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Conformably invariant restriction measures on Riemann surfaces.

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CONFORMALLY INVARIANT RESTRICTION MEASURES ON RIEMANN I: CHORDAL RESTRICTION MEASURES

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1. A (physically inspired) introduction to function theory on a Riemann surface

Felix Klein motivated his notion of "Complex function of position" by considering steady streamings of a fluid. We will begin by considering steady streamings in the plane.

Suppose a fluid in the (x, y)-plane has a velocity with x-component $P(x, y)$, and y-component $Q(x, y)$. Consider an infinitesimal square in the plane, parallel to the x and y-axis, with sides labeled I, II, III, IV, counter-clockwise, and side I parallel to the x-axis, below side III. Then

$$\begin{aligned} \text{flow across side } I \ &= Q(x, y)\ dx \\ \text{flow across side } III \ &= Q(x, y + dy)\ dx. \end{aligned}$$

So the difference $III - I$ is

$$\frac{\partial Q}{\partial y}\ dydx.$$

Similarly, the difference $II - IV$ is

$$\frac{\partial P}{\partial x}\ dxdy.$$

The streaming is steady if there are no sources or sinks (we also assume the fluid incompressible), i.e. there is as much fluid streaming into the infinitesimal square as there is streaming out:

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0 \quad \text{(zero divergence)}.$$

The circulation of the fluid around a closed curve C is

$$\int_C (P\ dx + Q\ dy).$$

The flow is irrotational if the circulation vanishes for all closed curves C. This implies that $P\ dx + Q\ dy$ is an *exact differential*, i.e. there exists a "velocity potential" $u(x, y)$ so that

$$P = \frac{\partial u}{\partial x} = u_x, \quad Q = \frac{\partial u}{\partial y} = u_y.$$

The zero divergence property then translates into

$$u_{xx} + u_{yy} = 0 \quad \text{(Laplace equation)},$$

so that u is harmonic.

A $(x(t), y(t))$ is an equipotential line if $u(x(t), y(t)) = \mathit{const}$. Then

$$u_x\dot{x} + u_y\dot{y} = 0,$$

so that an equipotential line that crosses the x-axis, crosses at an angle α, where

$$\frac{\dot{y}}{\dot{x}} = \tan\alpha = -\frac{u_x}{u_y}$$

if $u_x^2 + u_y^2 > 0$. On the other hand, the velocity vector of the fluid at that point forms an angle β with the x-axis, where

$$\frac{Q}{P} = \tan\beta = \frac{u_y}{u_x}.$$

Hence

$$|\alpha - \beta| = \pi/2$$

and the flow is perpendicular to the equipotential lines in the direction of increasing u. If we interchange streamlines with equipotential lines we get a "conjugate flow," whose velocity potential v needs to satisfy

$$v_x = -Q, \quad v_y = P \qquad \text{(or, alternatively, } v_x = Q, v_y = -P\text{)}.$$

This means u, v satisfy the *Cauchy-Riemann equations*

$$u_x = v_y, \quad u_y = -v_x.$$

If we set $z = x + iy$, and

$$f(z) = u(x, y) + iv(x, y),$$

then f is a holomorphic (analytic) function of z. Klein calls f a "complex function of position." It is also known as the complex potential of the flow.

1.1. **Steady streamings on a surface S.** Suppose S is given by the functions $x^i = x^i(\xi, \eta)$, $i = 1, 2, 3$ of the rectilinear parameters ξ, η and write $x = (x^1, x^2, x^3)$. A curve C on S is given by $x(\xi(t), \eta(t))$. Then the element of arc length along C is

$$\begin{aligned} ds^2 = dx \cdot dx &= (x_\xi \, d\xi + x_\eta \, d\eta) \cdot (x_\xi \, d\xi + x_\eta \, d\eta) \\ &= E \, d\xi^2 + 2F \, d\xi d\eta + G \, d\eta^2, \end{aligned}$$

with

$$E = x_\xi \cdot x_\xi, \quad F = x_\xi \cdot x_\eta, \quad G = x_\eta \cdot x_\eta.$$

Note that $ds^2 > 0$ implies

$$W^2 \equiv EG - F^2 > 0.$$

Suppose u is the velocity potential of a streaming on S. By that we mean that the streaming is orthogonal to the lines $u = const.$, with velocity $\partial u/\partial n$, where ∂n is the normal to the equipotential line. Again, the streaming is to be steady: Consider the patch in the surface S corresponding to an infinitesimal square in the (ξ, η)-parameter plane.

