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Stochastic Processes in Turbulent Transport

Krzysztof GAWEDZKI
ENS Lyon, Physique Theorique Enslapp, Lyon, France

Strada Costiera 11, 34014 Trieste, Italy - Tel. +39 040 2240 111; Fax +39 040 224 163 - sci_info@ictp.it, www.ictp.it

Stochastic processes in turbulent transport*

Krzysztof Gawędzki
Laboratoire de Physique, C.N.R.S., ENS-Lyon, Université de Lyon,
46 Allée d'Italie, 69364 Lyon, France

Abstract

This is a set of four lectures devoted to simple ideas about turbulent transport, a ubiquitous non-equilibrium phenomenon. In the course similar to that given by the author in 2006 in Warwick [45], we discuss lessons which have been learned from naive models of turbulent mixing that employ simple random velocity ensembles and study related stochastic processes. In the first lecture, after a brief reminder of the turbulence phenomenology, we describe the intimate relation between the passive advection of particles and fields by hydrodynamical flows. The second lecture is devoted to some useful tools of the multiplicative ergodic theory for random dynamical systems. In the third lecture, we apply these tools to the example of particle flows in the Kraichnan ensemble of smooth velocities that mimics turbulence at intermediate Reynolds numbers. In the fourth lecture, we extend the discussion of particle flows to the case of non-smooth Kraichnan velocities that model fully developed turbulence. We stress the unconventional aspects of particle flows that appear in this regime and lead to phase transitions in the presence of compressibility. The intermittency of scalar fields advected by fully turbulent velocities and the scenario linking it to hidden statistical conservation laws of multi-particle flows are briefly explained.

1 Turbulence and turbulent transport

Hydrodynamical turbulence, with its involved flow patterns of interwoven webs of eddies changing erratically in time [68], is a fascinating phenomenon occurring in nature from microscopic scales to astronomical ones and exposed to our scrutiny in everyday experience. Despite the ubiquity of turbulence, an understanding of this far-from-equilibrium phenomenon from first principles is a long-standing open problem, one of the few remaining theoretical challenges of classical physics. Although it seems still a too far-fetched goal, there has been a constant progress over years in the theoretical and practical knowledge of hydrodynamical turbulence and, even more so, of other non-equilibrium phenomena of a similar nature. This progress has been due to developments of experimental techniques and computer power, but also to new theoretical ideas inspired by studies of simple models of turbulence-related systems. The present lectures will discuss one circle of such ideas more relevant to the problem of transport properties in turbulent flows than to turbulence itself. They concern properties of stochastic processes underlying the turbulent mixing, see also [75, 39]. In refs. [60, 78, 73, 79, 76] one may find discussion of other theoretical, experimental, numerical and practical aspects of turbulent transport. For the use of stochastic processes in modeling of non-linear fluid dynamics itself, see e.g. [32].

It is believed that realistic turbulent flows are described by hydrodynamical equations, like the Navier-Stokes ones, that govern the evolution of the velocity field and, eventually, of other relevant fields like fluid density, temperature, etc. As any structure whose dynamics is governed by evolution equations, the turbulent flow may be viewed from a theoretical point as a dynamical system. Many attempts were made then to apply to turbulence ideas from the dynamical systems theory, see e.g. [22]. These ideas, developed for low-dimensional dynamical systems, although often successful in explaining the onset of turbulence, seem to miss important features when confronted with the fully developed turbulence. Some of them, however, prove very useful in the description of transport in turbulent flows, as we shall see in the sequel.

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1.1 Navier-Stokes equations

These are equations dating back to the work of Claude-Louis Navier from 1823 and of George Gabriel Stokes from 1843, completing Leonhard Euler's equations of hydrodynamics from 251 years ago. They have the form of the Newton equation for the mass times acceleration of the the fluid element:

$$\underset{\substack{\nearrow\\ \text{fluid}\\ \text{density}}}{\rho}(\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}) - \rho\underset{\substack{\uparrow\\ \text{kinematical}\\ \text{viscosity}}}{\nu}\boldsymbol{\nabla}^2\boldsymbol{v} \;=\; -\boldsymbol{\nabla}\underset{\substack{\nearrow\\ \text{pressure}}}{p} + \underset{\substack{\nwarrow\\ \text{force}\\ \text{density}}}{\boldsymbol{f}}\,.$$

and have to be supplemented with the continuity equation

$$\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0$$

and an equation of state $F(\rho, p) = 0$. For the incompressible fluid, the latter states that the density $\rho = \rho_0$ is constant and the Navier-Stokes equations may be rewritten in the form

$$\partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} \;-\; \nu\boldsymbol{\nabla}^2\boldsymbol{v} \;=\; \tfrac{1}{\rho_0}(-\boldsymbol{\nabla}p + \boldsymbol{f}) \tag{1.1}$$

and are supplemented with the incompressibility condition $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$. In the latter case, they may be viewed as an evolution equation on the (infinite dimensional) space $\mathscr{V}_0$ of the divergence-free vector fields describing an infinite-dimensional (non-autonomous) dynamical system

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \mathscr{X}(t, X) \qquad \text{for} \quad X \in \mathscr{V}_0. \tag{1.2}$$

In the theoretical modeling of the way in which the fluid motion is induced, one often assumes that the external force $\boldsymbol{f}$ is random. The right hand side $\mathscr{X}(t, X)$ will then be also random, describing an infinite-dimensional random dynamical system.

Eq. (1.1) is nonlinear. The strength of the non-linear advection term $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ relative to the linear one $\nu\boldsymbol{\nabla}^2\boldsymbol{v}$ describing the viscous dissipation depends on the length scale l and is measured by the running Reynolds number defined as

$$Re(l) = \frac{\Delta_l v \cdot l}{\nu}$$

where $\Delta_l v$ is the size of typical velocity difference across the distance scale l, with $\tau_l = \frac{l}{\Delta_l v}$ giving the typical turnover time of eddies of size l. In particular, if L is the integral scale corresponding to the size of the flow recipient, $Re(L) \equiv Re$ is the integral scale Reynolds number and τ_L is the integral time. On the other end, $Re(\eta) = 1$ for η equal to the viscous (or Kolmogorov) scale on which the non-linear and the dissipative term in the equation have comparable strength. The Kolmogorov time τ_η gives the turnover time of the smallest eddies. Phenomenological observations of hydrodynamical flows point to the following classification according to the size of the Reynolds number Re:

- $Re \lesssim 1$: laminar flows,
- $Re \sim 10$ to 10^2: onset of turbulence,
- $Re \gtrsim 10^3$: developed turbulence.

In he laminar regime, some explicit solutions are known [10]. At the onset of turbulence, the flow is driven by few unstable modes. The standard dynamical system theory studying temporal evolution of few degrees of freedom governed by ordinary differential equations or map iteration proved useful here e.g. in describing the scenarios of the appearance of chaotic motions, see e.g. [35]. Finally, in the fully developed turbulence, there are many unstable degrees of freedom. Kolmogorov's scaling theory [54] for this regime predicts that

$$\Delta_l v \;\propto\; l^{1/3} \qquad \text{for} \quad \eta \ll l \ll L$$

which is not very far from the observed behavior. The number of unstable modes may be estimated as given by $(L/\eta)^3 \propto Re^{9/4}$. New phenomena arise here that have to be addressed, like cascades with (approximately) constant energy flux, intermittency, etc.

1.2 Turbulent transport of particles and fields

One may view a turbulent flow as a dynamical system in another way, related to the transport of particles and fields by the fluid [22].

1.2.1 Transport of particles

The particles may be idealized fluid elements, called **Lagrangian particles**, or particles without inertia suspended in the fluid that also undergo molecular diffusion. We talk in the latter case of the

- **tracer particles**

  whose motion is governed by the evolution equation

$$\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = \boldsymbol{v}(t, \boldsymbol{R}) + \sqrt{2\kappa}\, \boldsymbol{\eta}(t)\,, \tag{1.3}$$

  where κ stands for the (molecular) diffusivity and $\boldsymbol{\eta}(t)$ is the standard vector-valued white noise. The Lagrangian particles correspond to the case $\kappa = 0$.

Small objects like water droplets in the air experience a friction force proportional to their relative velocity with respect to the fluid [61]. These are

- **particles with inertia**

  whose position $\boldsymbol{R}$ and velocity $\boldsymbol{U}$ satisfy in the simple case the equations of motion

$$\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = \boldsymbol{U}\,, \quad \frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t} = \tfrac{1}{\tau}\big(- \boldsymbol{U} + \boldsymbol{v}(t, \boldsymbol{R}) + \sqrt{2\kappa}\, \boldsymbol{\eta}(t)\big)\,, \tag{1.4}$$

  where τ, the Stokes time, accounts for time delay with respect to tracer particles whose evolution is recovered in the $\tau \to 0$ limit.

Finally, one may consider small objects with spatial structure suspended in the fluid, like

- **polymer molecules**

  whose motion is sometimes modeled [20] by the differential equations

$$\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = \boldsymbol{v}(t, \boldsymbol{R}) + \sqrt{2\kappa}\, \boldsymbol{\eta}(t), \qquad \frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} = (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{v}(t, \boldsymbol{R}) - \alpha \boldsymbol{B} + \sqrt{2\sigma}\, \boldsymbol{\xi}(t)\,, \tag{1.5}$$

  where $\boldsymbol{R}$ describes the position of one end of the polymer chain, $\boldsymbol{B}$ is the end-to-end separation vector, and $\boldsymbol{\eta}(t)$ and $\boldsymbol{\xi}(t)$ are independent white noises. The term $-\alpha\boldsymbol{B}$ describes the elastic counteraction to the stretching of the polymer.

All three cases provide examples of finite-dimensional *dynamical systems* if $\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\xi}$ *are given*. The dynamical systems are random if $\boldsymbol{v}, \boldsymbol{\eta}, \boldsymbol{\xi}$ are random *with given statistics*. They may be studied with tools of the dynamical systems theory to which they have also provided new inputs.

1.2.2 Transport of fields

Turbulent flows may also transport physical quantities described by fields changing continuously from point to point, like temperature, pollutant or dye concentration, magnetic field, etc. The evolution of such transported fields is described for small field intensity by linear advection-diffusion equations with variable coefficients related to the fluid velocity. Under the same assumption of small field intensity, one may ignore the influence of the transported field on the fluid dynamics, ending up with a *passive transport* approximation. The passive advection of a

- **scalar field** $\theta(t, \boldsymbol{r})$

  (e.g. temperature) is governed by the equation

$$\partial_t \theta + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\theta - \kappa \boldsymbol{\nabla}^2 \theta = g \tag{1.6}$$

  where κ stands for the diffusivity of the scalar and g is a scalar source.

Similarly, for a

- **density field** $n(t, \boldsymbol{r})$

  (e.g. of a pollutant), one has the partial differential equation

$$\partial_t n + \boldsymbol{\nabla} \cdot (n\boldsymbol{v}) - \kappa \boldsymbol{\nabla}^2 n = h \tag{1.7}$$

  that differs from that for the scalar field only for compressible velocities with $\boldsymbol{\nabla} \cdot \boldsymbol{v} \neq 0$.

For the passive transport of a

- **phase-space density** $n(t, \boldsymbol{r}, \boldsymbol{u})$
  (e.g. of an aerosol suspension of inertial particles) one can write the equality

$$\partial_t n + (\boldsymbol{u} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}})n + \boldsymbol{\nabla}_{\boldsymbol{u}} \cdot (n\frac{\boldsymbol{u}-\boldsymbol{v}}{\tau}) - \kappa \boldsymbol{\nabla}^2 n = h\,. \tag{1.8}$$

Finally, the passive transport of the

- **magnetic field** $\boldsymbol{B}(t, \mathrm{r})$
  satisfying $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ is governed by the evolution equation

$$\partial_t \boldsymbol{B} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{B} + (\boldsymbol{\nabla} \cdot \boldsymbol{v})\boldsymbol{B} - (\boldsymbol{B} \cdot \nabla)\boldsymbol{v} - \kappa \boldsymbol{\nabla}^2 \boldsymbol{B} = \boldsymbol{G}\,, \tag{1.9}$$

  where κ is the magnetic diffusivity and $\boldsymbol{G}$ is a source term.

