's change, starting from the trivial case $f(z) = z$.

2.1 Around Riemann's theorem

A *domain* $\mathbb{D}$ is a non empty connected and simply connected open set strictly included in the complex plane $\mathbb{C}$. Simple connectedness is a notion of purely topological nature which in two dimensions asserts essentially that a domain has no holes and is contractile: the domain has the same topology as a disc.

Riemann's theorem states that two domains $\mathbb{D}$ and $\mathbb{D}'$ are always conformally equivalent, i.e. there is an invertible holomorphic map $g : \mathbb{D} \mapsto \mathbb{D}'$

between them.

Riemann stated the theorem but his argument had many gaps. This was at least partly at the origin of the formidable development of functional analysis in the twentieth century but it took decades before a proof meeting modern standards was found.

Extending g to the boundary of $\mathbb{D}$ is impossible in general if the naïve notion of boundary is used, i.e. if the boundary of $\mathbb{D}$ is taken as the complement of $\mathbb{D}$ in its closure. As an example, take $\mathbb{D}$ to be the upper half plane $\mathbb{H}$ with the vertical line segment $]0, ia]$ removed and $\mathbb{D}' = \mathbb{H}$. The naïve boundary of $\mathbb{D}$ is the union of $\mathbb{R}$ and $]0, ia]$. The limits of $g(z)$ when z approaches a given point of the segment $]0, ia]$ from the left or from the right must be distinct. But another notion of boundary can be defined for which a continuous extension at the boundary is always possible. Intuitively, this more involved notion keeps track of the different sides from which a naïve boundary point can be approached. This is trivial in our simple example but the general case is involved and we shall not give a precise definition. We shall freely use the word "boundary" in the sequel, leaving to the reader the task of deciding from the context which kind of boundary we have in mind. In case when there is only one way to approach naïve boundary points the two notions coïncide.

In simple cases, the map f can be found in closed form. For instance, the upper-half plane $\mathbb{H}$ and the unit disc $\{z \in \mathbb{C}, |z| < 1\}$ centered on the origin are two domains. The conformal transformation $f(z) = i\frac{1-z}{1+z}$ maps the unit disc biholomorphically onto the upper half plane with $f(0) = i$ and $f(1) = 0$. But the general case is another matter.

The upper half plane has a three dimensional Lie group of conformal automorphisms, $\mathrm{PSL}_2(\mathbb{R})$, that also acts on the boundary of $\mathbb{H}$. This group is made of homographic transformations $f(z) = \frac{az+b}{cz+d}$ with a, b, c, d real and $ad - bc = 1$. To specify such map we have to impose three real conditions. Hence, there is a unique automorphism – possibly followed by a transposition – that maps any triple of boundary points to any other triple of boundary points. Similarly there is unique homographic transformation that maps any pair made of a bulk point and a boundary point to another pair of bulk and boundary points. By Riemann's theorem, any domain has a Lie group of conformal automorphisms isomorphic to $\mathrm{PSL}_2(\mathbb{R})$ and the

same normalization conditions can be used.

Riemann's theorem is used repeatedly in the rest of these notes. It is the starting point of many approaches to growth phenomena in two dimensions since it allows to code the shapes of growing domains in their uniformizing conformal maps. To make the description precise one has to choose a reference domain against which the growing domains are compared. By Riemann's theorem we may choose any domain as reference domain – and depending on the geometry of the problem some choices are more convenient than others. The unit disc and the upper half plane are often used as reference domains.

2.2 Hulls

One can be a more explicit when the domain $\mathbb{D}$ differs only locally from the upper half plane $\mathbb{H}$, that is if $\mathbb{K} = \mathbb{H} \setminus \mathbb{D}$ is bounded. Such a set $\mathbb{K}$ is called a hull. The real points in the closure of $\mathbb{K}$ in $\mathbb{C}$ form a compact set which we call $\mathbb{K}_{\mathbb{R}}$. In that case, $\mathbb{H}$ is the convenient reference domain. Let $g : \mathbb{D} \mapsto \mathbb{H}$ be a conformal bijection. For $z \in \mathbb{D}$ define $g(\bar{z}) \equiv \overline{g(z)}$. If z approaches a point x on the real axis while staying within $\mathbb{D}$, $g(z)$ has a real limit which we denote by $g(x)$. It follows that g extends to a holomorphic map on the connected open set $\mathbb{D} \cup \overline{\mathbb{D}} \cup (\mathbb{R} \setminus \mathbb{K}_{\mathbb{R}}) \cup \infty$ of the Riemann sphere, which contains a neighborhood of ∞. This is an illustration of the Schwartz reflection principle. One can use the automorphism group of $\mathbb{H}$ to ensure that $g(z) = z + O(1/z)$ for large z. This is called the *hydrodynamic normalization*. It involves three conditions : g maps ∞ to ∞, has unit derivative there, and has no constant term. These three condition are real because ∞ is on the boundary of the upper half plane seen within the Riemann sphere. There is no further freedom left. Thus any property of g is an intrinsic property of $\mathbb{K}$.

We shall denote this special representative by $g_{\mathbb{K}}$. The inverse map $f_{\mathbb{K}}$ is holomorphic on the full Riemann sphere except for cut along a compact subset of $\mathbb{R}$ across which its imaginary part has a positive discontinuity (in general this is a measure) $d\mu(x)$. Away from the cut, $f_{\mathbb{K}}$ has the standard representation

$$f_{\mathbb{K}}(w) = w - \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\mu(x)}{w - x}.$$

The coefficients of the expansion of $f_{\mathbb{K}}$ at infinity are essentially the moments of μ. I particular, they are real. Each of them quantify an intrinsic property of $\mathbb{K}$. The number $C_{\mathbb{K}} \equiv \frac{1}{\pi}\int_{\mathbb{R}} d\mu(x)$ is the total mass of μ. It is positive (or 0 is $\mathbb{K}$ is empty). Note that $f_{\mathbb{K}}(w) = w - C_{\mathbb{K}}/w + \cdots$ at large w and by inverting, $g_{\mathbb{K}}(z) = z + C_{\mathbb{K}}/z + \cdots$ at large z. The coefficient $C_{\mathbb{K}}$ plays an important role. It is called the capacity of $\mathbb{K}$ seen from ∞. It's positivity is intuitively related to the fact that one *removes* a piece from $\mathbb{H}$. Capacity is trivially translation invariant ($x + \mathbb{K}$, the translate of $\mathbb{K}$ by x units along the real axis, and $\mathbb{K}$ have the same capacity) and has weight 2 under dilations ($C_{s\mathbb{K}} = s^2 C_{\mathbb{K}}$ if s is a positive scale factor). Capacity has an additive property: simple series manipulations show that if $\mathbb{K}'$ and $\mathbb{K}''$ are two hulls and $\mathbb{K} = \mathbb{K}' \cup g_{\mathbb{K}'}^{-1}(\mathbb{K}'')$ (which is another hull) then $C(\mathbb{K}) = C(\mathbb{K}') + C(\mathbb{K}'')$. .

2.3 Basic examples

Example 1 *: The semidisc.*

Maybe the simplest example is when $\mathbb{K}$ is a semidisc $\{z \in \mathbb{H}, |z-b| \leq r\}$ for a real b and real positive r. Then $g_{\mathbb{K}}(z) = z + r^2/(z-b)$. Expansion at large z shows that $C_{\mathbb{K}} = r^2$.