The direction parameters of a ξ-curve, a curve $x(\xi,\eta)$ where ξ moves while η is fixed, are

$$(\lambda^1,\lambda^2)=(E^{-1/2},0) \text{ (unit length).}$$

The orthogonal unit vector, relative to the metric on S, is

$$(\mu^1,\mu^2)=(-F/(E^{1/2}W),E^{1/2}/W).$$

So the flow at right angle to a ξ-curve is

$$u_\xi\mu^1+u_\eta\mu^2=\frac{1}{E^{1/2}W}(-Fu_\xi+Eu_\eta),$$

while, by symmetry, the velocity of the flow at right angle to an η-curve is

$$\frac{1}{G^{1/2}W}(-Fu_\eta+Gu_\xi).$$

It follows that the flow across the coordinate curve extending from (ξ,η) to $(\xi+d\xi,\eta)$ is

$$\frac{1}{E^{1/2}W}(-Fu_\xi+Eu_\eta)\ E^{1/2}\ d\xi=\frac{1}{W}(-Fu_\xi+Eu_\eta)\ d\xi,$$

while the flow across the curve from $(\xi,\eta+d\eta)$ to $(\xi+d\xi,\eta+d\eta)$ is

$$\frac{1}{W}(-Fu_\xi+Eu_\eta)+\frac{\partial}{\partial\eta}\left(\frac{1}{W}(-Fu_\xi+Eu_\eta)\right)\ d\eta d\xi,$$

where the second term is the difference. Adding the difference from the flow across the other two boundary lines gives the *Beltrami equations* for the velocity potential of a steady streaming:

$$\frac{\partial}{\partial\xi}\left(\frac{Fu_\eta-Gu_\xi}{W}\right)+\frac{\partial}{\partial\eta}\left(\frac{Fu_\xi-Eu_\eta}{W}\right)=0.$$

Interchanging equipotential lines and streamlines we get a conjugate streaming. The velocity of the conjugate streaming in the ξ-direction should be the negative of the velocity of the original streaming perpendicular to ξ, i.e.

$$v_\xi\lambda^1+v_\eta\lambda^2=-(u_\xi\mu^1+u_\eta\mu^2),$$

so that

$$v_\xi=\frac{1}{W}(Fu_\xi-Eu_\eta). \tag{1}$$

Similarly, we obtain the companion equation

$$v_\eta=\frac{1}{W}(Gu_\xi-Fu_\eta). \tag{2}$$

Equations (1) and (2) generalize the Cauchy-Riemann equations to surfaces.

Remark 1. (i) That the curves $u = const.$ and $v = const.$ are (in general) orthogonal can also be seen by noting that

$$du^2 + dv^2 = \lambda ds^2$$

for some positive function λ (it is an easy exercise to calculate what λ is). This identity says that the complex function of position

$$p \in S \mapsto u + iv \in \mathbb{C}$$

is *conformal* (angle preserving).

(ii) If $x + iy$ is another complex function of position, then

$$dx^2 + dy^2 = \sigma ds^2$$

and the equations (1),(2) become

$$v_x = -u_y, \quad v_y = u_x,$$

since in this case the same calculation as above can be carried through, but now with $E = G = 1/\sigma, F = 0$. In other words, any complex function of position is an analytic function of any other complex function of position.

(iii) If Φ is conformal and orientation-preserving from S with element of arc-length ds^2 to the surface R with element of arc-length ds_1^2, then $ds^2 = \mu ds_1^2$, so that $du^2 + dv^2 = \lambda\mu ds_1^2$. It follows that $(u + iv) \circ \Phi^{-1}$ is a complex function of position on R.

(iv) The Beltrami equation can be solved under minimal assumptions. Thus we can always map a neighborhood of a point $p \in S$ conformally onto a domain of the plane. If (x, y) are the Euclidean coordinates of points in the neighborhood of p, then the map $z = x + iy$ is called a (local) *uniformizer* at p.

A *Riemann surface* is a surface which has uniformizers at each of its points, a pair of uniformizers valid over a common neighborhood being related by a conformal mapping.

Example 1 (Riemann sphere)**.** Let $S = \{x = (x^1, x^2, x^3) \in \mathbb{R}^3 : |x| = 1\}$ and identify $\{x^3 = 0\}$ with $\mathbb{C}$ by $(x^1, x^2, 0) \mapsto x^1 + ix^2$. Introduce the following functions: "Sight from north pole"

$$\Phi_1 : S\backslash\{(0,0,1)\} \to \mathbb{C}, \quad \Phi_1(x) = \frac{x^1 + ix^2}{1 - x^3},$$

and "sight from south pole"

$$\Phi_2 : S\backslash\{(0,0,-1)\} \to \mathbb{C}, \quad \Phi_2(x) = \frac{x^1 + ix^2}{1 + x^3}.$$

It is well known that these maps are conformal. Furthermore, for $x \in S \backslash \{(0,0,1),(0,0,-1)\}$

$$[\Phi_2(x)]^{-1} = \frac{x^1 - ix^2}{1 - x^3} = \overline{\Phi_1(x)},$$

so that $\Phi_1, \overline{\Phi_2}$ are compatible local uniformizers covering S.