The meaning of "small intensity" in the assumptions leading to the above equation depends on the physical situation (e.g. very small admixtures of polymers or chemical reactants may change the flow considerably).

1.2.3 Relations between transport of particles and fields

The dynamics of localized objects (particles, polymers) and passively transported fields listed above are closely related.

Let $\boldsymbol{R}(t; t_0, \boldsymbol{r}_0|\boldsymbol{\eta})$ denote the time t position of the tracer particle that at time t_0 passes through $\boldsymbol{r}_0$. It depends on the realization of the noise $\boldsymbol{\eta}$, see Eq. (1.3). Then the formula

$$\theta(t, \boldsymbol{r}) \;=\; \overline{\theta(t_0, \boldsymbol{R}(t_0; t, \boldsymbol{r}|\boldsymbol{\eta}))} \;=\; \overline{\int \theta(t_0, \boldsymbol{r}_0)\; \delta(\boldsymbol{r}_0 - \boldsymbol{R}(t_0; t, \boldsymbol{r}|\boldsymbol{\eta}))\; \mathrm{d}\boldsymbol{r}_0}\,, \tag{1.10}$$

where the overline stands for the average over the white noise $\boldsymbol{\eta}$ realizations, expresses the solution of the advection-diffusion equation (1.6) with vanishing source g in terms of the initial values $\theta(t_0, \boldsymbol{r})$ of the field. The meaning of this solution is that scalar field is conserved along the noisy trajectories.

Let us consider the matrix $W(t; t_0, \boldsymbol{r}_0|\boldsymbol{\eta}))$ that propagates the separations between two infinitesimally close tracer particle trajectories:

$$W^i_{\ j}(t; t_0, \boldsymbol{r}_0|\boldsymbol{\eta}) \;=\; \frac{\partial R^i(t; t_0, \boldsymbol{r}_0|\boldsymbol{\eta})}{\partial r_0^j}\,.$$

The determinant of matrix W describes the flow-induced change of the volume element along the tracer particle trajectory and it is equal to 1 for the incompressible flows. The formula

$$n(t, \boldsymbol{r}) \;=\; \overline{\det W(t_0; t, \boldsymbol{r}|\boldsymbol{\eta})\; n(t_0, \boldsymbol{R}(t_0; t, \boldsymbol{r}|\boldsymbol{\eta}))} \;=\; \overline{\int n(t_0, \boldsymbol{r}_0)\, \delta(\boldsymbol{r} - \boldsymbol{R}(t; t_0, \boldsymbol{r}_0|\boldsymbol{\eta}))\; \mathrm{d}\boldsymbol{r}_0} \tag{1.11}$$

gives the solution of the initial-value problem for the density evolution equation (1.7) with vanishing source h. Note that

$$W(t_0; t, \boldsymbol{r}|\boldsymbol{\eta}) \;=\; W(t; t_0, \boldsymbol{R}(t_0; t, \boldsymbol{r}|\boldsymbol{\eta})|\boldsymbol{\eta})^{-1}$$

so that the density changes inversely proportionally to the change of volume element along the Lagrangian trajectories. The density increases where the flow contracts the volume.

Similarly, denoting by $(\boldsymbol{R}(t; t_0, \boldsymbol{r}_0, \boldsymbol{u}_0|\boldsymbol{\eta}), \boldsymbol{U}(t; t_0, \boldsymbol{r}_0\, \boldsymbol{u}_0|\boldsymbol{\eta}))$ the phase-space trajectory of the inertial particle passing at time t_0 through $(\boldsymbol{r}_0, \boldsymbol{u}_0)$ and introducing the matrix $\boldsymbol{W}(t; t_0, \boldsymbol{r}_0, \boldsymbol{u}_0|\boldsymbol{\eta})$ propagating the phase-space separations between two infinitesimally close inertial particle trajectories, one obtains the formula

$$n(t, \boldsymbol{r}, \boldsymbol{u}) \;=\; \overline{\det \boldsymbol{W}(t_0; t, \boldsymbol{r}, \boldsymbol{u}|\boldsymbol{\eta})\; n(t_0, \boldsymbol{R}(t_0; t, \boldsymbol{r}, \boldsymbol{u}|\boldsymbol{\eta}), \boldsymbol{U}(t_0; t, \boldsymbol{r}, \boldsymbol{u}|\boldsymbol{\eta}))} \tag{1.12}$$

for the solution of the evolution equation (1.8) with $h = 0$.

Finally, the solution of the evolution equation (1.9) for the magnetic field with $\boldsymbol{G} = 0$ is given by the formula:

$$\boldsymbol{B}(t, \boldsymbol{r}) \;=\; \overline{(\det W(t_0; t, \boldsymbol{r}|\boldsymbol{\eta}))\; W(t_0; t, \boldsymbol{r}|\boldsymbol{\eta})^{-1}\, \boldsymbol{B}(t_0, \boldsymbol{R}(t_0; t, \boldsymbol{r}|\boldsymbol{\eta}))}\,. \tag{1.13}$$

The action of the matrix $W(t_0; t, \boldsymbol{r}|\boldsymbol{\eta})^{-1} = W(t; t_0, \boldsymbol{R}(t_0; t, \boldsymbol{r}|\boldsymbol{\eta})|\boldsymbol{\eta})$ stretches, contracts and/or rotates the magnetic field vector along trajectories. Note in passing that the solution of the second of Eqs. (1.5) for the polymer end-to-end separation in the linear approximation is given by

$$\boldsymbol{B}(t) \;=\; \mathrm{e}^{-\alpha(t-t_0)}\, W(t; t_0, \boldsymbol{r}_0|\boldsymbol{\eta})\, \boldsymbol{B}(t_0) \tag{1.14}$$

if the component σ of the polymer diffusivity vanishes.

The source terms providing inhomogeneous terms in the linear advection-diffusion equations for the fields may be taken into account by the standard "variation of constants" method. For example the solution (1.10) of the scalar evolution equation (1.6) picks up for non-zero source g an additional term

$$\int_{t_0}^{t} \overline{g(s, \boldsymbol{R}(s; t, \boldsymbol{r}|\boldsymbol{\eta}))}\, ds \tag{1.15}$$

on the right hand side.

Conclusion. *Passive transport of particles and fields by hydrodynamical flows, including turbulent ones, is described by simple equations closely related in both cases. The transport of particles will be studied below by the (random) dynamical system methods. We shall subsequently examine which attributes of the particle dynamics bear on which properties of the field advection, with the goal to capture and explain some essential features of the turbulent transport.*

2 Multiplicative ergodic theory

We shall be interested in a statistical description of particle dynamics in turbulent flows. For concreteness, we shall concentrate on the dynamics of Lagrangian particles carried by the a flow in a bounded region V. Their trajectories are solutions of the ODE

$$\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = \boldsymbol{v}(t,\boldsymbol{R})\,. \tag{2.1}$$

with random, not necessarily incompressible, velocity fields $\boldsymbol{v}(t,\boldsymbol{r})$, see Eq. (1.3). We shall assume that the statistics of the velocities is stationary. The deterministic time-independent velocity field will be also covered as a special case. Since Eq. (2.1) has a form of a general dynamical system, most of the considerations that follow will apply without much change also to other particle-transport equations. For reasons which were already mentioned in the introduction and will become clear in later, the considerations of the present and the next lecture, borrowing on the theory of differentiable dynamical systems, are relevant to the advection by turbulent flows at intermediate Reynolds numbers.

2.1 Natural invariant measures

In deterministic time-independent flows, one calls a measure $n(\mathrm{d}\boldsymbol{r})$ on the flow region V invariant if

$$\int\limits_V f(\boldsymbol{R}(t;\boldsymbol{r})\,n(\mathrm{d}\boldsymbol{r}) \;=\; \int\limits_V f(\boldsymbol{r})\,n(\mathrm{d}\boldsymbol{r}) \tag{2.2}$$

for bounded (measurable) functions f on V and all times t, with the shortened notation $\boldsymbol{R}(t;\boldsymbol{r}) \equiv \boldsymbol{R}(t;0,\boldsymbol{r})$. This may be generalized to the case of stationary random flows by considering a collection of measures $n(\mathrm{d}\boldsymbol{r}|\boldsymbol{v})$ on the flow region, parametrized by the velocity realizations, such that

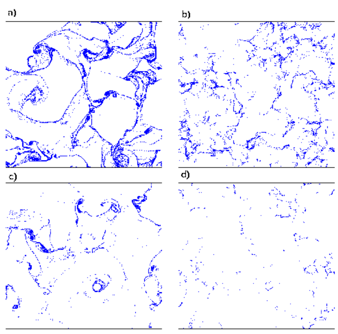
$$\Big\langle \int\limits_V f(\boldsymbol{R}(t;\boldsymbol{r}|\boldsymbol{v})|\boldsymbol{v}_t)\;n(\mathrm{d}\boldsymbol{r}|\boldsymbol{v})\Big\rangle \;=\; \Big\langle \int\limits_V f(\boldsymbol{r}|\boldsymbol{v})\;n(\mathrm{d}\boldsymbol{r}|\boldsymbol{v})\Big\rangle \;\equiv\; \int\limits_{V\times\mathscr{V}} f(\boldsymbol{r}|\boldsymbol{v})\;N(\mathrm{d}\boldsymbol{r}|\mathrm{d}\boldsymbol{v}) \tag{2.3}$$

where $\langle\,-\,\rangle$ denotes the average over the velocity ensemble and $\boldsymbol{v}_t(s,\boldsymbol{r}) = \boldsymbol{v}(s+t,\boldsymbol{r})$. Such a collection defines a measure $N(d\boldsymbol{r}|d\boldsymbol{v})$ on the product $V\times\mathscr{V}$ of the flow region V and the space $\mathscr{V}$ of velocity realizations. This measure is invariant under the 1-parameter group of transformations

$$V\times\mathscr{V} \ni (\boldsymbol{r}|\boldsymbol{v}) \;\longmapsto\; (\boldsymbol{R}(t;\boldsymbol{r}|\boldsymbol{v}),\boldsymbol{v}_t)\,.$$

It is called an invariant measure of the random dynamical system (2.1).

Let us seed the particles into the fluid at some past time t_0 with smooth normalized density $n_{t_0}(t_0,\boldsymbol{r}_0)$, e.g. the constant one. This density will evolve in time according to the advection equation $\partial_t n + \boldsymbol{\nabla}\cdot(n\boldsymbol{v}) = 0$, see Eq. (1.7). In an incompressible flow, the constant density $n_{t_0}(t,\boldsymbol{r}) = |V|^{-1}$ will stay constant and normalized also at all later times t. If the flow is compressible, however, then the particles will develop preferential concentrations and the densities $n_{t_0}(t,\boldsymbol{r}|\boldsymbol{v})$ will become rougher and rougher with time in typical velocity realizations $\boldsymbol{v}$, as illustrated by the snapshots taken form [21]:



Similar phenomenon occurs for the phase-space density of inertial particles, see e.g. [12, 13]. The probability measures $n(\mathrm{d}\boldsymbol{r}|\boldsymbol{v})$ on V are called natural measures if

$$\int\limits_V f(\boldsymbol{r})\,n(t,\mathrm{d}\boldsymbol{r}|\boldsymbol{v}) = \lim_{t_0\to-\infty}\frac{1}{|t_0|}\int_{t_0}^{0}\mathrm{d}s\int\limits_V f(\boldsymbol{r})\,n_s(0,\mathrm{d}\boldsymbol{r}|\boldsymbol{v})$$

for all continuous functions f and almost all velocity realizations $\boldsymbol{v}$. In particular, $\int f(\boldsymbol{r})\, n(\mathrm{d}\boldsymbol{r}|\boldsymbol{v}) = \lim\limits_{t_0\to-\infty} \int f(\boldsymbol{r})\, n_{t_0}(0,\boldsymbol{r}|\boldsymbol{v})\, d\boldsymbol{r}$ if the latter limits exist. In that case, the natural measures are just the time zero distributions of particles seeded with a smooth density at time $-\infty$. For compressible flows, the natural measures $n(\mathrm{d}\boldsymbol{r}|\boldsymbol{v})$ are concentrated on a random time- and velocity-field-dependent attractor. These measures have the property (2.3) and give rise to the natural invariant measure $N(\mathrm{d}\boldsymbol{r}|\mathrm{d}\boldsymbol{v})$. The latter permits to do statistics of the values of functions $f(\boldsymbol{r}|\boldsymbol{v})$ of points in the flow region and velocity realizations. Such statistics is often called "Lagrangian" since it may be calculated from time-averages along the Lagrangian trajectories.