Example 2 *: The vertical line segment.*

In the example when $\mathbb{K}$ is the vertical line segment $]0, ia]$, one gets $g_{\mathbb{K}}(z) = \sqrt{z^2 + a^2}$, a formula by which we mean that analytic continuation of the function $z\sqrt{1 + a^2/z^2}$ were the square root is defined by its usual power series around 1 when z is large. Expansion at large z shows that $2C_{\mathbb{K}} = a^2$.

Example 3 *: The oblique line segment.*

The case when $\mathbb{K}$ is an oblique line segment $]0, ae^{i\pi b}]$ making an angle πb with respect to the real positive axis ($b \in]0, 1[$) yields

$$z = (g_{\mathbb{K}}(z) - x_+)^b (g_{\mathbb{K}}(z) - x_-)^{1-b},$$

where the real parameters $x_- < 0 < x_+$ satisfy $bx_+ + (1-b)x_- = 0$ and $(-x_-)^b x_+^{1-b} = a$. Expansion at large z shows that $2C_{\mathbb{K}} = b(1-b)(x_+ - x_-)^2$.

The closer the line is to the real axis (i.e. the closer b is to 0 or π) and the larger a has to be to reach a given capacity.

Example 4 *: Arc of circle.*

An instructive example is when $\mathbb{K}$ is the arc $]r, re^{i\vartheta}]$ of a circle centered at 0 of radius r. Some of the following computations require to keep a precise track of the determination of the square root that appears in the formula for $g_{\mathbb{K}}$ because it is crucial for the interpretation. The map $f(w) = (w-r)/(w+r)$ sends the arc to the vertical line segment $]0, i\tan\vartheta/2]$, so that by the previous example, $w \mapsto \sqrt{f(w)^2 + \tan^2\vartheta/2}$ is a conformal map from $\mathbb{D}$ to $\mathbb{H}$. However, this map sends ∞ to $1/(\cos\vartheta/2)$, not to ∞. To get the hydrodynamic normalization, we have to compose with an appropriate automorphism of $\mathbb{H}$. This yields

$$g_{\mathbb{K}}(w) = r\frac{-(2-\cos^2\vartheta/2)\cos\vartheta/2\sqrt{\left(\frac{z-r}{z+r}\right)^2 + \tan^2\vartheta/2} + 2 - 3\cos^2\vartheta/2}{\cos\vartheta/2\sqrt{\left(\frac{z-r}{z+r}\right)^2 + \tan^2\vartheta/2} - 1},$$

whose expansion at ∞ starts like $g_{\mathbb{K}}(w) = w + (1-\cos^4\vartheta/2)r^2/w + O(1/w^2)$. Hence the capacity is $C_{\mathbb{K}} = (1-\cos^4\vartheta/2)r^2$.

The tip of the arc, $re^{i\vartheta}$ is mapped to $(3\cos^2\vartheta/2 - 2)r$ by $g_{\mathbb{K}}$. One checks that

$$(g_{\mathbb{K}}(w) - g_{\mathbb{K}}(re^{i\vartheta}))\frac{\partial g_{\mathbb{K}}(w)}{\partial\vartheta} = 2r^2\sin\vartheta/2\ \cos^3\vartheta/2,$$

which is w-independent.

Moreover $\lim_{w\mapsto r^-} g_{\mathbb{K}}(w) = r(1-2\sin\vartheta/2-\sin^2\vartheta/2)$ and $\lim_{w\mapsto r^+} g_{\mathbb{K}}(w) = r(1+2\sin\vartheta/2-\sin^2\vartheta/2)$. The behavior of $g_{\mathbb{K}}$ when $\vartheta \mapsto \pi^-$ is interesting. In this limit, $\mathbb{K}$ becomes a semicircle. Let $\tilde{\mathbb{K}} = \{w \in \mathbb{H}, |w| \leq r\}$ be the corresponding semidisc. The points w inside $\tilde{\mathbb{K}}$ are cut away from ∞ when $\vartheta \mapsto \pi^-$, and one checks that $\lim_{\vartheta\mapsto\pi^-} g_{\mathbb{K}}(w) = -2r$ for these points, i.e. they are swallowed in the limit. However, the points $\{w \in \mathbb{H}, |w| > r\}$ are mapped to $\lim_{\vartheta\mapsto\pi^-} g_{\mathbb{K}}(w) = w + r^2/w = g_{\tilde{\mathbb{K}}}(w)$.

2.4 Iteration of conformal maps

With Riemann's theorem at our disposal, we can start to encode growth processes. Suppose than the initial domain is the upper half plane and that

a small amount of matter is removed at each time step (so that in fact it is the lower half plane that grows). At time step n, a certain $\mathbb{K}_n$ has been removed from $\mathbb{H}$. Let $g_n \equiv g_{\mathbb{K}_n}$ denote the corresponding map and f_n its inverse. Then $g_n(\mathbb{K}_{n+1} \setminus \mathbb{K}_n)$ describes a small amount of matter removed to $\mathbb{H}$. If $g_n(\mathbb{K}_{n+1} \setminus \mathbb{K}_n)$ has typical size s and is located in the neighborhood of point x on the real axis, $\mathbb{K}_{n+1} \setminus \mathbb{K}_n$, which is what is really removed at time $n+1$ has typical size $s|f_n'(x)|$.

Example 5 *: Simple iteration.*

Choose a small number ε. Let b_n, $n > 0$ be an independent sequence drawn from some chosen probability distribution. At time step $n+1$ take $g_n(\mathbb{K}_{n+1} \setminus \mathbb{K}_n)$ to be the semidisc $\{z \in \mathbb{H}, |z - b_{n+1}||f_n'(b_{n+1})| \leq \varepsilon\}$, so that

$$g_{n+1}(z) = g_n(z) + \frac{\varepsilon^2}{|f_n'(b_{n+1})|^2 (g_n(z) - b_{n+1})}.$$

This defines a random growth process were at each time step a small semidisc-like grain of matter of size $\sim \varepsilon$ is removed. Despite its simplicity, little is known (at least to the author) about this process.

Many other (probabilistic or deterministic) rules can be invented, but the resulting processes are mostly impossible to study analytically at the moment. Let us simply note to conclude that the samples obtained by methods (but using the disc geometry) look strikingly like DLA. Fig.2.1 is obtained by iteration of conformal maps, compare with Fig.1.9.

2.5 Continuous time growth processes

Our aim is to motivate the introduction of Loewner chains.

If $\mathbb{K}$ is not simply a semidisc, but an union of well-separated small semidiscs of radii r_α centered at b_α, a moment of thought leads to realize that

$$g_{\mathbb{K}}(z) \sim z + \sum_\alpha \frac{r_\alpha^2}{z - b_\alpha}.$$

The large z expansion yields $C_{\mathbb{K}} \sim \sum_\alpha r_\alpha^2$, a positive number as expected.

