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# School on Stochastic Geometry, the Stochastic Lowener Evolution, and Non-Equilibrium Growth Processes

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## From Critical to Off-Critical Interfaces (Statistical interfaces and SLE)

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# FROM CRITICAL TO OFF-CRITICAL INTERFACES [redacted] (07/2008)

D. Bernard
with M. Bauer

## I/ statistical interfaces and SLE

1. Interfaces
2. statistical mechanics and curves
3. statistical martingales
4. SLE/CFT correspondance

## II/ SLE/CFT and applications

1. Loewner eq, (algebraic version)
2. CFT martingales
3. Application 1: coupling to Gaussian field.
4. Application 2: SLE($\kappa,\rho$) and generalisation.

( 5. : N-SLE

## III/ Off-critical SLE

1. (Warm-up) random walk / Girsanov theorem.
2. S.A.W as an example (of off-critical curve)
3. Off-critical drift and partition $f^{ct}$.
   " SLE and field theory
4. Example 1: $\left[\begin{array}{l} c=1+\text{mass} \\ SLE_4+\text{''} \end{array}\right.$ (massive free field)
5. Example 2: off-critical LERW

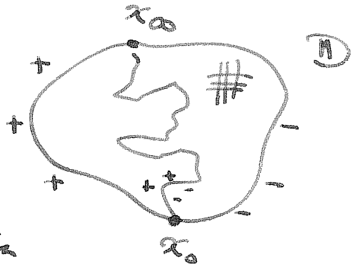
# I. Statistical interfaces and SLE [redacted]

## 1/ What we describe? interfaces

Some of these have been described by M. Bauer, let us me just recall few example to fix the ideas.

### i.) Ising model (spin clusters)

in domain $\mathbb{D}$
lattice $\Lambda \subset \mathbb{D}$




(simple curve)

fix appropriate boundary cond to impose that for each realisation there is an interface $x_0 \to x_\infty$

at $T_c$, in the limit mesh $\to 0$ $\Rightarrow$ SLE ($\kappa = 3$) chordal.

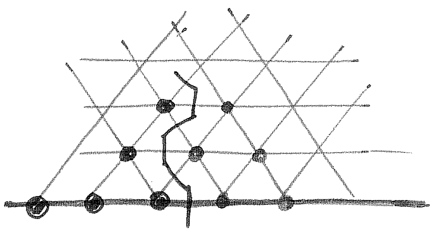
we can play with different boundary condition
e.g

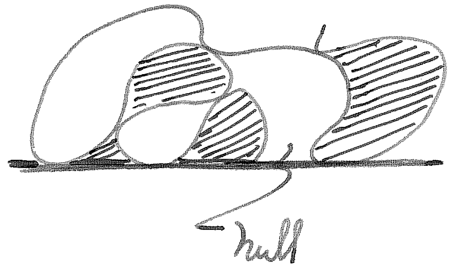

convergence towards SLE dipolar (but locally the same)

### ii) Percolation

E.g on hexagonal in the upper half plane

simple curve on the lattice but not in the continuum limit $\Rightarrow$ double points ($\infty \neq$)



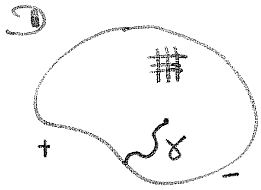


at criticality $p = p_c$ $(= 1/2)$

hull which is $\neq$ from cluster

## 2/ statistical mechanics and curves

i) stat. mech / Boltzman weight $\Rightarrow$ induces a measure on curve.



local degree of freedom $\to$ configuration $c \in C$

Boltzman weight $w_c$ (for each configuration)

Proba of a config $P_c = \frac{1}{Z_{\mathbb{D}}} w_c$

$$Z_{\mathbb{D}} = \sum_{c \in C} w_c$$

$\to$ measure on (portion of) curves.

fix a curve $\gamma$ (view as a portion of an interface induced by the b.c.)