2.2 Tangent flow

Recall that the matrix with the elements

$$W^i_{\ j}(t;t_0,\boldsymbol{r}|\boldsymbol{v}) \;=\; \frac{\partial R^i(t;t_0,\boldsymbol{r}|\boldsymbol{v})}{\partial r^j}$$

propagates infinitesimal separations $\delta\boldsymbol{R}$ between Lagrangian trajectories in a given velocity realization:

$$\delta\boldsymbol{R}(t;t_0,\boldsymbol{r}|\boldsymbol{v}) \;=\; W(t;t_0,\boldsymbol{r}|\boldsymbol{v})\,\delta\boldsymbol{r}\,.$$

We shall abbreviate $W(t;0,\boldsymbol{r}|\boldsymbol{v}) \equiv W(t;\boldsymbol{r}|\boldsymbol{v})$ dropping also the other dependences in W from the notation whenever they are not necessary. Diagonalizing the matrix W^TW and writing its positive eigenvalues in the exponential form as $\mathrm{e}^{2\rho_1} \geq \mathrm{e}^{2\rho_2} \geq \cdots \geq \mathrm{e}^{2\rho_d}$, we obtain the so called "stretching exponents" $\rho_i(t;\boldsymbol{r}|\boldsymbol{v})$ with the ordering

$$\rho_1 \geq \rho_2 \geq \cdots \geq \rho_d\,.$$

Due to the cumulative effect of short-time stretchings, contractions and rotations of the separation vectors of infinitesimally closed Lagrangian trajectories, the stretching exponents typically grow in absolute value in time. We shall be interested in their long-time asymptotics. Let us call the ratios $\sigma_i = \frac{\rho_i}{|t|}$ the "stretching rates". They are sometimes also called "finite-time Lyapunov exponents". For fixed time t, the functions $\sigma_i(t;\boldsymbol{r}|\boldsymbol{v})$ may be treated as random variables on the product space $V\times\mathscr{V}$ equipped with the natural invariant measure $N(\mathrm{d}\boldsymbol{r}|\mathrm{d}\boldsymbol{v})$. Their joint probability density function (PDF) is then given by the formula:

$$P_t(\vec{\sigma}) = \int\limits_{V\times\mathscr{V}} \delta(\vec{\sigma}-\vec{\sigma}(t;\boldsymbol{r},\boldsymbol{v}))\, N(\mathrm{d}\boldsymbol{r}|\mathrm{d}\boldsymbol{v})\,.$$

The PDF's $P_t(\vec{\sigma})$ and $P_{-t}(\vec{\sigma})$ are very simply related:

$$P_t(\vec{\sigma}) \;=\; P_{-t}(-\bar{\vec{\sigma}})\,, \tag{2.4}$$

where $\bar{\vec{\sigma}} = (\sigma_d,\ldots,\sigma_1)$ if $\vec{\sigma} = (\sigma_1,\ldots,\sigma_d)$. This follows from the relation $\boldsymbol{R}(-t;\boldsymbol{R}(t;\boldsymbol{r}|\boldsymbol{v})|\boldsymbol{v}_t) = \boldsymbol{r}$ implying by the chain rule that $W(t;\boldsymbol{r}|\boldsymbol{v}) = W(-t;\boldsymbol{R}(t,\boldsymbol{r}|\boldsymbol{v})|\boldsymbol{v}_t)^{-1}$ and that

$$\vec{\sigma}(t;\boldsymbol{r}|\boldsymbol{v}) \;=\; -\bar{\vec{\sigma}}(-t;\boldsymbol{R}(t,\boldsymbol{r}|\boldsymbol{v})|\boldsymbol{v}_t) \tag{2.5}$$

and from the invariance (2.3) of the measure $N(\mathrm{d}\boldsymbol{r}|\mathrm{d}\boldsymbol{v})$. A very general result, the backbone of the multiplicative ergodic theory, assures that the stretching rates approach constant limiting values for long times. Denote $\ln_+ x = \max(0,\ln x)$.

- **Multiplicative Ergodic Theorem** (Oseledec 1968 [66], Ruelle 1979 [70]).

  If the measure invariant measure $N(\mathrm{d}\boldsymbol{r}|\mathrm{d}\boldsymbol{v})$ is ergodic and $\int \ln_+ \|W(t;\boldsymbol{r}|\boldsymbol{v})\| N(\mathrm{d}\boldsymbol{r},\mathrm{d}\boldsymbol{v}) < \infty$ for some $t>0$ then

  $$\lim_{t\to\infty} P_t(\vec{\sigma}) \;=\; \delta(\vec{\sigma}-\vec{\lambda}) \tag{2.6}$$

  for a vector $\vec{\lambda} = (\lambda_1,\ldots,\lambda_d)$ of "Lyapunov exponents" such that $\lambda_1 \geq \cdots \geq \lambda_d$.

This may be viewed as a particular case of a multiplicative version of the law of large numbers [5]. For typical $\boldsymbol{r}$, $\boldsymbol{v}$ and $\delta\boldsymbol{r}$,

$$\lambda_1 \;=\; \lim_{t\to\infty} \frac{1}{t}\,\ln\|W(t;\boldsymbol{r}|\boldsymbol{v})\,\delta\boldsymbol{r}\|\,.$$

The strict positivity of λ_1 signals the sensitive dependence on the initial conditions and is taken as a definition of "chaos". Similarly, for $1\leq n\leq d$ and typical $\delta\boldsymbol{r}_1,\ldots,\delta\boldsymbol{r}_n$,

$$\lambda_1+\cdots+\lambda_n \;=\; \lim_{t\to\infty} \frac{1}{2t}\,\ln\det\Big(\delta\boldsymbol{r}_i\cdot W^TW(t;\boldsymbol{r}|\boldsymbol{v})\,\delta\boldsymbol{r}_j)_{1\leq i,j\leq n}\Big). \tag{2.7}$$

It is important to know how the limit (2.6) is approached. If $\lambda_1 > \cdots > \lambda_d$ then one may expect that the fluctuations of the stretching rates about their limiting values behave according to

- **Multiplicative central limit** *(not covered by a general theorem)* asserting that

$$\lim_{t\to\infty} \; t^{-d/2} P_t(\vec{\lambda} + t^{-1/2}\vec{\tau}) \;=\; \frac{e^{-\frac{1}{2}\vec{\tau}\cdot C^{-1}\vec{\tau}}}{\det(2\pi C)^{1/2}}\,.$$

Finally, as argued in [7], see also [31], again under the assumption that $\lambda_1 > \cdots > \lambda_d$, one may expect occurrence of the regime of

- **Multiplicative large deviations** *(again not covered by a general theorem)*, where

$$-\lim_{t\to\infty} \tfrac{1}{t}\,\ln P_t(\vec{\sigma}) \;=\; H(\vec{\sigma})$$

for a "rate function" $H \geq 0$ such that $H(\vec{\lambda}) = 0$.

For incompressible velocities where $\sum \sigma_i \equiv 0$, the rate function $H(\vec{\sigma})$ is infinite unless $\vec{\sigma}$ satisfies the latter condition. Loosely speaking, the existence of the regime of large deviations means that

$$P_t(\vec{\sigma}) \;\;\approx\;\; \mathrm{e}^{-tH(\vec{\sigma})}$$

for large t. As in the additive case, the existence of such regime with the rate function $H(\vec{\sigma})$ twice differentiable around the minimum implies the central limit behavior with

$$(C^{-1})^{ij} = \frac{\partial^2 H}{\partial\sigma_i\partial\sigma_j}(\vec{\lambda})$$

There are partial results about existence of the multiplicative large deviation regime for deterministic dynamical systems [50] and for random dynamical systems with decorrelated velocities [11, 31]. The latter case will be discussed in the next lecture. The large deviation regime for the stretching rates in realistic flows becomes accessible numerically and even in experiments [23, 9].

Let us denote by $\boldsymbol{v}'$ the time-reversed velocity field: $\boldsymbol{v}'(t,\boldsymbol{r}) = -\boldsymbol{v}(t,\boldsymbol{r})$. The large deviations rate function $H(\vec{\sigma})$ and $H'(\vec{\sigma})$, the latter one pertaining to the flow in the time-reversed velocities, should satisfy

- **Multiplicative fluctuation relation** (of Gallavotti-Cohen type)

$$H'(-\bar{\sigma}) = H(\vec{\sigma}) - \sum_{i=1}^{d} \sigma_i \tag{2.8}$$

In particular, for time-reversible velocities when the fields $\boldsymbol{v}$ and $\boldsymbol{v}'$ have the same distribution, the two rate function coincide: $H' = H$. The original fluctuation relations studied the large deviation regime of $s = -\sum_{i=1}^{d} \rho_i$ equal to the (phase-)space contraction exponent. The latter has been interpreted [71, 72] as the entropy production of the dynamical system, with $\sigma = s/t$ standing for the entropy production rate. For deterministic uniformly hyperbolic (discrete-time) systems that are time-reversible, Gallavotti and Cohen [44] established that the rate function $h(\sigma)$ of large deviations of s satisfies the relation

$$h(-\sigma) = h(\sigma) + \sigma\,.$$

This was generalized recently in [24] to random uniformly hyperbolic systems. Since

$$h(\sigma) = \min_{-\sum \sigma_i = \sigma} H(\vec{\sigma})\,,$$

the relation (2.8) implies Eq. (2.2).

Let us prove here following [6], see also [39], another, simpler, relation for a modified joint PDF of the time t stretching rates defined by

$$\tilde{P}_t(\vec{\sigma}) = \left\langle \int \delta(\vec{\sigma} - \vec{\sigma}(t;\boldsymbol{r}|\boldsymbol{v}))\,\frac{\mathrm{d}\boldsymbol{r}}{|V|} \right\rangle \tag{2.9}$$

Note that $\tilde{P}_t$ employs $\frac{\mathrm{d}\boldsymbol{r}}{|V|}$ to average over $\boldsymbol{r}$ rather than the natural measures $n(\mathrm{d}\boldsymbol{r}|\boldsymbol{v})$ used in P_t. Numerically, it is in general simpler to attain the PDF $\tilde{P}_t$ than P_t as the latter requires performing longer Lagrangian averages.

- **Transient multiplicative fluctuation relation** (of the Evans-Searls type)

$$\tilde{P}'_t(-\bar{\sigma}) \;=\; \tilde{P}_t(\vec{\sigma})\;\mathrm{e}^{|t|\sum_i\sigma_i}\,, \tag{2.10}$$

where $\tilde{P}'(\vec{\sigma})$ pertains to the time-reversed flow.