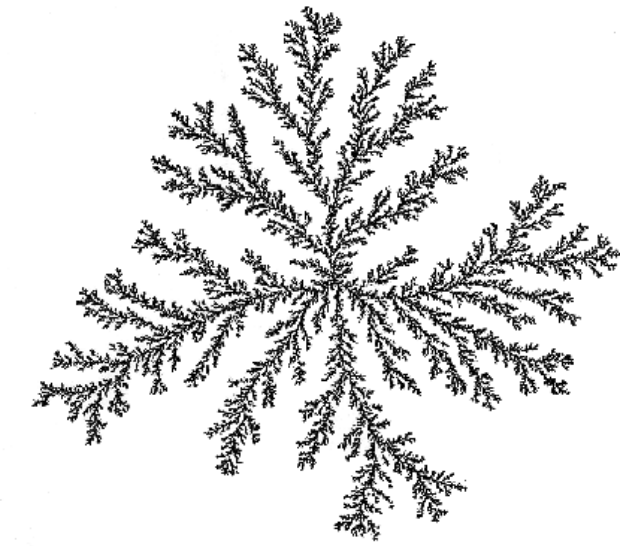

Figure 2.1: A shape produced by iteration of random conformal maps.

Taking a naïve limit, one gets that if ε is a small positive number, $\nu(x)$ is a nonnegative function on $\mathbb{R}$ and $\mathbb{K} = \{z = x + iy \in \mathbb{H}, y \leq \varepsilon\nu(x)\}$ then

$$g_{\mathbb{K}}(z) \sim z + \frac{\varepsilon}{\pi} \int_{\mathbb{R}} \frac{\nu(u)du}{z-u}.$$

Indeed, using that, if $\nu(x) \neq 0$, $\lim_{\varepsilon \mapsto 0^+} \Im m\,(x + i\varepsilon\nu(x) - u)^{-1} = \pi\delta(u-x)$ one checks that $\Im m\, \frac{1}{\pi} \int_{\mathbb{R}} \frac{\nu(u)du}{x+i\varepsilon\nu(x)-u} \sim -\nu(x)$ so that to first order in ε $g_{\mathbb{K}}(z)$ is real when z is on the boundary of $\mathbb{K}$. Even more generally, one could replace the positive measure $\nu(u)du$ by any positive measure $d\rho(u)$. A naïve large z expansion, certainly valid if the function ν (or more generally the measure $d\rho$) has compact support and finite mass, gives $C_{\mathbb{K}} \sim \frac{\varepsilon}{\pi} \int_{\mathbb{R}} \nu(u)du$ (more generally $C_{\mathbb{K}} \sim \frac{\varepsilon}{\pi}\rho(\mathbb{R})$, again a a positive number.

Now think about a continuous time growth process for which $\mathbb{K}_t$ has been removed from $\mathbb{H}$ at time t. Let $g_t \equiv g_{\mathbb{K}_t}$ denote the corresponding map and f_t its inverse. Fix t and a small positive ε. Then $g_t(\mathbb{K}_{t+\varepsilon} \setminus \mathbb{K}_t)$ describes a small amount of matter removed to $\mathbb{H}$. We could take as a definition of continuous time growth that the associated map $g_{t+\varepsilon} \circ f_t$ is described by a nonnegative function $\nu_t(u)$ or more generally a positive measure $d\rho_t(u)$ as above. Taking the limit $\varepsilon \mapsto 0^+$ leads to

$$\frac{\partial g_t(z)}{\partial t} = \frac{1}{\pi}\int_{\mathbb{R}} \frac{d\rho_t(u)}{g_t(z)-u}. \tag{2.1}$$

Such an evolution equation is called a Loewner chain with reference domain $\mathbb{H}$. The analogous equations with reference domain the unit disc can be obtained straightforwardly by the same arguments. The large z expansion yields

$$\frac{dC_{\mathbb{K}_t}}{dt} = \frac{1}{\pi}\rho_t(\mathbb{R}).$$

So if hulls are constructed little by little by a growth process, the positivity of capacity is obvious.

In principle, if the family of measures ρ_t is given, one can solve for $g_t(z)$ with the initial condition $g_0(z) = z$. Again, ρ_t can be random or deterministic. We should note that Loewner chains are in some sense kinematic equations that give a general framework to encode growth processes. But in a real dynamical problem ρ_t has to be specified. It may depend explicitly on g_t. For instance $d\rho_t(u) = |f_t'(u)|^{-2}du$ is related to Laplacian growth, though the unit disc geometry is the relevant one in that case. The exponent -2, which we already interpreted for discrete iteration, ensures that the size of $\mathbb{K}_t$ grows linearly with time. But other exponents between 0 and -2 are interesting too. Note that DLA provides a discrete analogue of Laplacian growth. The particle size plays the role of an ultraviolet cutoff.

2.6 Geometric interpretation

One can give a the following geometric interpretation of Loewner chains. Set $g_t(z) \equiv z_t$, view z_t as the position of a fluid particle as time goes by, and suppose for simplicity that $d\rho_t(u) = \nu_t(u)du$ so that the Loewner chain becomes

$$\frac{dz_t}{dt} = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\nu_t(u)du}{z_t-u}.$$

Hence $\frac{1}{\pi}\int_{\mathbb{R}} \frac{\nu_t(u)du}{z-u}$ plays the role of a time dependent holomorphic vector field on the manifold with boundary $\mathbb{H}$. At point $z = x + iO^+$ i.e. close to the the real axis (the boundary of $\mathbb{H}$) this vector field has imaginary part $-\nu(x)$, so that when x is away from the support of ρ_t, (that is, when $\nu_t(.) = 0$ in a neighborhood of x), the vector field is real, i.e. tangent to the boundary. However, if x is on the support of ρ_t the vector field has a

finite negative imaginary part, which means that some fluid particles that started inside $\mathbb{H}$ can be swallowed by the boundary. In fact $\mathbb{K}_t$ is nothing but the set of fluid particles which where in $\mathbb{H}$ at $t = 0$ but have hit the boundary before time t.

The reader is urged to review the examples 1-4 in this light. For the semidisc case, take r as time, either with $b = 0$ or with $b = r$. For the case of line segments, take a constant b and use a as time. For the arc of circle, using ϑ as time, with special care in the limit $\vartheta \mapsto \pi^-$. It is instructive to compute the measure ρ_t in each case and to check that the above interpretation of $\mathbb{K}_t$ is correct.

Another, more abstract, geometric interpretation is also possible. Let N_- be the group of series of the form $z + \sum_{m \leq -1} g_m z^{m+1}$ with real coefficients and convergent for large z (the domain of convergence may depend on the series, so N_- is made of "germs", and is in fact the group of germs of holomorphic functions fixing ∞ and with derivative 1 at ∞). In the same spirit, let O_∞ be the space of germs of holomorphic functions at infinity. We let N_- act on O_∞ by composition, $\gamma_g \cdot F \equiv F \circ g$. Observe that $\gamma_{g_1 \circ g_2} = \gamma_{g_2} \cdot \gamma_{g_1}$ so this is an anti-representation.

Note that the g_t's of a Loewner chain with bounded $\mathbb{K}_t$ belong to N_-. If $F \in O_\infty$ and if z is large enough, $F(z)$ is well defined as well as $F(z_t)$ for small t (where the meaning of small may depend on z and F) and

$$\frac{dF(z_t)}{dt} = \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\rho_t(u)}{z_t - u} \frac{\partial F}{\partial z}(z_t),$$

which can be rewritten

$$\frac{d}{dt}(\gamma_{g_t} \cdot F) = \gamma_{g_t} \cdot (\nu_t \cdot F)$$

where $\nu_t(z) \equiv \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\rho_t(u)}{z-u} \frac{\partial}{\partial z}$ is a germ of vector field.