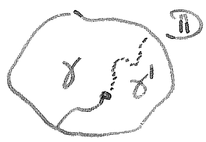
this selects a subset of configuration $C[\gamma]$ because 'spins' on the side of the curve are fixed

By Boltzman rule $P_{\mathbb{D}}[\gamma] = \frac{Z_{\mathbb{D}}[\gamma]}{Z_{\mathbb{D}}}$ , $Z_{\mathbb{D}}[\gamma] = \sum_{c \in C[\gamma]} w_c$

Rks: By construction, this is a probability measure ($\sum p[\gamma] = 1$)

because summing over all curves (say of a given length, call it $T$) reproduces the full sum: $\sum_{\gamma_T} Z_{\mathbb{D}}[\gamma_T] = Z_{\mathbb{D}}$ i.e. $\bigcup_{\gamma_T} C[\gamma_T] = C$

ii) domain Markov property



what to compare? $\begin{cases} P_{\mathbb{D}}[\gamma'|\gamma] & \text{in } \mathbb{D}\text{, conditioned on } \gamma \\ P_{\mathbb{D}\setminus\gamma}[\gamma'] & \text{in } \mathbb{D}\setminus\gamma \end{cases}$

(both measure of identical support)

$$P_{\mathbb{D}}[\gamma'|\gamma] = \frac{P_{\mathbb{D}}[\gamma\gamma']}{P_{\mathbb{D}}[\gamma]} = \frac{Z_{\mathbb{D}}[\gamma\gamma']}{Z_{\mathbb{D}}} \cdot \frac{Z_{\mathbb{D}}}{Z_{\mathbb{D}}[\gamma]} = \frac{Z_{\mathbb{D}}[\gamma\gamma']}{Z_{\mathbb{D}}[\gamma]}$$

$$P_{\mathbb{D}\setminus\gamma}[\gamma'] = \frac{Z_{\mathbb{D}\setminus\gamma}[\gamma']}{Z_{\mathbb{D}\setminus\gamma}}$$

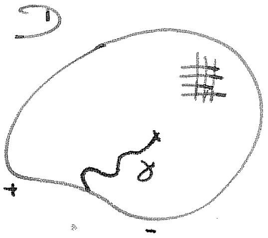
but $\begin{cases} Z_{\mathbb{D}\setminus\gamma}[\gamma'] = e^{E_\gamma} Z_{\mathbb{D}}[\gamma\gamma'] \\ Z_{\mathbb{D}\setminus\gamma} = e^{E_\gamma} Z_{\mathbb{D}}[\gamma] \end{cases}$ because

iii) <u>continuum limit</u>

As explained in the previous week
domain Markov prop.
$\oplus$ conf. transport $\Big\} \to$ SLE

## 3/ Statistical martingale

The aim is to show that statistical correlation $f^{ct}$ are martingales for the process describing the growth of the curve.




O: observables are maps: $c \to O_c$
example
correlation $f^{ct}$ are expectation

$$\langle O \rangle_D \equiv \mathop{E}_D[O] = \frac{1}{Z_D} \sum_c w_c O_c$$

consider now this expectation but conditioned on $\gamma$ (of length $T$, say)
(As for the Markov prop, it is easy to see that ~~conditioni~~ the conditioned expectation is the same as the expectation in the cut domain)

$$\langle O \rangle_{D|\gamma_T} = \langle O \rangle_{D \setminus \gamma_T} = \frac{1}{Z_{D|\gamma_T}} \sum_{c \in C[\gamma]} w_c O_c$$

Since $\gamma_T$ is random (with law $P[\gamma_T]$), these are random number
We have

$$E[\langle O \rangle_{D|\gamma_T}] = \langle O \rangle_D$$

i.e. ‖ Reorganisation of the statistical sum

average over curve $\gamma_T$ (the proof is easy ....)

<u>More</u>, specifying $\gamma_T$ of a given length, partition the configuration space; increasing the length amounts to consider finer and finer partition. $\Rightarrow$ we thus define a filtration (increase the knowledge)

So we have

$$\langle \mathcal{O} \rangle_{\mathcal{D} \setminus \gamma_T} = E[\mathcal{O} | \gamma_T]$$

$$\Rightarrow \begin{cases} E[\langle \mathcal{O} \rangle_{\mathcal{D} \setminus \gamma_S} | \gamma_T] = \langle \mathcal{O} \rangle_{\mathcal{D} \setminus \gamma_T} & \text{for } S > T \\ \text{ie } T \mapsto \langle \mathcal{O} \rangle_{\mathcal{D} \setminus \gamma_T} \text{ is a martingale} \end{cases}$$