A *closed Riemann surface* is a sphere with a finite number of handles attached.

A *finite Riemann surface* is a closed Riemann surface with a finite number of disks removed. Points on the boundary are covered by *boundary uniformizers*.

2. Chordal Schramm-Loewner evolution

Consider the upper half-plane $\mathbb{H}$ with a simple curve $t \mapsto \gamma_t$ in it, growing from a point $x \in \mathbb{R}$ to ∞. For each $t \geq 0$ we can use a streaming, regular except for a dipole singularity at ∞, to map $\mathbb{H}\backslash\gamma[0,t]$ onto $\mathbb{H}$: The streaming is parallel to the boundary, and we set the potential equal to zero there. We normalize by demanding that the map has expansion $z + a/z + o(1/|z|)$ as $z \to \infty$. The tip γ_t of the curve segment $\gamma[0,t]$ is mapped to a point on $\mathbb{R}$. In fact, that point is the value of the conjugate velocity potential at the tip of the slit. If the curve grows randomly, then the value of the conjugate velocity potential at the tip is a real-valued random motion β_t. If the random curve γ satisfies the conformal Markov property (plus one reflection symmetry) then it was shown by Schramm that $\beta_t = \sqrt{\kappa}B_t$, where B_t is a linear standard Brownian motion ($E[B_t^2] = t$).

If $B_0 = x \in \mathbb{R}$, then we call the random curve γ_t chordal SLE$_\kappa$ in $\mathbb{H}$ from x to ∞. For existence and many of its basic results, see Lawler, and Werner. For $\kappa = 8/3$, and only for this value, does chordal SLE possess the following property: If H is a simply connected domain in $\mathbb{H}$ so that $\mathbb{H}\backslash H$ is bounded and bounded away from 0, then chordal SLE$_{8/3}$ in the upper half-plane from 0 to ∞ conditioned on $\gamma \subset \overline{H}$ has the same law as the image of chordal SLE$_{8/3}$ in the upper half-plane from 0 to ∞ under a map Φ which sends $\mathbb{H}$ to H and fixes 0 and ∞. Note that two such maps differ by a scaling factor and that chordal SLE is scale invariant. If we define chordal SLE$_{8/3}$ in H from 0 to ∞ as the image of chordal SLE$_{8/3}$ in $\mathbb{H}$ from 0 to ∞ under a map Φ as above (again, this is well-defined by scale invariance). Then the restriction property says that SLE$_{8/3}$ conditioned to stay in the smaller domain H is SLE$_{8/3}$ in that smaller domain.

More generally, suppose we have a family $\{P^D_{p,q}\}$ of probability laws of random cross-cuts (simple curves connecting two distinct boundary points), where the family is indexed by simply connected non-degenerate domains D and boundary points $p, q \in \partial D$. Then we say that the family satisfies the restriction property if whenever $D' \subset D$, $p, q \in \partial D' \cap \partial D$ so that $D\backslash D'$ is bounded away from p and q, and γ has law $P^D_{p,q}$, then the law of γ conditioned on $\gamma \subset D'$ is equal to $P^{D'}_{p,q}$.

If now $D'' \subset D' \subset$, $p, q \in \partial D'' \cap \partial D$ so that these points are bounded away from $D\backslash D''$, then

$$
\begin{aligned}
P^D_{p,q}(\gamma \subset D'') &= P^D_{p,q}(\gamma \subset D'' | \gamma \subset D') P^D_{p,q}(\gamma \subset D') \\
&= P^{D'}_{p,q}(\gamma \subset D'') P^D_{p,q}(\gamma \subset D'). \qquad (3)
\end{aligned}
$$

If we identify pairs of simply connected domains $D' \subset D$ with the (equivalence class of) conformal maps $f^{D',D}$ from D' onto D fixing p, q, then we can write (3) as

$$F(f^{D'',D}) = F(f^{D'',D'})F(f^{D',D})$$

for some function F.

If, in addition, the family of probability laws is also conformally invariant in the sense that for any conformal map Φ on D

$$\Phi_* P^D_{p,q} = P^{\Phi(D)}_{\Phi(p),\Phi(q)},$$

then we can fix a reference domain, D, denote by $f^{D'}$ the conformal map from D' onto D fixing p, q, and conclude from (3)

$$P^D_{p,q}(\gamma \subset D'') = P^D_{p,q}(\gamma \subset f^{D'}(D''))P^D_{p,q}(\gamma \subset D'),$$

I.e. we may express as

$$G(f^{D''}) = G(f^{f^{D'}(D'')})G(f^{D'}).$$

Since

$$f^{D''} = f^{f^{D'}(D'')} \circ f^{D'}$$

we have a homomorphism of semigroups and Lawler, Schramm, and Werner used this semigroup to show that then

$$P^D_{p,q}(\gamma \subset D') = [(f^{D'})'(p)(f^{D'})'(q)]^\alpha$$

for some $\alpha > 0$, and that in the particular case of $\mathrm{SLE}_{8/3}$ where, in addition, the domain Markov property holds, $\alpha = 5/8$.