So the Loewner chain equation can be viewed as a flow on N_-

$$\frac{d}{dt}\gamma_{g_t} = \gamma_{g_t} \cdot \nu_t.$$

The group N_- has an interesting representation theory, related to that of the Virasoro algebra, which can be used as a probe for this flow.

2.7 Local growth

Suppose that as time goes by the measures ρ_s are δ-peaks of height $\pi a_s/2$ (the factor 2 is purely historical) at position ξ_s. In the upper half plane reference geometry, the growth process will be described by an equation of the type

$$\frac{\partial g_s(z)}{\partial s} = \frac{2a_s}{g_s(z) - \xi_s}.$$

Note that examples 2-4 fall in this category. The formula was given for example 4 if $s = \vartheta$ and the other cases lead to simple computations left to the reader.

If one is interested only in the growth of the hull, but not in the way the evolution is parameterized, one can make change the time variable without arm. The statement that ξ_s changes quickly or slowly makes sense only compared with the changes in a_s. For instance, suppose that the function a_s vanishes in some interval, while ξ_s keeps on changing so that it has a different value at the beginning and at the end of the interval. During that interval g_s has not changed but when a_s starts moving again, the place at which the hull resumes growth can be far from the place where it was growing before the pause. This is a limiting case of what happens when variations of ξ_s are large with respect to those of a_s. This means that if, at s_0, ξ_s starts to move very fast with respect to a_s, the growth takes place very near $\mathbb{K}_{s_0}$ or the real axis. This conclusion is supported by example 3.

We also infer that to have *local growth*, i.e. to have the position where the hull grows vary continuously, we need to impose that ξ_s stops if a_s does. To make this statement precise, it is convenient to go to a special time parameterization. The capacity of the hull at time s is $C_{\mathbb{K}_s} = 2\int_0^s ds'\, a_{s'}$, a non-decreasing function of s. Define $t = \int_0^s ds'\, a_{s'}$, take t to be the new time variable and by abuse of notation write ξ_t for $\xi_{s(t)}$, $\mathbb{K}_t$ for $\mathbb{K}_{s(t)}$ and so on. Then by construction $C_{\mathbb{K}_t} = 2t$ and the equation reads

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \xi_t} \tag{2.2}$$

We take as a definition of *local growth* that ξ_t is continuous function of t. The function ξ_t is often called the *driving function* of the Loewner evolution. It is sometimes convenient to normalize ξ_t by $\xi_0 = 0$ or what

amounts to the same to impose that the hull starts growing from point 0.

A broad class of growing hulls that can be described by such an equation is given by continuous simple curves started on the boundary of $\mathbb{H}$ and staying in $\mathbb{H}$ thereafter. Let $\gamma_{[0,\infty]}$ be a parameterized simple continuous curve from 0 to ∞ in $\mathbb{H}$ and assume that the capacity parameterizations has been chosen, so that $\mathbb{K}_t \equiv \gamma_{]0,t]}$ is a hull with capacity $2t$. When ε is small, $\mathbb{K}_{\varepsilon,t} \equiv g_t(\gamma_{]t,t+\varepsilon]})$ is a tiny piece of a curve. The support of the discontinuity measure $d\rho_{f_{\varepsilon,t}}$ is small and becomes a point when ε goes to 0. Measures supported at a point are δ functions, so there is a point ξ_t such that, as a measure, $d\rho_{f_{\varepsilon,t}}/dx \sim 2\varepsilon\delta(x-\xi_t)$ as $\varepsilon \to 0^+$.

For a general local Loewner growth process, one defines $\gamma_t = f_t(\xi_t + i0^+) \equiv \lim_{\varepsilon\mapsto 0^+} f_t(\xi_t + i\varepsilon)$ (remember f_t is the inverse map of g_t). We shall often use the shorthand notation $\gamma_t = f_t(\xi_t)$. The set $\gamma_{]0,t]} \equiv \cup_{s\in]0,t]}\gamma_s$ is called *the trace* of the growth process. If the hull is a simple curve, the notation is consistent. Whether the trace is a curve (simple or not) in general is highly non obvious, but this will be the case for all examples in these notes, though proving it can be a formidable task.

At time t, growth takes place at point ξ_t in the g_t plane i.e. at point γ_t in the original "physical" plane. Thus it is tempting to conclude that $\mathbb{K}_t$ coincides with $\gamma_{]0,t]}$. Though this picture works nicely for examples 2-3, it is slightly too naïve and fails in the example 4 when the trace, which is an arc of circle closes to a semicircle and the corresponding semidisc completes the hull.

For a given z with $\Im m z \geq 0$ and $z \neq \xi_0$, the local existence and uniqueness of solutions to eq.(2.2) is granted by general theorems on ordinary differential equations, but problems may arise if a time τ_z (depending on z in general) exists for which $g_{\tau_z}(z) = \xi_{\tau_z}$. One possibility is to declare $g_t(z)$ undefined for $t \geq \tau_z$. But it is often the case that, as suggested by examples 2-3, the two limits $\lim_{x\mapsto\xi^{\pm}_{\tau_z}} g_t \circ f_{\tau_z}(x)$ exist, allowing to think that after τ_z, $g_t(z)$ has split in two real trajectories.

There is regularity criterion on the function $\xi_\cdot$ that guaranties that if $x \neq \xi_0$ is real, τ_x is infinite. It is sufficient that for each t,

$$\lim_{s\mapsto t^-} \sup_{t'\in[s,t[} \frac{|\xi_t - \xi_{t'}|}{|t-t'|^{1/2}} < 4. \tag{2.3}$$

To prove this criterion, it is convenient to consider $X_t \equiv g_t(x) - \xi_t$, a

continuous function which satisfies the integral equation $X_t = x - \xi_t + \int_0^t \frac{2ds}{X_s}$. As this implies that $\xi_\tau - \xi_t = X_t - X_\tau + \int_t^\tau \frac{2ds}{X_s}$, we can see $\xi_.$ as a functional of $X_.$. The task is to control its behavior if X_t has a given sign, say positive, on $[0,\tau[$ and vanishes at τ. It is clear that the two terms in $X_t + \int_t^\tau \frac{2ds}{X_s}$ vary in opposite directions, in that the faster X_t goes to 0 , the slower is the vanishing of $\int_t^\tau \frac{2ds}{X_s}$ at $t = \tau$. So the mildest behavior of the sum as t goes to τ is when the two terms have a similar behavior. A detailed analysis requires some care, but a quick and dirty way to retrieve the criterion is to impose that the two terms be equal, which gives $X_t = 2\sqrt{\tau - t}$ hence $\xi_\tau - \xi_t = 4\sqrt{\tau - t}$ as announced.