The proof of the relation (2.10) employs a simple change-of-variables argument similar to the ones used to obtain transient fluctuation relations [36, 37, 53]. First, using the definition (2.9) and the stationarity of the velocity ensemble, we obtain

$$\tilde{P}_{-t}(-\bar{\sigma}) \;=\; \left\langle \int \delta(-\bar{\sigma}-\vec{\sigma}(-t;\boldsymbol{R}|\boldsymbol{v}))\,\frac{\mathrm{d}\boldsymbol{R}}{|V|}\right\rangle \;=\; \left\langle \int \delta(\vec{\sigma}+\bar{\sigma}(-t;\boldsymbol{R}|\boldsymbol{v}_t))\,\frac{\mathrm{d}\boldsymbol{R}}{|V|}\right\rangle. \tag{2.11}$$

Upon the substitution $\boldsymbol{R}=\boldsymbol{R}(t;\boldsymbol{r}|\boldsymbol{v})$ with the Jacobian

$$\frac{\partial(\boldsymbol{R})}{\partial(\boldsymbol{r})} \;=\; \det W(t;\boldsymbol{r}|\boldsymbol{v}) \;=\; e^{|t|\sum_{i=1}^{d}\sigma_i(t;\boldsymbol{r}|\boldsymbol{v})} \tag{2.12}$$

this gives

$$\begin{aligned}\tilde{P}_{-t}(-\bar{\sigma}) \;&=\; \left\langle \int \delta(\vec{\sigma}+\bar{\sigma}(-t;\boldsymbol{R}(t;\boldsymbol{r}|\boldsymbol{v})|\boldsymbol{v}_t))\;\mathrm{e}^{|t|\sum_{i=1}^{d}\sigma_i(t;\boldsymbol{r}|\boldsymbol{v})}\,\frac{\mathrm{d}\boldsymbol{r}}{|V|}\right\rangle\\ &=\; \left\langle \int \delta(\vec{\sigma}-\vec{\sigma}(t;\boldsymbol{r}|\boldsymbol{v}))\;\mathrm{e}^{|t|\sum_{i=1}^{d}\sigma_i(t;\boldsymbol{r}|\boldsymbol{v})}\,\frac{\mathrm{d}\boldsymbol{r}}{|V|}\right\rangle \;=\; \tilde{P}_t(\vec{\sigma})\;\mathrm{e}^{|t|\sum_{i=1}^{d}\sigma_i}\end{aligned}$$

where we have use the relation (2.5) to obtain the second equality. Now, the straightforward relation $R(-t;\boldsymbol{r}|\boldsymbol{v})=R(t;\boldsymbol{r}|\boldsymbol{v}')$ implies that

$$\tilde{P}_{-t}(\vec{\sigma}) \;=\; \tilde{P}'_t(\vec{\sigma})\,. \tag{2.13}$$

and the identity (2.10) follows.

For long positive times $t\gg 0$, the Lagrangian particles approach the (time-varying) attractor and their statistics should not depend much on whether their initial points were distributed with the uniform measure or with the attractor measure. One may then expect that

$$\tilde{P}_t(\vec{\sigma}) \;\approx\; P_t(\vec{\sigma}) \tag{2.14}$$

at long positive times, and similarly for $\tilde{P}'_t$ and P'_t. If this holds and the PDF's P_t and P'_t exhibit large deviations regime then (2.10) implies (2.8). Both assumptions are not granted, however, and require a careful analysis in specific situations.

2.3 Applications of multiplicative large deviations

Although concerning relatively rare events, large deviations play an important role in turbulent transport phenomena. Indeed, rare fluctuations of concentrations of toxic emissions or chemical agents may determine the nocivity and/or the reaction rates. Suppose that we put at time zero a blob of density $n_0(\boldsymbol{r}_0)$ into a compressible fluid. Then, for vanishing diffusivity[1], the blob density at later times is

$$n(t,\boldsymbol{r}) \;=\; \det W(0;t,\boldsymbol{r})\;n_0(\boldsymbol{R}(0;t\boldsymbol{r})\,, \tag{2.15}$$

see Eq. (1.11). Along the trajectory $\boldsymbol{R}(t;\boldsymbol{r}_0)$,

$$n(t) \;\equiv\; n(t,\boldsymbol{R}(t;\boldsymbol{r}_0)) \;=\; \det W(t;\boldsymbol{r}_0)^{-1}n_0(\boldsymbol{r}_0) \;=\; \mathrm{e}^{-\sum_i\rho_i(t;\boldsymbol{r}_0)}n_0(\boldsymbol{r}_0)$$

and

$$\langle n(t)^q\rangle \;=\; n_0(\boldsymbol{r}_0)^q\int \mathrm{e}^{-qt\sum\sigma_i}\,\tilde{P}_t(\vec{\sigma})\,\mathrm{d}\vec{\sigma} \;\approx\; n_0(\boldsymbol{r}_0)^q\int \mathrm{e}^{-qt\sum\sigma_i-tH(\vec{\sigma})}\,\mathrm{d}\vec{\sigma} \quad\propto\quad \mathrm{e}^{\gamma_q t} \tag{2.16}$$

for large t, where

$$-\gamma_q \;=\; \min_{\sigma_1\geq\cdots\geq\sigma_d}\; q\sum_i\sigma_i+H(\vec{\sigma})\,. \tag{2.17}$$

As we see, the asymptotic behavior of the moments of the density of the blob along a Lagrangian trajectory is determined by the rate function $H(\vec{\sigma})$. Of course, $\gamma_0=0$. Eq. (2.8) implies, in turn, that $\gamma_{-1}=0$ since the minimum of $H'(-\bar{\sigma})$ is equal to zero. In the incompressible flow, all γ_q vanish.

There are other quantities that may be expressed via the rate function $H(\vec{\sigma})$ of the large deviations of the stretching exponents including:

[1] At long distances and not too long times, the effect of molecular diffusivity my be disregarded in practical situations.

- rates of decay of moments of the tracer in the presence of diffusivity [7]
- multifractal dimensions of the natural attractor measure [50, 16]
- evolution of the moments of the polymer stretching $\boldsymbol{B}$ and the onset of drag reduction in polymer solutions [8, 26]

Conclusion. *In smooth dynamical systems cumulation of local stretchings, contractions and rotations leads to the linear growth of the stretching exponents with the asymptotic rates equal to the Lyapunov exponents. If the Lyapunov exponents are all different then one expects that the statistics of the stretching exponents exhibits at long but finite times a large deviations regime characterized by the rate function $H(\vec{\sigma})$. Such a function contains more information about the short-distance long-time properties of the flow than the Lyapunov exponents and it enters the determination of several asymptotic transport properties of turbulent velocities at intermediate Reynolds numbers. The rate functions H for the direct and time-reversed flows are related by the Gallavotti-Cohen type symmetry (2.8).*

3 Particles in smooth Kraichnan velocities

In 1968 Robert H. Kraichnan initiated the study of turbulent transport in a Gaussian ensemble of velocities decorrelated in time. The statistics of such velocities is determined by the mean $\langle v^i(t,\boldsymbol{r})\rangle$, that we shall take equal to zero, and by the covariance of the form

$$\big\langle v^i(t,\boldsymbol{r})\, v^j(t',\boldsymbol{r}')\big\rangle \,=\, \delta(t-t')\, D^{ij}(\boldsymbol{r},\boldsymbol{r}')\,.$$

To imitate turbulent flows far from boundaries, one may assume:

- homogeneity $\qquad D^{ij}(\boldsymbol{r},\boldsymbol{r}') \,=\, D^{ij}(\boldsymbol{r}-\boldsymbol{r}')\,,$
- isotropy $\qquad D^{ij}(\boldsymbol{r}) \,=\, \delta^{ij}\, D_1(|\boldsymbol{r}|) \,+\, r^i r^j\, D_2(|\boldsymbol{r}|)\,,$
- scaling $\qquad D^{ij}(\boldsymbol{r}) - D^{ij}(\boldsymbol{0}) \,=\, \begin{cases} O(\boldsymbol{r}^2) & \text{for } |\boldsymbol{r}| \ll \eta, \\ O(|\boldsymbol{r}|^\xi) & \text{for } \eta \ll |\boldsymbol{r}| \ll L. \end{cases}$

The range where $D^{ij}(\boldsymbol{r}) - D^{ij}(\boldsymbol{0})$ is approximately quadratic is called the "Batchelor regime" and is used to mimic flows at intermediate Reynolds numbers, dominated by viscous effects, with the spatial velocity increments approximately linear in the point separation. The range where $D^{ij}(\boldsymbol{r}) - D^{ij}(\boldsymbol{0})$ scales with power ξ between 0 and 2 mimics the developed "inertial range" turbulence at high Reynolds numbers, where the spatial velocity increments scale like a fractional power of the distance between the points (the Kolmogorov scaling is rendered by $\xi = 4/3$). Although not very realistic, the Kraichnan model of turbulent velocities has provided a very useful playground for theoretical and numerical studies of transport properties in random flows. It allowed to test old ideas about turbulent transport and, even more importantly, to come up with important new ideas [39].

3.1 Lagrangian particles and tangent flow in Kraichnan model

In the Kraichnan ensemble, velocities are white noise (so distributional) in time and the Lagrangian trajectories are defined by the stochastic differential equation (SDE)

$$\mathrm{d}\boldsymbol{R} = \boldsymbol{v}(t,\boldsymbol{R})\,\mathrm{d}t$$

that we shall interpret with the Itô convention [69, 64], i.e. as equivalent to the integral equation

$$\boldsymbol{R}(t) \,=\, \boldsymbol{R}(t_0) \,+\, \lim_{t_0<t_1<\cdots<t_n<t} \sum_i \int_{t_i}^{t_{i+1}} \boldsymbol{v}(s,\boldsymbol{R}(t_i))\,\mathrm{d}s\,,$$

where the limit of the Riemann sums defines the Itô stochastic integral $\int_{t_0}^t \boldsymbol{v}(s,\boldsymbol{R}(s))\,\mathrm{d}s$. Note that the average value of the Itô integral vanishes since $\langle \boldsymbol{v}(s,\boldsymbol{R}(t_i))\rangle = 0$ for $s \geq t_i$. The Stratonovich convention would use the Riemann sums with $\boldsymbol{v}(s,\boldsymbol{R}(t_i))$ replaced by $\boldsymbol{v}(s,\boldsymbol{R}(\frac{1}{2}(t_i+t_{i+1})))$ but it gives the same result if $\partial_i D^{ij}(\boldsymbol{r},\boldsymbol{r}) \equiv 0$, which holds e.g. for homogeneous and isotropic velocities.

For a regular function f, the Itô stochastic calculus gives the SDE for the composed process with an additional (Itô) term as compared to the standard chain rule:

$$\mathrm{d}f(\boldsymbol{R}) \,=\, \boldsymbol{v}(t,\boldsymbol{R})\cdot\boldsymbol{\nabla} f(\boldsymbol{R})\,\mathrm{d}t + \tfrac{1}{2}D^{ij}(\boldsymbol{0})\nabla_i\nabla_j f(\boldsymbol{R})\,\mathrm{d}t$$

with the assumption of homogeneity. For the averages, one obtains the differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\langle f(\boldsymbol{R})\big\rangle \,=\, \big\langle \tfrac{1}{2}D^{ij}(0)\nabla_i\nabla_j f(\boldsymbol{R})\big\rangle$$

which may be rewritten in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f(\boldsymbol{r})\,\big\langle\delta(\boldsymbol{r}-\boldsymbol{R}(t;t_0,\boldsymbol{r}_0))\big\rangle\,\mathrm{d}\boldsymbol{r} \,=\, \int \tfrac{1}{2}D^{ij}(\boldsymbol{0})\,f(\boldsymbol{r})\,\big\langle\delta(\boldsymbol{r}-\boldsymbol{R}(t;t_0,\boldsymbol{r}_0))\big\rangle\,\mathrm{d}\boldsymbol{r}$$

or, stripping the last identity from arbitrary functions f, as

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\langle\delta(\boldsymbol{r}-\boldsymbol{R}(t;t_0,\boldsymbol{r}_0))\big\rangle \,=\, \tfrac{1}{2}D^{ij}(\boldsymbol{0})\,\nabla_{r^i}\nabla_{r^j}\big\langle\delta(\boldsymbol{r}-\boldsymbol{R}(t;t_0,\boldsymbol{r}_0))\big\rangle\,.$$