3. Chordal restriction on finite Riemann surfaces

Let S be a finite bordered Riemann surface, $p, q \in \partial S$ distinct boundary points, γ a cross-cut connecting p and q, and U a "strip" connecting p, q, i.e. a simply connected domain in S whose boundary is the disjoint union of two cross-cuts and two open segments of the boundary of S, one containing p, the other containing q. Let u, v be local uniformizers at p and q, respectively.

Define a measure on cross-cuts connecting p and q by

$$\mu^S_{u,v}(p,q)[\gamma \subset U] = (\Phi'_U(p)\Phi'_U(q))^{5/8},$$

where

$$\Phi_U : U \to \mathbb{H}, \quad \{p,q\} \mapsto \{0,\infty\}$$

and $\Phi'(p), \Phi'(q)$ are defined relative to uniformizers u, v at p, q, and $z, -1/z$ at $0, \infty$.

If x, y is another pair of uniformizers at p, q, then

$$\mu^S_{x,y}(p,q)[\gamma \subset U] = \mu^S_{u,v}(p,q)[\gamma \subset U] \left(\frac{du}{dx}\frac{dv}{dy}\right)^{5/8}.$$

This behavior under change of uniformizer means that the invariant object is

$$\mu^S_{u,v}(p,q)[\gamma \subset U](dudv)^{5/8},$$

a *boundary bi-differential* of weight 5/8.

Does this indeed define a measure? Countable additivity?

Consider all strips U' connecting p, q which are contained in a strip U connecting p, q. Then

(4)

$$\begin{aligned}
\mu^U_{u,v}(p,q)[\gamma \subset U'] &= (\Phi'_{U'}(p)\Phi'_{U'}(q))^{5/8} \\
&= \left[(\Phi_{U'} \circ \Phi_U^{-1} \circ \Phi_U)'(p)\ (\Phi_{U'} \circ \Phi_U^{-1} \circ \Phi_U)'(q)\right]^{5/8} \\
&= [\Phi'_U(p)\Phi'_U(q)]^{5/8}\ \ P^{\mathbb{H}}_{0,\infty}(\gamma \subset \Phi_U(U')).
\end{aligned}$$

This shows that $\mu^U_{u,v}(p,q)$ is a measure on the σ-algebra generated by the events $\{\gamma \subset U'\}$

It is not hard to show that there is a countable collection $\{U_n\}$ of strips such that any strip is an increasing union of a sub-collection of $\{U_n\}$. Also, any cross-cut connecting p and q is contained in at least one U_n. By standard measure theory, the family $\{\mu^{U_n}_{u,v}(p,q)\}_{U_n}$ defines a unique measure $\mu^S_{u,v}(p,q)$ such that μ^{U_n} is the restriction of μ^S to $\{\gamma \subset U_n\}$, IF the family is compatible:

$$\text{if } U' \subset U \cap \tilde{U}, \text{ then } \mu^U_{u,v}(p,q)[\gamma \subset U'] = \mu^{\tilde{U}}_{u,v}(p,q)[\gamma \subset U'].$$

But this follows from the first line of (4).

Remark 2. (i) The measures μ^S satisfy the conformal restriction property: I. If $\Phi : S \to T$ is conformal, then the pull-back of μ^T is μ^S, where the pullback acts on both, the measure and the differential form. II. If $S \subset T$, then μ^T restricts to μ^S.

(ii) Up to a multiplicative constant, this is the unique conformal restriction family.

(iii) If A is an event in the σ-algebra generated by the events $\gamma \in U$, U a strip in S, then $\mu^S[A]$ is a positive boundary bi-differential which at the boundary points p, q takes the value $\mu^S_{u,v}(p,q)[A]$ in the uniformizers u, v.

(iv) If f is an observable on the space of cross-cuts, then $\int f\ d\mu^S$ is a boundary differential of weight 5/8 on S. Here f needs to be $\mu^S_{u,v}(p,q)$-integrable for some uniformizers u, v at each p, q.

References for Riemann surfaces

For an introduction to the classical aspects of the theory with some good motivation,

(1) Felix Klein, An introduction to algebraic differentials and their integrals,

(2) Schiffer and Spencer, Functionals of finite Riemann surfaces,

(3) G. Springer, Introduction to Riemann surfaces.

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