Example 6 *: Square root driving term.*

The Loewner equation when $\xi_\tau - \xi_t = 4\alpha\sqrt{\tau - t}$ can be solved in closed form for any α though the formulæ are cumbersome. We normalize ξ_t so that $\xi_0 = 0$, i.e. take $\xi_t = 4\alpha(\sqrt{\tau} - \sqrt{\tau - t})$. By left-right symmetry, we can assume that $\alpha \geq 0$. For $\alpha \in [1, +\infty[$ it is convenient to set $\alpha \equiv \cosh\eta$, $\eta \in [0, +\infty[$. One parameterizes time as

$$\frac{2e^{-\eta \coth \eta} \sinh \eta}{\sin(2\vartheta \sinh \eta)} \frac{(\sin(\vartheta e^{\eta}))^{(\coth \eta + 1)/2}}{(\sin(\vartheta e^{-\eta}))^{(\coth \eta - 1)/2}} = \sqrt{\frac{\tau - t}{\tau}},$$

with $\vartheta \in [0, \pi e^{-\eta}]$. As a function of ϑ, the hull builds the curve

$$\left\{ 2\sqrt{\tau} \left(e^{-\eta} - \frac{2 \sinh \eta \sin(\vartheta e^{-\eta})}{\sin(2\vartheta \sinh \eta)} e^{i\vartheta e^{\eta}} \right) \right\}_{\vartheta \in [0, \pi e^{-\eta}]}.$$

For $\vartheta = \pi e^{-\eta}$ the curve closes a whole domain, just as in the arc of circle example 4, which in fact is the special case $\alpha = 3\sqrt{2}$.

For $\alpha \in [0, 1[$ it is convenient to set $\alpha \equiv \cos\varphi$, $\varphi \in]0, \pi/2]$. The formulæ can be obtained by analytic continuation $\eta \mapsto i\varphi$, this time with a parameter $\vartheta \in [0, \infty]$. The hulls remain simple curves even for $\vartheta = \infty$.

Fig.2.7 illustrates the different behaviors.

The very same criterion on the behavior of the function $\xi_.$ is also sufficient for to ensure that the hull $\mathbb{K}_t$ is a simple continuous curve, say $\{\gamma_s, s \in]0, t]\}$, and $\gamma_t = f_t(\xi_t)$, i.e. that our naïve expectation $\mathbb{K}_t = \cup_{s \in]0,t]} f_s(\xi_s)$ is fulfilled.

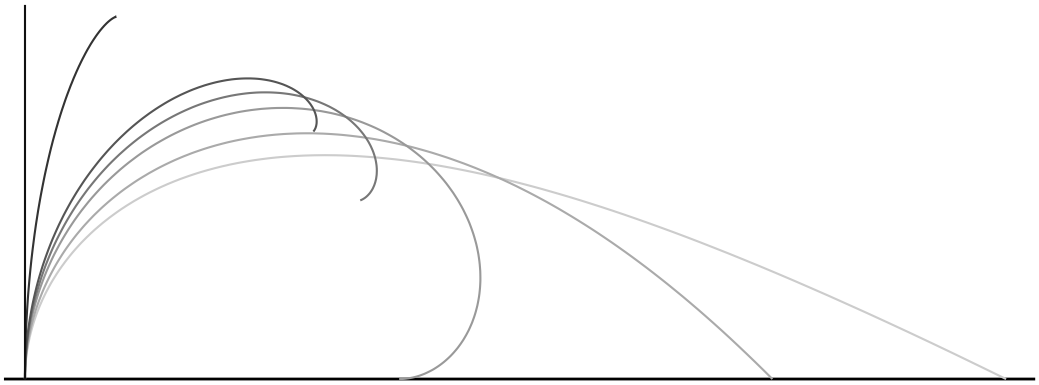

Figure 2.2: The hull at time τ for $\alpha = 0, 1, 2, 3, 4, 5, 6$

The two properties – "$g_t(x))$ for real x does not hit ξ_t" and "the hull is a simple curve"– are in fact equivalent. The intuitive reason is the following. The fact that $g_t(x))$ for real x hits ξ_t at some time τ is the sign that at time τ the hull "swallows a whole piece of $\mathbb{H}$". The previous example illustrates this relationship when the hull hits the real axis. But from the point of view of iteration, if $s \geq 0$ is fixed, it is obvious that when $t \geq 0$ varies the function $\tilde{g}_{t,s}(z) \equiv g_{t+s} \circ f_s(z + \xi_s) - \xi_s$ satisfies the Loewner equation (2.2) with driving function $\tilde{\xi}_t \equiv \xi_{t+s} - \xi_s$. So if the driving function $\tilde{\xi}_t \equiv \xi_{t+s} - \xi_s$ leads to a hull hitting the real axis, the driving function ξ_t leads to a hull hitting itself or the real axis. This discussion also explains why, if the trace is a continuous curve, it can have double points but no crossings.

Chapter 3

Stochastic Loewner evolutions

Stochastic Loewner evolutions were introduced by Schramm in 1999 as a general framework to study random curves satisfying certain properties. His specific interest was to prove that loop erased random walks (in short lerw's, the precise definition is irrelevant here) on a two dimensional lattice have a conformally invariant continuum limit. Schramm observed that these walks have on the lattice the so-called domain Markov property (to be defined below) a property that can that can be rephrased in the continuum. Though he was not able at that time to prove the existence of a conformally invariant limit of lerw's, he recognized that conformal invariance and the domain Markov property brought together would have remarkable consequences, and was able to prove that the probability measures on random curves in the continuum satisfying at the same time conformal invariance and the domain Markov property formed a one parameter family. Crucial to the proof and the explicit description of these measures was the idea of viewing curves as hulls and to use Loewner evolutions. That in this context the most useful description of a curve is by encoding it into a growth process via a Loewner chain is at first sight very surprising and may explain why physicists who had understood the importance of conformal invariance to study many examples of random curves in the early 1980's failed to "produce Schramm's argument before Schramm".

The general idea is to impose properties relating different members in a family of probability measures on continuous curves without crossings, but possibly with multiple points. Let us note that curves here are considered modulo reparameterizations, but not simply as subsets of the plane. For simple curves, this would essentially make no difference, but curves with

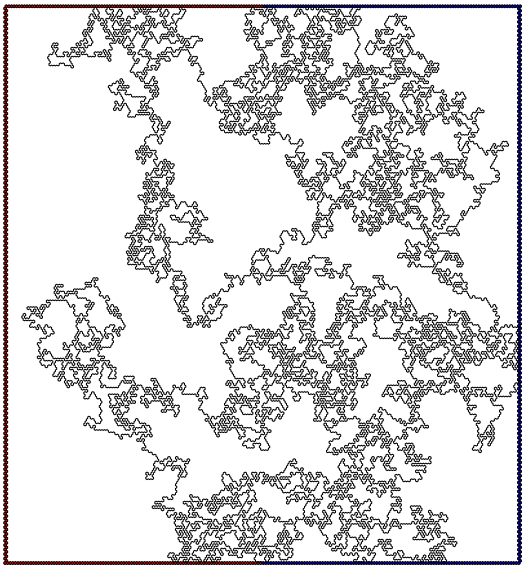

Figure 3.1: A percolation sample

multiple points require more care.

In the discrete setting, it is a fact that interfaces on appropriate lattices are simple curves, so why bother to deal with non simple curves ? The answer is that even if at the scale of the lattice spacing the interface is simple, when one tries to take a continuum limit by looking at a macroscopic scale while taking a smaller and smaller lattice spacing, a curve that makes a large excursion and then comes back close to itself, say a few lattice spacings away, has a double point from the macroscopic viewpoint. While in some models –like lerw's, Schramm's initial motivation– the interface remain simple when the lattice spacing gets smaller, some other models –like percolation– clearly exhibit multiple points in the continuum limit. This is clearly seen on samples, see Fig.3.