With the additional assumption of isotropy, $\frac{1}{2}D^{ij}(\boldsymbol{0}) = D_0\,\delta^{ij}$, and we infer that the probability to find the Lagrangian trajectory at time t at point $\boldsymbol{r}$ is that of the diffusing particle:

$$P(t_0,r_0;t,r) \,\equiv\, \big\langle\delta(\boldsymbol{r}-\boldsymbol{R}(t;t_0,\boldsymbol{r}_0))\big\rangle \,=\, \mathrm{e}^{|t-t_0|D_0\boldsymbol{\nabla}^2}(\boldsymbol{r}_0,\boldsymbol{r}) \,=\, \frac{1}{(4\pi D_0|t-t_0|)^{d/2}}\mathrm{e}^{-\frac{(\boldsymbol{r}-\boldsymbol{r}_0)^2}{4D_0|t-t_0|}}\,.$$

The constant D_0 is called the "eddy diffusivity". One infers that in mean, the Kraichnan turbulence causes diffusion. In particular, for the mean density and the mean scalar,

$$\begin{aligned}\langle n(t,\boldsymbol{r})\rangle &= \Big\langle \int n(t_0,\boldsymbol{r}_0)\;\delta(\boldsymbol{r}-\boldsymbol{R}(t;t_0,\boldsymbol{r}_0))\;\mathrm{d}\boldsymbol{r}_0\Big\rangle \;=\; \int n(t_0,\boldsymbol{r}_0)\;\mathrm{e}^{(t-t_0)D_0\boldsymbol{\nabla}^2}(\boldsymbol{r}_0,\boldsymbol{r})\;\mathrm{d}\boldsymbol{r}_0\,,\\ \langle \theta(t,\boldsymbol{r})\rangle &= \Big\langle \int \theta(t_0,\boldsymbol{r}_0)\;\delta(\boldsymbol{r}_0-\boldsymbol{R}(t_0;t,\boldsymbol{r}))\;\mathrm{d}\boldsymbol{r}_0\Big\rangle \;=\; \int \theta(t_0,\boldsymbol{r}_0)\;\mathrm{e}^{(t-t_0)D_0\boldsymbol{\nabla}^2}(\boldsymbol{r},\boldsymbol{r}_0)\;\mathrm{d}\boldsymbol{r}_0\end{aligned}$$

for $t \geq t_0$, see Eqs. (1.11) and (1.10). An initial blob of scalar or density diffuses in mean as watched in the laboratory frame. Of coarse, the evolution of the average density does not contain information about the dynamical behavior of the density fluctuations. A way to catch a glimpse of the latter is to look at the evolution of the blob from the point of view that moves with the fluid, as was discussed in Sect. 2.3.

To this end, let us study the dynamics of the small (infinitesimal) separation between the Lagrangian trajectories that satisfies the SDE

$$\mathrm{d}\delta\boldsymbol{R} \;=\; (\delta\boldsymbol{R}\cdot\boldsymbol{\nabla}\boldsymbol{v})(t,\boldsymbol{R}(t))\,\mathrm{d}t \;\equiv\; \Sigma(t)\,\delta\boldsymbol{R}\,\mathrm{d}t\,, \tag{3.1}$$

where $\Sigma^i{}_j(t) = (\nabla_j v^i)(t,\boldsymbol{R}(t))$. Again, we shall consider this equation with the Itô convention but the Stratonovich convention would give again the same result under the assumption that $\partial_i D^{ij}(\boldsymbol{r},\boldsymbol{r}) \equiv 0$. For a function of $\delta\boldsymbol{R}$, the Itô calculus gives:

$$\mathrm{d}f(\delta\boldsymbol{R}) \;=\; (\delta\boldsymbol{R}\cdot\boldsymbol{\nabla}\boldsymbol{v})(t,\boldsymbol{R}(t))\cdot\nabla f(\delta\boldsymbol{R})\,\mathrm{d}t \;-\; \tfrac{1}{2}\delta R^k\delta R^l\,(\nabla_k\nabla_l D^{ij})(\boldsymbol{0})\,(\nabla_i\nabla_j f)(\delta\boldsymbol{R})\,\mathrm{d}t$$

in the homogeneous case and for its average,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\langle f(\delta\boldsymbol{R})\big\rangle \;=\; \big\langle \tfrac{1}{2}\delta R^k\delta R^l\,C^{ij}_{kl}\,(\nabla_i\nabla_j f)(\delta\boldsymbol{R})\big\rangle\,, \tag{3.2}$$

where $C^{ij}_{kl} = -\nabla_k\nabla_l D^{ij}(0)$. The last result is the same as if $\delta\boldsymbol{R}(t)$ satisfied the linear Itô SDE

$$\mathrm{d}\delta\boldsymbol{R} \;=\; S(t)\,\delta\boldsymbol{R}\,\mathrm{d}t \tag{3.3}$$

with the matrix-valued white noise $S(t)$ such that $\langle S(t)\rangle = 0$ and $\langle S^i{}_k(t)\,S^j{}_l(t')\rangle = C^{ij}_{kl}\,\delta(t-t')$. Here taking the Itô prescription is essential since although the Itô and Stratonovich prescriptions give the same result for Eq. (3.1), they do not agree for Eq. (3.3) with $S^i{}_j(t)$ being the white noise! Similarly, the statistics of the tangent process $W(t;\boldsymbol{r}_0)$ for a fixed initial point $\boldsymbol{r}_0$ may be obtained by solving the linear Itô SDE

$$\mathrm{d}W \;=\; S(t)\,W\,\mathrm{d}t \tag{3.4}$$

with the white noise $S(t)$ and $W(0) = Id$. The decoupling of the Lagrangian trajectories $\boldsymbol{R}(t;\boldsymbol{r}_0))$ from these equations is the great simplification of the Kraichnan model!

The further simplification is the decoupling of the natural measures from the statistics of $W(t;\boldsymbol{v}|\boldsymbol{v}))$ for the homogeneous Kraichnan flows on a periodic box V. Due to the homogeneity, such measures $n(\mathrm{d}\boldsymbol{r}|\boldsymbol{v})$ have to satisfy the relation

$$\big\langle n(\mathrm{d}\boldsymbol{r}|\boldsymbol{v})\big\rangle = \frac{\mathrm{d}\boldsymbol{r}}{|V|}$$

Now, since $n(\mathrm{d}\boldsymbol{r}|\boldsymbol{v})$ depends on the past velocities and, for $t \geq 0$, $W(t,\boldsymbol{r}|\boldsymbol{v})$ depends on the future ones (independent of the past ones in the Kraichnan ensemble), we infer that

$$\Big\langle \int_V f(W(t;\boldsymbol{r}|\boldsymbol{v}))\;n(\mathrm{d}\boldsymbol{r}|\boldsymbol{v})\Big\rangle = \int_V \big\langle f(W(t;\boldsymbol{r}|\boldsymbol{v}))\big\rangle\,\big\langle n(\mathrm{d}\boldsymbol{r}|\boldsymbol{v})\big\rangle = \int_V \Big\langle f(W(t;\boldsymbol{r}|\boldsymbol{v}))\,\frac{\mathrm{d}\boldsymbol{r}}{|V|}\Big\rangle = \big\langle f(W(t;\boldsymbol{r}_0|\boldsymbol{v}))\big\rangle\,.$$

It follows that the statistics of $W(t;\boldsymbol{r}|\boldsymbol{v})$ for $t \geq 0$ (and hence also of $\vec{\rho}(t;\boldsymbol{r}|\boldsymbol{v})$) with $(\boldsymbol{r},\boldsymbol{v})$ distributed with the natural invariant measure $N(\mathrm{d}\boldsymbol{r},\mathrm{d}\boldsymbol{v})$ coincides with that of the solution of Eq. (3.4) with the white noise $S(t)$ and $W(0) = Id$ (and of the corresponding $\vec{\rho}(t)$). In other words,

$$P_t(\vec{\sigma}) \;=\; \tilde{P}_t(\vec{\sigma}) \qquad \text{for} \qquad t \geq 0\,.$$

Since the Kraichnan ensemble is time reversible, we also infer from Eq. (2.10) that

$$P_t(-\bar{\sigma}) \;=\; P_t(\vec{\sigma})\;\mathrm{e}^{t\sum_{i=1}^{d}\sigma_i}$$

for all $t \geq 0$.

3.2 Multiplicative large deviations in Kraichnan velocities

3.2.1 Isotropic case

The specific form of $P_t(\vec{\sigma})$ depends on the covariance C^{ij}_{kl} reflecting the single-point statistics of the velocity gradients. In the isotropic case,

$$C^{ij}_{kl} = \beta(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k) + \gamma\delta^{ij}\delta_{kl}$$

with the right hand side specified, up to normalization, by the "compressibility degree"

$$\wp \;=\; \frac{\langle(\nabla_i v^i)^2\rangle}{\langle(\nabla_j v^i)^2\rangle} \;=\; \frac{C^{ij}_{ij}}{C^{ii}_{jj}} \;=\; \frac{(d+1)\beta+\gamma}{2\beta+\gamma d}\,. \tag{3.5}$$

In general, $0 \le \wp \le 1$. Vanishing $\wp$ corresponds to incompressible velocities, whereas $\wp = 1$ to gradient ones. The normalization of C^{ij}_{kl} is set by the time scale τ_η that we shall define by the relation

$$\tau_\eta \;=\; \frac{2d\,\delta(0)}{\langle(\nabla_i v^j)^2\rangle} \;=\; \frac{2d}{C^{ii}_{jj}} \;=\; \frac{2}{2\beta+\gamma d}\,. \tag{3.6}$$

It may be thought of as mimicking the Kolmogorov time scale in the Kraichnan flow. By the Itô calculus, for $W(t)$ solving the SDE (3.4), one obtains the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle f(W)\rangle \;=\; \Big\langle \tfrac{1}{2} W^m_i W^n_j C^{ij}_{kl}\,\frac{\partial}{\partial W^m_k}\frac{\partial}{\partial W^n_l} f(W)\Big\rangle\,. \tag{3.7}$$

Any function $\tilde{f}(\vec{\rho})$ of the stretching exponents may be viewed as a function f on the matrix group $GL(d)$ such that

$$f(\mathcal{O}'W\mathcal{O}) \;=\; f(W) \qquad\text{for}\qquad \mathcal{O}',\mathcal{O}\in O(d)\,.$$

To calculate $\frac{\mathrm{d}}{\mathrm{d}t}\langle\tilde{f}(\vec{\rho})\rangle$, it is enough to compute the operator inside the average on the right hand side of Eq. (3.7) in the action on such functions. A straightforward although tedious calculation gives [19]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\langle\tilde{f}(\vec{\rho})\rangle \;=\; \langle\mathscr{L}\tilde{f}(\vec{\rho})\rangle$$

for the generator

$$\mathscr{L} \;=\; \frac{\beta+\gamma}{2}\Big(\sum_i\frac{\partial^2}{\partial\rho_i^2} + \sum_{i\ne j}\coth(\rho_i-\rho_j)\frac{\partial}{\partial\rho_i}\Big) + \frac{\beta}{2}\Big(\sum_i\frac{\partial}{\partial\rho_i}\Big)^2 - \frac{(d+1)\beta+\gamma}{2}\sum_i\frac{\partial}{\partial\rho_i}$$

After the conjugation by the function $\mathcal{F}(\vec{\rho}) = \mathrm{e}^{-\frac{1}{2}\sum_i\rho_i}(\prod_{i<j}\sinh(\rho_i-\rho_j))^{1/2}$, the operator $\mathscr{L}$ becomes the Calogero-Sutherland-Moser Hamiltonian of d quantum particles on the line [65]:

$$\mathcal{F}\mathscr{L}\mathcal{F}^{-1} \;=\; \frac{\beta+\gamma}{2}\Big(\sum_i\frac{\partial^2}{\partial\rho_i^2} + \tfrac{1}{2}\sum_{i<j}\frac{1}{\sinh^2(\rho_i-\rho_j)}\Big) + \frac{\beta}{2}\Big(\sum_i\frac{\partial}{\partial\rho_i}\Big)^2 + \text{const.} \;\equiv\; -\mathscr{H}_{\rm CSM}\,.$$