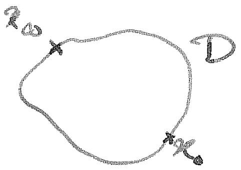
## 4. SLE/CFT correspondance

Identification between CFT/SLE comes by taking the continuous limit of the previous formulae. eg

$$E[\langle \mathcal{O} \rangle_{\mathcal{D} \setminus \gamma_T}] = \langle \mathcal{O} \rangle_{\mathcal{D}} \quad \text{in the continuum}$$

measure curve ie SLE $\xleftrightarrow{\text{link}}$ statistical sum at criticality ie CFT

a) $\langle \mathcal{O} \rangle_{\mathcal{D}}$ as ratio of partition function
as ratio of CFT correlation $f^{ct}$



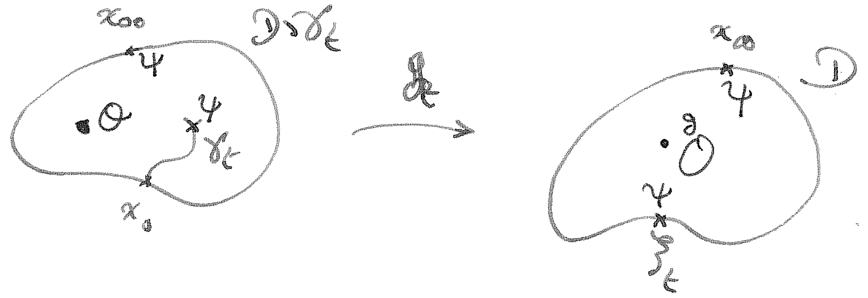

need to code for the bdry cond. and their change
$\Rightarrow$ insertion of bdry cond. changing operator $\psi(x_0)$ and $\psi(x_\infty)$

$$\langle \mathcal{O} \rangle_{\mathcal{D}} \xrightarrow[\text{continuum}]{\text{in the}} \frac{\langle \psi(x_\infty)\, \mathcal{O}\, \psi(x_0) \rangle_{\mathcal{D}}}{\langle \psi(x_\infty) \psi(x_0) \rangle_{\mathcal{D}}} \quad \text{(here cft correlation)}$$

represents some kind of the continuum limit of the partition function.

(*) gives the SLE/CFT correspondance.
ie this ratio of cft correlator but in $\mathcal{D} \setminus \gamma_t$ are <u>martingales</u>

The advantages is that the (because of conformal invariance) the transformation from $D\setminus\gamma_t$ to $D$ is purely kinematical.




$g_t(\gamma_t) = \xi_t$
$g_t(x_\infty) = x_\infty$
(b.d cond )

Then $\langle \psi(x_\infty)\,\mathcal{O}\,\psi(\gamma_t)\rangle_{D\setminus\gamma_t} = |g_t'(x_\infty)|^h\,|g_t'(\gamma_t)|^h\,\langle \psi(x_\infty)\,{}^g\mathcal{O}\,\psi(\xi_t)\rangle_D$

with $h$ the scaling dim of the operator $\psi$.

math

$\Rightarrow$ The process

$$t \to \frac{\langle \psi(x_\infty)\,{}^{g_t}\mathcal{O}\,\psi(\xi_t)\rangle_D}{\langle \psi(x_\infty)\,\psi(\xi_t)\rangle_D}$$

are (chordal) SLE martingales

Rk: the Jacobian $|g'(\gamma_t)|^h$ (which) are infinite cancel

Rk: ${}^g\mathcal{O}$ = transform of operator $\mathcal{O}$ by conformal map.

ii) CFT/SLE (chordal)

The martingale relation is true iff. we chose correctly the CFT and the operator $\psi$.

CFT: with central charge $c = \frac{(6-\kappa)(3\kappa-8)}{2\kappa} < 1$

$\psi$: with dimension $h = \frac{6-\kappa}{2\kappa}$

and "degenerate" at level 2.

iii) Examples of CFT/SLE

$c = 1$ Gaussian free field $\kappa = 4$

$c = 1/2$ Ising $\kappa = 3$ spin cluster

$\kappa = 16/3$ FK (high temp$^{\circ}$) cluster

$c = 0$ $\kappa = 6$ percolation

$\kappa = 8/3$ S.A.W.