In the following three sections, we suppose that we are given a family of probability measures $\{\mathbf{P}_{\mathbb{D},a,b}\}$ indexed by triples consisting of a domain $\mathbb{D}$ and two distinct boundary points a, b of $\mathbb{D}$. For a given triple $(\mathbb{D}, a, b)$, $\mathbf{P}_{\mathbb{D},a,b}$ is a a measure on $\Omega_{\mathbb{D},a,b}$, the set of continuous curves without crossings within $\overline{\mathbb{D}}$ –the union of $\mathbb{D}$ and its boundary (in the refined sense alluded too in section 2.1)– joining a to b (it is understood that a and b are not multiple points).

First, we want do define what it means for the family $\{\mathbf{P}_{\mathbb{D},a,b}\}$ to be conformally invariant and to have the domain Markov property.

3.1 Conformal invariance

By Riemann's theorem, if $(\mathbb{D}, a, b)$ and $(\mathbb{D}, a', b')$ are any two triples, there is a conformal map $g : \mathbb{D} \mapsto \mathbb{D}'$ such that $g(a) = a'$ and $g(b) = b'$. It is clear that g induces a bijection, which we call $\check{g}$, from $\Omega_{\mathbb{D},a,b}$ to $\Omega_{\mathbb{D}',a',b'}$. Conformal invariance of the family $\{\mathbf{P}_{\mathbb{D},a,b}\}$ is the statement that $\check{g}$ is measurable and the image measure $\mathbf{P}_{\mathbb{D},a,b} \circ \check{g}^{-1}$ coincides with $\mathbf{P}_{\mathbb{D}',a',b'}$, i.e. if C' is a measurable subset of $\Omega_{\mathbb{D}',a',b'}$ then $\check{g}^{-1}(C')$ is a measurable subset of $\Omega_{\mathbb{D},a,b}$ and $\mathbf{P}_{\mathbb{D},a,b}(\check{g}^{-1}(C')) = \mathbf{P}_{\mathbb{D}',a',b'}(C')$.

Conformal invariance by itself is a rather weak constraint. Indeed, suppose that a probability $\mathbf{P}_{\mathbb{D}_0,a_0,b_0}$ on $\Omega_{\mathbb{D}_0,a_0,b_0}$ has been defined for a single triple $\mathbb{D}_0, a_0, b_0$ and that it is invariant under the conformal transformations of $\mathbb{D}_0$ fixing a_0 and b_0. Such transformations form a group with one real parameter. Then the direct image $\mathbf{P}_{\mathbb{D}_0,a_0,b_0}$ by any conformal transformation g will define unambiguously $\mathbf{P}_{g(\mathbb{D}_0),g(a_0),g(b_0)}$. By the Riemann mapping theorem, this defines $\mathbf{P}_{\mathbb{D},a,b}$ for any triple, and the resulting family of probabilities is clearly conformally invariant.

To get a more rigid situation, one has to impose another constraint on the family $\{\mathbf{P}_{\mathbb{D},a,b}\}$. Schramm translated in the continuum a property that holds for loop erased random walks in the discrete setting : the domain Markov property, to which we turn our attention now.

Before doing so, let us remark that this strategy is rather typical. If continuous curves without crossings are replaced by general hulls joining a to b in $\mathbb{D}$ the notion of domain Markov property does not make sense but another one, restriction, turns out to be fruitful and allow for another complete classification. We shall have little to say about these nice "restriction measures" in the sequel.

3.2 Domain Markov property

Fix a triple $(\mathbb{D}, a, b)$ and consider an element $\gamma \in \Omega_{\mathbb{D},a,b}$. If a real continuous parameter along γ is given and s is any intermediate value of the parameter, the past and the future of s split γ in two (not necessarily dis-

joint) curves without crossings. The curve corresponding to the past of s starts at a and is called an initial segment of γ. The curve corresponding to the future of s ends at b and is called a final segment of γ. The final segment starts at some point $c \in \mathbb{D}$ which is also the end of the initial segment. We use the notation $\gamma_{]a,c]}$ for such an initial segment with point c included and $\gamma_{]c,b]}$ for the final segment. Beware that the notation is a bit ambiguous, because of possible multiple points on γ.

Several curves γ' share the same initial segment $\gamma_{]a,c]}$, and the discussion that follows focuses on the question : if an initial segment is given, what is the distribution of the final segment?

Making sense of this question is not so obvious. First, there should be enough measurable sets in $\Omega_{\mathbb{D},a,b}$. We shall for a while assume that this is so. But even in that case, the event "γ' starts exactly with$\gamma_{]a,c]}$" is more than likely to occur with probability 0. Vaguely, what may have a nontrivial probability is the event "γ' has an initial segment that is close (in some quantified sense) to $\gamma_{]a,c]}$". Probabilists have invented so called conditional expectations and regular conditional probabilities just to deal with that kind of situations. Starting from $\mathbf{P}_{\mathbb{D},a,b}$ this allows to define new probability measures, denoted $\mathbf{P}_{\mathbb{D},a,b}(\quad|\gamma_{]a,c]})$, read "conditional probability given the initial segment $\gamma_{]a,c]}$", that can be manipulated just as conditional probabilities when the state space is discrete[1].

The set of points in $\mathbb{D}$ that cannot be joined to b by a continuous curve in $\mathbb{D}$ without hitting the initial segment form a set that we call a hull[2] and denote by $\mathbb{K}_c$. This notation is again slightly ambiguous. Note that $\mathbb{D} \setminus \mathbb{K}_c$ is again a domain. If the initial segment is $\gamma_{]a,c]}$, the final segment starts at c and never enters inside $\mathbb{K}_c$. So the support of the conditional probability $\mathbf{P}_{\mathbb{D},a,b}(\quad|\gamma_{]a,c]})$ is included in $\Omega_{\mathbb{D}\setminus\mathbb{K}_c,c,b}$. But on this set we have another probability measure, namely $\mathbf{P}_{\mathbb{D}\setminus\mathbb{K}_c,c,b}$, and the two can be compared.

We say that a set $\{\gamma_{]a,c]}\}$ of curves in $\mathbb{D}$ without crossings starting at a is a set of distinct representatives if any curve in $\Omega_{\mathbb{D},a,b}$ has exactly one of its initial segments in $\{\gamma_{]a,c]}\}$. For instance, for the triple $(\mathbb{H}, 0, \infty)$, the

[1] There is a small price to pay, however. For instance, the definition of this conditional probability may fail or be ambiguous for certain $\gamma_{]a,c]}$ but these nasty initial segments form altogether a set of probability 0 for $\mathbf{P}_{\mathbb{D},a,b}$.

[2] If $(\mathbb{D}, a, b) = (\mathbb{H}, 0, \infty)$, this is consistent with our initial definition, and with the new definition, conformal maps send hulls to hulls

initial segments whose associated hull has capacity t form a set of distinct representatives. Intuitively, to get the expectation of a random variable on $\Omega_{\mathbb{D},a,b}$, one can compute its conditional expectation on $\gamma_{]a,c]}$, and then integrate over $\gamma_{]a,c]}$ in a system of distinct representatives.