Since, with the initial value $\vec{\rho}(0)=0$,

$$\langle\tilde{f}(\vec{\rho}(t))\rangle \;=\; \int\tilde{f}(t\vec{\sigma})\;P_t(\vec{\sigma})\;\mathrm{d}\vec{\sigma}\,,$$

it follows that

$$P_t(\vec{\sigma}) \;=\; t\,\mathrm{e}^{t\mathscr{L}}(\vec{0},t\vec{\sigma}) \;=\; t\lim_{\vec{\rho}_0\to 0}\;\mathcal{F}(\vec{\rho}_0)^{-1}\;\mathrm{e}^{-t\mathscr{H}_{\rm CSM}}(\vec{\rho}_0,t\vec{\sigma})\;\mathcal{F}(t\vec{\sigma})\,.$$

The spectral representation of the heat kernel $\mathrm{e}^{-t\mathscr{H}_{\rm CM}}$ is known explicitly [65]. Its saddle point calculation gives the large deviation form of $P_t(\vec{\rho})$. The latter may be also found directly [7] since

$$P_t(\vec{\sigma}) \;\simeq\; t\,\mathrm{e}^{t\mathscr{L}_\infty}(0,t\vec{\sigma}) \;=\; \text{const.}\;\mathrm{e}^{-tH(\vec{\sigma})}\,,$$

where $\mathscr{L}_\infty$ is the second order differential operator with constant coefficients obtained from $\mathscr{L}$ by replacing $\cosh(\rho_i-\rho_j)$ by ±1 for $i\lessgtr j$, respectively. An easy calculation gives then:

$$H(\vec{\sigma}) \;=\; \frac{\tau_\eta}{4(d+\wp(d-2))}\Big[(d-1)(d+2)\sum_i(\sigma_i-\lambda_i)^2 \;+\; (\wp^{-1}-d)\Big(\sum_i(\sigma_i-\lambda_i)\Big)^2\Big]$$

with the Lyapunov exponents

$$\lambda_i\tau_\eta \;=\; \frac{(d+\wp(d-2))(d-2i+1)}{(d-1)(d+2)} \;-\; \wp$$

equally spaced. In particular, $\lambda_1 > 0$ for $\wp < d/4$ (the chaotic phase) and $\lambda_1 < 0$ for $\wp > d/4$. The change of sign of λ_1 induces a change in the transport properties (direct cascade of the the scalar for $\lambda_1 > 0$ versus inverse cascade for $\lambda_1 < 0$ [29]). The rate function H satisfies the Gallavotti-Cohen relation (2.8) with $H' = H$, as may be easily checked.

The exponents γ_q of Eq. (2.17) determining the asymptotic behavior of the density fluctuations in the Lagrangian frame are easily computed to be

$$\gamma_q \tau_\eta \;=\; \wp d\, q(q+1)\,.$$

Note that $\gamma_q > 0$ for $q > 0$ and $\wp > 0$ so that in the compressible homogeneous isotropic Kraichnan flow the positive moments of the density grow exponentially in the Lagrangian frame.

3.2.2 Non-isotropic case

The equal spacing of the Lyapunov exponents and the Gaussian form of the large deviations of the stretching rates is particular for the Kraichnan flow that is homogeneous and isotropic at short distances. On a periodic box, this is not a generic situation. For example, on a periodic square, typically, the covariance of the matrix-valued white noise $S(t)$ will only have symmetries of the square and will be given by the expression

$$C^{ij}_{kl} \;=\; 2\alpha\delta^{ij}_{kl} + \beta(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k) + \gamma\delta^{ij}\delta_{kl}\,, \tag{3.8}$$

where δ^{ij}_{kl} is equal to 1 if $i = j = k = l$ and vanishes otherwise. The compressibility degree (3.5) and the Kolmogorov time (3.6) are now given by

$$\wp \;=\; \frac{2\alpha + 3\beta + \gamma}{2(\alpha + \beta + \gamma)}\,, \qquad \tau_\eta \;=\; \frac{1}{\alpha + \beta + \gamma}$$

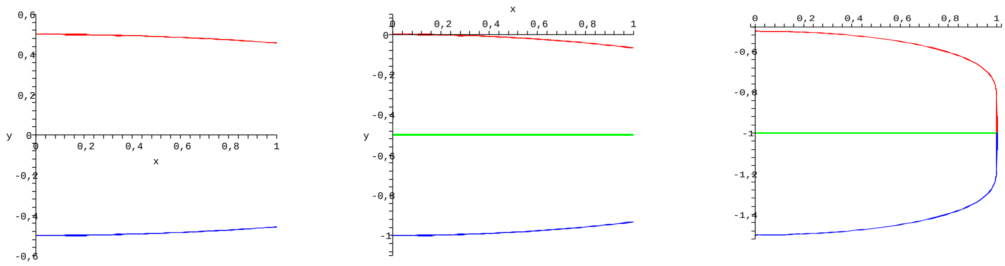
and one may introduce the "anisotropy degree"

$$\omega \;=\; |\alpha|\tau_\eta\,.$$

The Lyapunov exponents may be expressed in this case [31] by the complete elliptic integrals [49]:

$$\lambda_1\tau_\eta \;=\; -1 \,+\, \frac{3 - 2\wp + \omega}{2}\,\frac{\boldsymbol{E}(k)}{\boldsymbol{K}(k)}\,, \qquad \lambda_2\tau_\eta \;=\; 1 \,-\, 2\wp \,-\, \frac{3 - 2\wp + \omega}{2}\,\frac{\boldsymbol{E}(k)}{\boldsymbol{K}(k)}\,,$$

where $k^2 = \frac{2\omega}{3 - 2\wp + \omega}$, depicted below as functions of $0 \le \omega \le 1$ for three values of the compressibility degree $\wp = 0$, $\wp = \frac{1}{2}$ and $\wp = 1$ (the middle green lines give $\frac{1}{2}(\lambda_1 + \lambda_2)\tau_\eta = -\wp$).



(a) $\wp = 0$ (b) $\wp = \frac{1}{2}$ (c) $\wp = 1$

If $\lambda_1 > \lambda_2$ then the large deviations of the stretching exponents are controlled by the rate function[31]

$$\begin{aligned} H(\sigma_1, \sigma_2) \;&=\; \tfrac{1}{8}\tau_\eta\wp^{-1}(\sigma_1 + \sigma_2 + 2\wp\tau_\eta^{-1})^2 \\ &+\; \max_{\nu}\Big[\nu(\sigma_1 - \sigma_2) - (1+\omega)\tau_\eta^{-1}\nu(\nu+1) + (3 - 2\wp + \omega)\tau_\eta^{-1}\,E_{\nu,0}(k^2)\Big], \end{aligned} \tag{3.9}$$

where $E_{\nu,0}(k^2)$ is the groundstate energy of the Lamé-Hermite one-dimensional Schrödinger operator

$$-\frac{\mathrm{d}^2}{\mathrm{d}u^2} \,+\, \nu(\nu+1)\,k^2\,\mathrm{sn}^2(u,k) \tag{3.10}$$

acting on periodic functions of u with period $2\boldsymbol{K}(k)$. $\mathrm{sn}(u,k)$ is the Jacobian sine-amplitude function [49]. If $\lambda_1 = \lambda_2$, which happens for $\omega = 1 = \wp$, then

$$H(\sigma_1, \sigma_2) \;=\; \tfrac{1}{8}\tau_\eta(\sigma_1 + \sigma_2 + 2\tau_\eta^{-1})^2 \,+\, \tfrac{1}{8}\tau_\eta(\sigma_1 - \sigma_2)^2 \tag{3.11}$$

and cannot be obtained as the limit of the previous formula (the time-scales at which the large deviation regime sets in diverge as $(\lambda_1 - \lambda_2)\tau_\eta \to 0$ [31]). Except for the last case, the large deviations of $(\rho_1 - \rho_2)$ are non-Gaussian in the presence of anisotropy, i.e. for $\omega > 0$.

3.3 Inertial particles in Kraichnan model

The inertial particles dynamics in the Kraichnan velocities is given for the vanishing diffusivity κ by the SDE

$$d\boldsymbol{R}\,=\,\boldsymbol{U}\,dt\,,\qquad d\boldsymbol{U}\,=\,\tfrac{1}{\tau}\big(-\boldsymbol{U}+\boldsymbol{v}(t,\boldsymbol{R})\big)\,dt\,, \tag{3.12}$$

see Eq. (1.4). Both Itô or Stratonovich conventions give for it the same result. Small phase-space separations between two inertial particles satisfy, in turn, the equations

$$d\boldsymbol{\delta}R\,=\,\boldsymbol{\delta}V\,dt\,,\qquad d\delta\boldsymbol{U}\,=\,\tfrac{1}{\tau}\big(-\delta\boldsymbol{U}+\Sigma(t)\,\delta\boldsymbol{R}\big)\,dt\,, \tag{3.13}$$

where, as before, $\Sigma^i{}_j(t)=\nabla_j v^i(t,\boldsymbol{R}(t))$. The matrix $\boldsymbol{W}(t;\boldsymbol{r},\boldsymbol{u})$ propagating $(\delta\boldsymbol{R},\delta\boldsymbol{U})$ along the trajectory starting at $(\boldsymbol{r},\boldsymbol{u})$ satisfies then the SDE

$$d\boldsymbol{W}\,=\,\begin{pmatrix}0 & Id\\ \frac{1}{\tau}\Sigma(t) & -\frac{1}{\tau}\end{pmatrix}\boldsymbol{W}\,dt\,. \tag{3.14}$$

As before, if the velocity statistics is homogeneous then $\Sigma(t)$ may be replaced by the matrix-valued white noise $S(t)$ as long as we are interested in the statistics of $\boldsymbol{W}(t;\boldsymbol{r},\boldsymbol{u})$ for fixed $(\boldsymbol{r},\boldsymbol{u})$. The same is true if we average over the latter points with the natural measures $n(d\boldsymbol{r},d\boldsymbol{u})$ since the latter average decouples due to the independence of the past and future velocities). For the homogeneous and isotropic Kraichnan velocities, on a periodic box V, the ensemble average of the natural measures is equal to the Maxwell distribution:

$$\big\langle n(d\boldsymbol{r},d\boldsymbol{u})\big\rangle\,=\,\frac{\tau^{d/2}}{(2\pi D_0)^{d/2}|V|}\,e^{-\frac{\tau}{2D_0}\boldsymbol{U}^2}\,d\boldsymbol{r}\,d\boldsymbol{u}\,.$$

Note that Eq. (3.14) implies that $\det\boldsymbol{W}(t)=e^{-dt/\tau}$ so that the corresponding stretching rates $\sigma_i(t)$, $i=1,\dots,2d$, satisfy the relation

$$\sum_{i=1}^{2d}\sigma_i(t)\,=\,-\frac{d}{\tau}\,. \tag{3.15}$$

A simple argument [40] shows that the Gallavotti-Cohen type symmetry (2.8) for the large deviations rate functions reduces now to the identity

$$H(\tau^{-1}\vec{1}-\vec{\sigma})\,=\,H(\vec{\sigma}) \tag{3.16}$$

with $\vec{1}\equiv(1,\dots,1)$ holding for $\vec{\sigma}$ satisfying the relation (3.15).