The family $\{\mathbf{P}_{\mathbb{D},a,b}\}$ is said to have the *domain Markov property* if, for any triple $(\mathbb{D}, a, b)$, $\mathbf{P}_{\mathbb{D},a,b}(\quad|\gamma_{]a,c]}) = \mathbf{P}_{\mathbb{D}\setminus\mathbb{K}_c,c,b}$ except maybe for a set of initial segments whose intersection with any system of distinct representatives is of measure 0 for $\mathbf{P}_{\mathbb{D},a,b}$.

This expression of the domain Markov property is more intuitive on the lattice in the discrete setting –because the interfaces are simple curves and because conditional probabilities have a much simpler definition– and it holds in many examples. It is vaguely related to the notion of locality in physics. The reader can check it straightforwardly for the exploration process. Eq.(1.1) makes the domain Markov property plain for loop-erased random walks as well, whereas a direct proof using the original definition is more cumbersome.

3.3 Schramm's argument

Our aim is to explore the interplay between conformal invariance and the domain Markov property of the family $\{\mathbf{P}_{\mathbb{D},a,b}\}$.

First, by conformal invariance, we may concentrate on the triple $(\mathbb{H}, 0, \infty)$. We choose a parameterization of curves in $\Omega_{\mathbb{H},0,\infty}$ in such a way that the hull $\mathbb{K}_t \equiv \mathbb{K}_{\gamma_t}$ associated with the initial segment $\gamma_{]0,t]} \equiv \gamma_{]0,\gamma_t]}$ of $\gamma \in \Omega_{\mathbb{H},0,\infty}$ has capacity $2t$. Because of the underlying continuous curve γ, the growth of $\mathbb{K}_t$ is local, and the associated g_t satisfies a Loewner equation $\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z)-\xi_t}$ for some continuous function ξ_t. The probability $\mathbf{P}_{\mathbb{H},0,\infty}$ on $\Omega_{\mathbb{H},0,\infty}$ induces a a random process on the set of initial segments $\gamma_{]0,t]}$, hence on the set of hulls $\mathbb{K}_t$, and on the set of continuous functions ξ_t.

Our next aim is to derive consequences for the stochastic process ξ_t of the domain Markov property and conformal invariance.

First for fixed $(\mathbb{H}, 0, \infty)$ there is a remnant of conformal invariance : dilations. Hence for $\lambda > 0$, the hull $\frac{1}{\lambda}\mathbb{K}_{\lambda^2 t}$ must have the same distribution as a $\mathbb{K}_t$. The corresponding Loewner map is $\frac{1}{\lambda}g_{\lambda^2 t}(\lambda z)$, whose driving function is $\frac{1}{\lambda}\xi_{\lambda^2 t}$. Hence the processes ξ_t and $\frac{1}{\lambda}\xi_{\lambda^2 t}$ have the same law. We say that ξ_t has dimension $1/2$

Given K_t, the domain Markov property states that $\gamma_{]t,\infty]}$ is distributed according to $\mathbf{P}_{\mathbb{D}\setminus\mathbb{K}_t,\gamma_t,\infty}$. The conformal transformation $g_t(z) - \xi_t$ maps $\mathbb{D}\setminus\mathbb{K}_t$ to $\mathbb{H}$, γ_t to 0 and ∞ to ∞. By conformal invariance, $g_t(\gamma_{]t,\infty]}) - \xi_t$ is distributed according to $\mathbf{P}_{\mathbb{H},0,\infty}$. In particular for $s \geq 0$ $g_t(\gamma_{]t,t+s]}) - \xi_t$ has the same distribution as a $\gamma_{]0,s]}$ hence is independent of $\gamma_{]0,t]}$. But the Loewner map for $g_t(\gamma_{]t,t+s]}) - \xi_t$ is $g_{s+t} \circ f_t(z + xi_t) - \xi_t$ (remember f_t is the inverse of g_t), whose driving function is $\xi_{t+s} - \xi_t$. We infer that the random function $\xi_.$ is such that for any $t, s \geq 0$, $\xi_{t+s} - \xi_t$ is independent of $\{xi_{t'}\}$, $t' \in [0, t]$ and distributed like a ξ_s.

To resume our knowledge, the random process $\xi_.$ has continuous samples, independent identically distributed increments and dimension 1/2. By a deep general result, a random process with continuous samples and independent identically distributed increments is of the form $\sqrt{\kappa}B_t + \rho t$ for some nonnegative κ and some real ρ. Obviously it has dimension 1/2 if and only if $\rho = 0$.

To conclude, Schramm's argument shows that if a family of probabilities $\{\mathbf{P}_{\mathbb{D},a,b}\}$ on curves without crossing indexed by triples $(\mathbb{D}, a, b)$ is conformally invariant and has the domain Markov property, the law induced by $\mathbf{P}_{\mathbb{H},0,\infty}$ on initial hulls of capacity $2t$ by is described by a stochastic Loewner evolution

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \sqrt{\kappa}B_t} \tag{3.1}$$

for some $\kappa \geq 0$ and some normalized Brownian motion B_t.

A priori, this does not show that each κ is realized via some family $\{\mathbf{P}_{\mathbb{D},a,b}\}$ (because the Loewner evolution deals wit hulls, not with curves).

3.4 Basic properties

The first important property is a kind of converse to Schramm's result. If $\kappa \geq 0$ is a real number, and B_t a continuous realization of a normalized Brownian motion, a deep theorem states that the trace associated to the stochastic Loewner evolution eq.(3.1) is almost surely a continuous curve joining 0 to ∞. This curve is simple and stays in $\mathbb{H}$ if $\kappa \in [0, 4]$, has double points and hits the real axis if $\kappa \in]4, 8[$ and is spacefilling if $\kappa \in [8, +\infty[$.

At the time Schramm introduced stochastic Loewner evolutions, this very hard theorem was not known (he contributed to prove it later).

As explained before, a continuous trace cannot have crossings. Thus for any $\kappa \geq 0$, the stochastic Loewner evolution defines a probability measure $\mathbf{P}_\kappa$ on continuous curves without crossings joining 0 to ∞ in $\mathbb{H}$. This measure is scale invariant. Hence, for each κ, conformal transformations can be used to define in a consistent way a family of probabilities $\{\mathbf{P}^\kappa_{\mathbb{D},a,b}\}$. This family is trivially conformally invariant, and it is easy to check that is satisfies the domain Markov property.

This finishes the complete classification.

Taking the existence of a curve for granted, the change of behavior from simple curves to curves with double points at $\kappa = 4$ can be understood as follows. First, the necessary condition (negation of eq.(2.3) for the existence of multiple points is fulfilled for all values of κ, though in some kind of marginal way, for if $\xi_t = \sqrt{\kappa} B_t$ where B_t is a normalized Brownian motion, the law of the iterated logarithm states that, with probability one

$$\lim_{s \mapsto t^-} \sup_{t' \in [s,t[} \frac{|\xi_t - \xi_{t'}|}{|t-t'|^{1/2} \log\log|t-t'|^{-1}} = \sqrt{2\kappa}.$$

So the stochastic Loewner source is wilder by a $\log\log|t-t'|^{-1}$ than the criterion. The fact that for $\kappa \leq 4$ the Loewner trace is a simple curve shows that, as should be expected, the criterion is only necessary, but not sufficient. Intuitively, Brownian motion is more singular than necessary, but for $\kappa \leq 4$ with too little correlation time to behave consistently for long enough periods to produce multiple points.