3.3.1 One-dimensional case

For the one-dimensional Kraichnan velocities, setting $\delta r(t)=\psi(t)\,e^{t/(2\tau)}$ permits to rewrite Eqs. (3.13) in the second order form [80]

$$-\ddot{\psi}\,+\,V(t)\psi\,=\,-\frac{1}{4\tau^2}\psi \tag{3.17}$$

with $V(t)=\frac{1}{\tau}\Sigma(t)$. Again, $V(t)$ may be replaced by a white noise with the covariance

$$\big\langle V(t)\,V(t')\,\big\rangle\,=\,\frac{2}{\tau^2\tau_\eta}\,\delta(t-t') \tag{3.18}$$

Upon viewing t as a space variable, Eq. (3.17) becomes the one-dimensional stationary Schrödinger equation in the delta-correlated potential $V(t)$ at the energy $E=-\frac{1}{4\tau^2}$, a well known model for the one-dimensional Anderson localization. This is an old problem whose solution goes back to [51], see also [59]. One considers the process $x(t)=\dot{\psi}(t)/\psi(t)$ that satisfies the SDE

$$dx\,=\,-(x^2+E-V(t))\,dt\,. \tag{3.19}$$

Its trajectories reach $-\infty$ in a finite time but reappear immediately at $+\infty$. Such jumps correspond to a passage of $\psi(t)$ through zero with $\dot{\psi}\neq 0$ or to a crossing of close inertial particles in space. The jumping process possesses the stationary probability measure

$$n(dx)\,=\,\frac{1}{Z}\,e^{-\tau^2\tau_\eta(\frac{1}{3}x^2+Ex)}\Big(\int_{-\infty}^{x}e^{\tau^2\tau_\eta(\frac{1}{3}y^2+Ey)}dy\Big)\,dx\,, \tag{3.20}$$

with the density $\sim\ x^{-2}$ for large $|x|$ (Z is the normalization constant). The localization Lyapunov exponent λ, equal to the mean value of $x(t)=\frac{\mathrm{d}}{\mathrm{d}t}\ln(|\psi(t)|)$, is always positive signaling the permanent localization in one dimension. The Lyapunov exponents for the inertial particles are $\lambda_1=\lambda-\frac{1}{2\tau}$ and $\lambda_2=-\lambda-\frac{1}{2\tau}$ (recall the relation between $\psi(t)$ and $\delta r(t)$). λ can be expressed [59] by the Airy functions [49] and one obtains:

$$\lambda_{1,2}\tau_\eta\ =\ 4u^2\Big(-u\pm\frac{\mathrm{Ai}'(u^2)\,\mathrm{Ai}(u^2)\ +\ \mathrm{Bi}'(u^2)\,\mathrm{Bi}(u^2)}{\mathrm{Ai}(u^2)\ +\ \mathrm{Bi}(u^2)}\Big)\,, \tag{3.21}$$

for $u=\frac{1}{2}St^{-1/3}$ and the Stokes number $St=\frac{\tau}{\tau_\eta}$. For large St, λ_1 is positive and decreases as $St^{-2/3}$. For St small, it is negative and approaches when $St\to0$ the value -1 for the Lagrangian particles.

3.3.2 Two-dimensional case

For the two-dimensional homogeneous and isotropic Kraichnan velocities, one may treat $\delta\boldsymbol{r}(t)\,\mathrm{e}^{t/(2\tau)}=\psi(t)$ as a complex-valued process which still satisfies Eq. (3.17) with $V(t)$ a complex valued white noise with the covariance [67, 62]

$$\big\langle V(t)\,V(t')\big\rangle\ =\ \frac{2\wp-1}{\tau^2\tau_\eta}\,\delta(t-t')\,,\qquad \big\langle V(t)\,\overline{V(t')}\big\rangle\ =\ \frac{2}{\tau^2\tau_\eta}\,\delta(t-t')\,. \tag{3.22}$$

For the complex-valued process $z(t)=\dot\psi(t)/\psi(t)$ one obtains now the SDE

$$\mathrm{d}z\ =\ -(z^2+E-V(t))\,\mathrm{d}t \tag{3.23}$$

which may be shown to possess an invariant probability measure, this time without an explicit analytic form but decaying like $|z|^{-4}$ for large $|z|$. The top Lyapunov exponent λ_1 is equal to the expectation of $\mathrm{Re}\,z$ in this measure. It was studied numerically in [62, 52, 14]. It seems again to behave as $St^{-2/3}$ for large Stokes numbers. The reference [14] contains also the numerical results about large deviations of $\mathrm{Re}\,z(t)$.

Conclusion. *In smooth Kraichnan velocities with symmetry properties, the tangent process for Lagrangian particles is related to integrable quantum-mechanical systems. This relation gives an analytic access to the large deviation rate functions of the stretching exponents. For inertial particles, similar relations to one-dimensional models of the Anderson localization permit to find analytically the Lyapunov exponents in one-dimensional flows and facilitate their analysis in two (and three) dimensions.*

4 Particles and fields in rough Kraichnan velocities

The previous discussion pertained to the transport of particles in the Batchelor regime with smooth velocities and small particle separations. We shall pass now to the study of particle dynamics at separations in the inertial range of scales where velocities become rough. We shall concentrate on the motion of Lagrangian particles. For recent ideas and numerics about the motion of inertial particles in rough flows, see [15, 41].

Let us start by considering the simultaneous motion of N Lagrangian particles $\boldsymbol{R}_n(t)$ in the homogeneous Kraichnan velocities. A function $f(\underline{\boldsymbol{R}})$ of their joint positions $\underline{\boldsymbol{R}}(t) \equiv (\boldsymbol{R}_1(t), \ldots, \boldsymbol{R}_N(t))$ evolves accordingly to the Ito stochastic equation

$$\mathrm{d}f(\underline{\boldsymbol{R}}) \;=\; \sum_{n=1}^{N} v^i(t, \boldsymbol{R}_i)\, \nabla_{R_n^i} f(\underline{R})\, \mathrm{d}t \;+\; \underbrace{\frac{1}{2} \sum_{m,n=1}^{N} D^{ij}(\boldsymbol{R}_m - \boldsymbol{R}_n) \nabla_{R_m^i} \nabla_{R_m^j} f(\underline{R})\, \mathrm{d}t}_{\mathcal{M}_N}\,. \tag{4.1}$$

For the expectations, this gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\langle f(\underline{\boldsymbol{R}})\big\rangle \;=\; \big\langle(\mathcal{M}_N f)(\underline{\boldsymbol{R}})\big\rangle\,. \tag{4.2}$$

4.1 Particle dispersion

First, let us study the evolution of a function of the separation $\Delta\boldsymbol{R}(t) \equiv \boldsymbol{R}_1(t) - \boldsymbol{R}_2(t)$ between two trajectories:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\langle f(\Delta\boldsymbol{R})\big\rangle \;=\; \big\langle(D^{ij}(\boldsymbol{0}) - D^{ij}(\Delta\boldsymbol{R}))\nabla_i\nabla_j f(\Delta\boldsymbol{R})\big\rangle\,. \tag{4.3}$$

In particular, we shall look at functions of the inter-particle distance $\Delta \equiv |\Delta\boldsymbol{R}|$, called the "particle dispersion". In the Batchelor regime,

$$D^{ij}(\boldsymbol{0}) - D^{ij}(\Delta\boldsymbol{R}) \;\approx\; \frac{1}{2}\Delta R^k \Delta R^l\, \nabla_k\nabla_l D^{ij}(\boldsymbol{0})$$

and Eq. (4.3) reduces to the same equation as for $\big\langle f(\delta\boldsymbol{R})\big\rangle$, see the relation (3.2). For functions of the particle dispersion $\Delta \equiv |\Delta\boldsymbol{R}|$, one obtains after a short calculation, assuming also isotropy of the velocity ensemble:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle f(\Delta)\rangle \;=\; \Big\langle\Big[\frac{2\beta+\gamma}{2}\Big(\frac{\partial}{\partial \ln\Delta}\Big)^2 + \lambda_1 \frac{\partial}{\partial\ln\Delta}\Big] f(\Delta)\Big\rangle,$$

where $\lambda_1 = \frac{d-2}{2}\gamma - \beta$ is the first Lyapunov exponent. For the PDF of the distance Δ this gives

$$\big\langle\delta(\Delta - |\Delta\boldsymbol{R}(t)|)\big\rangle \;=\; \frac{1}{\sqrt{2\pi(\beta+\gamma)t}}\,\mathrm{e}^{-\frac{(\ln\frac{\Delta}{\Delta_0} - \lambda_1 t)^2}{2(\beta+\gamma)t}}\,\frac{1}{\Delta}\,,$$

i.e. a log-normal distribution with $\Delta_0 \equiv |\Delta\boldsymbol{R}(0)|$. Note that

$$\lim_{\Delta_0\to 0}\; \big\langle\delta(\Delta - |\Delta R(t)|)\big\rangle = \delta(\Delta)$$

so that when the initial distance Δ_0 of the trajectories tends to zero, so does their time t distance. The latter behavior is imposed by the uniqueness of the Lagrangian trajectories in each velocity realization, given their initial point (and assuming that the molecular diffusion is absent). It is consistent with both exponential separation (when $\lambda_1 > 0$) and exponential contraction (when $\lambda_1 < 0$) of close trajectories.

Now, let us consider the Kraichnan model in the inertial range. Here $D^{ij}(\boldsymbol{0}) - D^{ij}(\boldsymbol{r}) = O(r^\xi)$ and the isotropy imposes the form of the tensor:

$$D^{ij}(\boldsymbol{0}) - D^{ij}(\boldsymbol{r}) \;=\; \beta\, r^i r^j |\boldsymbol{r}|^{\xi-2} + \frac{1}{2}\gamma\, \delta^{ij} r^\xi$$

in the way fixed, up to normalization, by the compressibility degree

$$\wp \;=\; \lim_{r\to 0}\frac{\nabla_i\nabla_j D^{ij}(\boldsymbol{r})}{\nabla_j\nabla_j D^{ii}(\boldsymbol{r})} \;=\; \frac{2\xi^{-1}(d-1+\xi)\beta+\gamma}{2\beta+d\gamma}$$

between 0 and 1 (taking $\xi = 2$ reproduces the smooth case formula (3.5)). The equation for the evolution of the two-particle dispersion Δ takes now the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle f(\Delta)\rangle \;=\; \Big\langle\frac{2\beta+\gamma}{2}\Delta^{\xi-a}\frac{\partial}{\partial\Delta}\Delta^a\frac{\partial}{\partial\Delta}f(\Delta)\Big\rangle \;\equiv\; \big\langle(M_a f)(\Delta)\big\rangle$$

for $a = \frac{(d-1)\gamma}{2\beta+\gamma} = \frac{d+\xi}{1+\wp\xi} - 1$. The distance process $\Delta(t)$ is the one-dimensional diffusion with the generator M_a and the transition PDF

$$\big\langle \delta(\Delta - \Delta(t;\Delta_0))\big\rangle \,=\, \mathrm{e}^{tM_a}(\Delta_0,\Delta)\,.$$

It is instructive to change variables from Δ to $x = \Delta^{1-\xi/2}$ since the process $x(t)$ is the Bessel diffusion with the generator proportional to $x^{1-D_{\rm eff}}\partial_x x^{D_{\rm eff}-1}\partial_x$, i.e. to the radial Laplacian in $D_{\rm eff}$ dimensions, where

$$D_{\rm eff} \,=\, \frac{2(a+1-\xi)}{2-\xi}$$

In other words, $x(t)$ behaves as the norm of the Brownian motion in (continuous) dimension $D_{\rm eff}$. The short-distance behavior of the Brownian motion below and above two dimensions is different. In particular, for $0 < D_{\rm eff} < 2$, i.e. for

$$p_c^1 \,\equiv\, \frac{d-2}{2\xi} + \frac{1}{2} \,<\, \wp \,<\, \frac{d}{\xi^2} \,\equiv\, p_c^2\,,$$

the generator of the Bessel process admits different boundary conditions at $x = 0$. They correspond to different behaviors of Lagrangian particles when they collide.

4.2 Phases of Lagrangian flow

There are three different phases with trajectory behavior and different transport properties in the inertial range, depending on the level of compressibility [48, 33, 57].