This fact is related to another well studied question : recurrence of Brownian motion. If space dimension d is 1, Brownian motion passes infinitely many times at any point, if $d = 2$, it passes infinitely many times in the any neighborhood of any point, but not exactly at any given point, and if $d \geq 3$, it has a nonzero probability to remain at a given finite distance of any point. So dimension 2 is somehow a marginal case. Now let R_t be the norm of a d-dimensional Brownian motion. Assume $R_0 > 0$. One can show using stochastic calculus that $W_t \equiv -R_t + \frac{d-1}{2}\int_0^t \frac{ds}{R_s}$ is a standard 1-dimensional Brownian motion. In this equation, d appears as a an explicit parameter, and one can reverse the logic : given a standard 1-dimensional Brownian motion W_t what are the properties of R_t, called the d-dimensional Bessel process mathematics. Setting $\kappa = 4/(d-1)$ one sees that $X_t \equiv \sqrt{\kappa}(R_t + W_t)$ satisfies the equation $\frac{dX_t}{dt} = \frac{2}{X_t - \sqrt{\kappa} W_t}$ so first, indeed from W_t on can retrieve R_t by solving a differential equation and

second, the Bessel process is essentially a stochastic Loewner evolutions but looking only at the boundary of $\mathbb{H}$. For general d, the Bessel processes behave with respect to visits to 0 just like the recurrence properties of Brownian motion for integer d suggest: the d-dimensional Bessel process hits the origin infinitely many times if $d < 2$, but never if $d \geq 2$. Equivalently, if $\kappa \leq 4$, $X_t - \sqrt{\kappa}W_t$ never vanishes, but vanishes infinitely many times if $\kappa > 4$. But we already know that the vanishing of $X_t - \sqrt{\kappa}W_t$ is the sign that the growing curve hits itself or the real axis.

Another very hard result is the fractal dimension : the measures $\mathbf{P}^{\kappa}_{\mathbb{D},a,b}$ is concentrated on curves with fractal dimension $\min\{1 + \kappa/8, 2\}$.

Two additional properties have been used to constraint further the situation.

The first one is locality. Let $\mathbb{L}$ be a hull in $\mathbb{D}$ bounded away from a and b. To each curve in $\Omega_{\mathbb{D},a,b}$ we can associate its smallest initial segment that hits the boundary of $\mathbb{L}$ (we take this initial segment to be the curve itself if it never hits $\mathbb{L}$. These initial segments form a system Σ of distinct representatives both in $\Omega_{\mathbb{D},a,b}$ and in $\Omega_{\mathbb{D}\setminus\mathbb{L},a,b}$. Thus both $\mathbf{P}_{\mathbb{D},a,b}$ and $\mathbf{P}_{\mathbb{D}\setminus\mathbb{L},a,b}$ induce a probability measure on Σ. The property of locality is the statement that these two measures coincide. In a more mundane way, if $\mathbb{L}$ is a hull in $\mathbb{D}$ bounded away from a and b, the distribution of curves up to the first hitting of $\mathbb{L}$ are the same in $\mathbb{D}$ and in $\mathbb{D} \setminus \mathbb{L}$. Stochastic calculus can be used to show that the family $\{\mathbf{P}^{\kappa=6}_{\mathbb{D},a,b}\}$ is the only one to have the locality property. Let us note that it is no surprise that a value of κ satisfying locality is > 4. Indeed, if $\kappa \leq 4$, the traces are simple curves that do not hit the boundary. Then no trace touches $\mathbb{L}$ for $\mathbf{P}_{\mathbb{D}\setminus\mathbb{L},a,b}$, but hitting $\mathbb{L}$ for $\mathbf{P}_{\mathbb{D},a,b}$ has a finite probability if $\mathbb{L}$ is nontrivial, so that the supports of the two probability measures induced on Σ are not the same.

On the lattice, percolation is modelled by coloring each site with one of two colors, independently of the other sites. The associated interfaces have obviously the locality property, so the only candidate if percolation has a conformally invariant continuum limit is the family with $\kappa = 6$. That percolation has a conformally invariant continuum limit has indeed been proved.

The second one is the restriction property. Again, let $\mathbb{L}$ be a hull in $\mathbb{D}$ bounded away from a and b. Consider the subset $\Gamma_{\mathbb{L}}$ of $\Omega_{\mathbb{D},a,b}$ made

of curves that do not hit $\mathbb{L}$, and the associated conditional probability $\mathbf{P}_{\mathbb{D},a,b}(\quad|\Gamma_{\mathbb{L}})$. Note that $\Gamma_{\mathbb{L}}$ is a subset of $\Omega_{\mathbb{D}\setminus\mathbb{L},a,b}$. The restriction property is the statement that $\mathbf{P}_{\mathbb{D},a,b}(\quad|\Gamma_{\mathbb{L}}) = \mathbf{P}_{\mathbb{D}\setminus\mathbb{L},a,b}$. Stochastic calculus can be used to show that the family $\{\mathbf{P}^{\kappa=8/3}_{\mathbb{D},a,b}\}$ is the only one to have the restriction property. This restriction property was alluded to before. Indeed, properly defined restriction measures form a one parameter family of measures on hulls, which intersect the SLE family at $\kappa = 8/3$. Let us note that the filling of a Brownian excursion is another example of restriction measure. Again, it is no surprise that a value of κ satisfying restriction is ≤ 4. Indeed, if $\kappa > 4$, the probability of hitting $\mathbb{L}$ under $\mathbf{P}_{\mathbb{D}\setminus\mathbb{L},a,b}$ would be nonzero, so that $\Gamma_{\mathbb{L}}$ would not have full measure in $\Omega_{\mathbb{D}\setminus\mathbb{L},a,b}$.

On the lattice, the weight of a self avoiding walk is given in the plane, and then the same weight is used for this curve in any domain that contains it. So self avoiding walks on the lattice have the restriction property. So the only candidate if self avoiding walks have a conformally invariant continuum limit is the family with $\kappa = 8/3$. But this time a proof of the existence of a continuum limit of self avoiding walks is still to come.

Recently, two important conjectures on SLE have been proven.

One of them is reversibility. The treatment of random curves by a Loewner evolution is quite asymmetric by definition. However interfaces between two points in physics (i.e. in statistical mechanics models) quite generally make no difference between the two ends. So it was conjectured very early that interfaces generated by an SLE process were reversible. One difficulty is with the parameterization. Take an SLE sample in $\mathbb{H}$ from 0 to infinity, parameterize it with capacity. Apply the transformation $z \mapsto -1/z$ and parameterize the inversed sample with capacity. Now any point on the curve has two parameters attached to it. One of the troubles is that the relationship between the two parameters is extremely wild. Anyway, reversibility is now a theorem.

The second one is duality. Take an SLE_κ sample with $\kappa > 4$ and look at the boundary of $\mathbb{K}_t$. This is simple curve, and one can expect that its distribution is conformally invariant in some sense. So it is natural to ask if and how it fits in the SLE framework. It was conjectured by physicists that it is related in some sense to an $\mathrm{SLE}_{16/\kappa}$, and that in particular it has dimension $1 + 2/\kappa$. Though this is correct, the precise recent theorem that

gives an explicit description involves nontrivial extensions of SLE where the driving function is $\sqrt{16/\kappa}B_t$ plus some rather complicated drift terms.