4.2.1 Weakly compressible phase

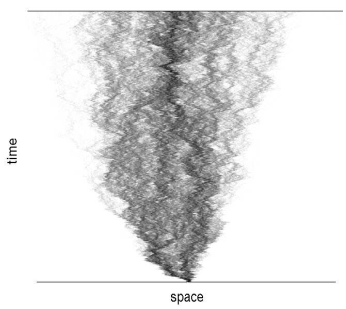
For $0 \le \wp \le p_c^1$, i.e. for $D_{\rm eff} > 2$, there is only one possible boundary condition at zero:

$$(\Delta^a \frac{\partial}{\partial\Delta} f)(0) \,=\, 0$$

and it renders M_a self-adjoint on the Hilbert space $L^2(\mathbb{R}_+, \Delta^{a-\xi}\mathrm{d}\Delta)$. The process $\Delta(t)$ has a reflecting behavior at zero and

$$\lim_{\Delta_0\to 0}\langle\delta(\Delta - \Delta(t;\Delta_0))\rangle \,=\, \text{const.}\,\frac{\Delta^{a-\xi}}{t^{\frac{a+1-\xi}{2-\xi}}}\,\mathrm{e}^{-\frac{2\Delta^{2-\xi}}{(2\beta+\gamma)(2-\xi)^2 t}}\,.$$

The formula says that the distance between two particles at time t is smoothly distributed even if their initial distance is zero: the trajectories separate in finite time (unlike in the chaotic regime with exponential separation where close particles take longer and longer to attain a sizable separation)! This implies the “spontaneous randomness” of the Lagrangian flow [19]: the sharp distribution $\delta(\boldsymbol{r} - \boldsymbol{R}(t;t_0,\boldsymbol{r}_0|\boldsymbol{v})$ of the time t position of the trajectory starting at a fixed point in a fixed velocity field realization is replaced by a diffused distribution $P(t_0,\boldsymbol{r}_0;t,\boldsymbol{r}|\boldsymbol{v})$ that is not concentrated at one point, as illustrated below:



This is possible because for $D(\mathbf{0}) - D(\boldsymbol{r}) \propto |r|^\xi$ the typical velocity realizations $\boldsymbol{v}(t,\boldsymbol{r})$ are only Hölder continuous in space, with exponent smaller than $\xi/2$. In such velocity fields, the solutions of the trajectory equation $\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = \boldsymbol{v}(t,\boldsymbol{R})$ in a fixed $\boldsymbol{v}$ are not uniquely determined by the initial condition but form a “generalized flow”: a stochastic (Markov) process with transition probabilities $P(t_0,\boldsymbol{r}_0;t,\boldsymbol{r}|\boldsymbol{v})$. The latter have been constructed rigorously by Le Jan and Raimond in [57].

The spontaneous randomness of the Lagrangian flow implies the persistence of dissipation in the scalar transport. Note that for the incompressible flow where

$$\int P(t,\boldsymbol{r};t_0,\boldsymbol{r}_0|\boldsymbol{v})\ \mathrm{d}\boldsymbol{r}_0\ =\ \int P(t,\boldsymbol{r};t_0,\boldsymbol{r}_0|\boldsymbol{v})\ \mathrm{d}\boldsymbol{r}\ =\ 1$$

and the unforced scalar evolves according to the equation

$$\theta(t,\boldsymbol{r})\ =\ \int P(t,\mathrm{r};\mathrm{t}_0,\mathrm{r}_0)\ \theta(\mathrm{t}_0,\mathrm{r}_0)\ \mathrm{dr}_0\,, \tag{4.4}$$

the inequality

$$0\leq\int P(t,\boldsymbol{r};t_0,\boldsymbol{r}_0|\boldsymbol{v})\ (\theta(t_0,\boldsymbol{r}_0)-\theta(t,\boldsymbol{r}))^2\,\mathrm{d}\boldsymbol{r}_0\,\mathrm{d}\boldsymbol{r}\ =\ \int\theta(t_0,\boldsymbol{r}_0)^2\,\mathrm{d}r_0-\int\theta(t,\boldsymbol{r})^2\,\mathrm{d}r$$

implies that the L^2 norm of θ (the scalar "energy") cannot grow. Besides, it is conserved if and only if, for each $\boldsymbol{r}$, $\theta(t_0,\boldsymbol{r}_0)=\theta(t,\boldsymbol{r})$ on the support of $P(t,\boldsymbol{r};t_0,\cdot|\boldsymbol{v})$, so if and only if the flow is deterministic. The persistent dissipation of the scalar energy in generalized flows is a short-distance phenomenon. It leads to the direct scalar-energy cascade where the scalar energy supplied at long distances is transfered without loss towards shorter and shorter scales and finally dissipated on the infinitesimal ones (in the limit of vanishing molecular diffusivity κ) [55, 39].

4.2.2 Strongly compressible phase

For $\wp>p_c^2$, i.e. for $D_{\mathrm{eff}}<0$, the only possible boundary condition for M_a is $f(0)=0$ and it again renders M_a self-adjoint in $L^2(\mathbb{R}_+,\Delta^{a-\xi}\mathrm{d}\Delta)$. Now, $\Delta(t)$ is absorbed at zero and

$$\big\langle\delta(\Delta-\Delta(t;\Delta_0))\big\rangle\ =\ \mathrm{regular}\ +\ \mathrm{const.}\,\delta(\Delta)\,,$$

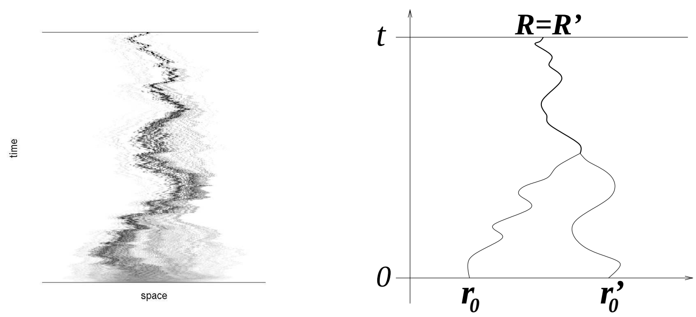
see [48]. In this phase there is a positive probability for the Lagrangian trajectories that start at positive distance Δ_0 to collapse together by time t. When $\Delta_0\to 0$ then the contact term dominates:

$$\lim_{\Delta_0\to 0}\ \big\langle\delta(\Delta-\Delta(t,\Delta_0))\big\rangle\ =\ \delta(\Delta)$$

which signals that the trajectories are deterministic:

$$P(t_0,\boldsymbol{r}_0;t,\boldsymbol{r}|\boldsymbol{v})\ =\ \delta(\boldsymbol{r}-\boldsymbol{R}(t;t_0,\boldsymbol{r}_0|\boldsymbol{v}))$$

for single trajectories $\boldsymbol{R}(t;t_0,\boldsymbol{r}_0|\boldsymbol{v})$ collapsing for different $\boldsymbol{r}_0$ at later times, as on the illustration:



This is again a non-standard behavior. Since trajectories are unique, there is no persistent dissipation of the scalar energy and, in the presence of a steady source, the latter exhibits an inverse cascade (towards long distances) [48].

4.2.3 Intermediate compressibility phase

For $p_c^1<\wp<p_c^2$, i.e. $0<D_{\mathrm{eff}}<2$, both reflecting and absorbing boundary conditions are possible, the first one is selected by adding very small diffusivity, the second one by considering velocities smeared at very small distances (small viscosity) [33]. The other permitted boundary conditions are the "sticky" ones [46]:

$$\mu\Big(\Delta^{\xi-a}\frac{\partial}{\partial\Delta}\Delta^a\frac{\partial}{\partial\Delta}f\Big)(0)\ =\ \Big(\Delta^a\frac{\partial}{\partial\Delta}f\Big)(0)$$

for μ the amount of "glue", $0 < \mu < \infty$. These conditions render M_a self-adjoint on the space $L^2(\mathbb{R}_+, (\Delta^{a-\xi} + \mu\delta(\Delta))\mathrm{d}\Delta)$. The particles stick together and separate immediately, never spending a finite interval of time together. Nevertheless, they spend a finite portion of the total time glued together. The persistent dissipation is still present and leads to a direct scalar-energy cascade of the forced scalar. It induces however an anomalous scaling of the 2-point scalar structure function $\big\langle(\theta(t,\boldsymbol{r})-\theta(t,\boldsymbol{0}))^2\big\rangle \propto r^{1-a}$ (instead of the dimensional result $\propto r^{2-\xi}$, see below). The sticky condition corresponds to fine-tuned small diffusivity and small viscosity [34, 46].

For $p_c^1 < \wp < p_c^2$, the transition probabilities $P(t_0, \boldsymbol{r}_0; t, \boldsymbol{r}|\boldsymbol{v})$ were constructed rigorously by Le Jan and Raimond[57, 58], for the reflecting boundary condition, the absorbing one and, in one dimension, also for the sticky ones. In the first case, they depend on $\boldsymbol{v}$ only, whereas in the second and third cases, on the (white noise) $\boldsymbol{v}$ and on the additional "black" (non-standard) noise[77] that decides what the particles do when they meet. Constructing the transition probabilities of the generalized flow for the sticky boundary conditions in more than one dimension is still an open problem.

4.3 Zero-mode mechanism for scalar intermittency

The statistics of fully turbulent velocities exhibits the phenomenon of intermittency, characterized by frequent appearance of strong signals interwoven with weak ones both in time and in space domain [42]. Similar, but even more pronounced effects characterize the statistics of advected scalars [3, 63]. One of the motivations behind studying scalar advection in Kraichnan velocities, whose Gaussian statistics does not exhibit any intermittency, was a search for a mechanism of this enhanced scalar intermittency [56]. We shall briefly describe such a mechanism discovered subsequently [28, 47, 74] and its interpretation in terms of the related flow of Lagrangian particles.

4.3.1 Passive scalar in Kraichnan velocities

For the scalar advected by smooth velocities,

$$\theta(t, \boldsymbol{R}(t; t_0, \boldsymbol{r}_0)) \;=\; \theta(t_0, \boldsymbol{r}_0) \;=\; \mathrm{const.}$$

In the Kraichnan velocity ensemble this means that the scalar field has to satisfy the Itô stochastic PDE

$$\mathrm{d}_t\theta(t,\boldsymbol{r}) \;=\; -\big(\boldsymbol{v}(t,\boldsymbol{r})\mathrm{d}t\big)\cdot\boldsymbol{\nabla}\theta(t,\boldsymbol{r}) \;+\; \tfrac{1}{2}D_0\,\boldsymbol{\nabla}^2\theta(t,\boldsymbol{r})\,\mathrm{d}t\,.$$

Indeed, the Itô rule gives then

$$\begin{aligned}\mathrm{d}_t\theta(t,\boldsymbol{R}(t)) \;=&-\,\big(\boldsymbol{v}(t,\boldsymbol{R}(t))\mathrm{d}t\big)\cdot\boldsymbol{\nabla}\theta(t,\boldsymbol{R}(t)) \;+\; \tfrac{1}{2}D_0\boldsymbol{\nabla}^2\theta(t,\boldsymbol{R}(t))\,\mathrm{d}t\\ &+\;\big(\boldsymbol{v}(t,\boldsymbol{R}(t))\mathrm{d}t\big)\cdot\boldsymbol{\nabla}\theta(t,\boldsymbol{R}(t)) \;+\; \tfrac{1}{2}D_0\boldsymbol{\nabla}^2\theta(t,\boldsymbol{R}(t))\,\mathrm{d}t\\ &-\,D_0\boldsymbol{\nabla}^2\theta(t,\boldsymbol{R}(t))\,\mathrm{d}t \;=\; 0\end{aligned}$$

(the last term in the intermediate equality came from the mixed increments of $\theta(t)$ and of $\boldsymbol{R}(t)$). The corresponding Stratonovich stochastic PDE would be

$$\mathrm{d}_t\theta(t,\boldsymbol{r}) \;=\; -\big(\boldsymbol{v}(t,\boldsymbol{r})\mathrm{d}t\big)\cdot\boldsymbol{\nabla}\theta(t,\boldsymbol{r})$$

(without the diffusion term). Similarly, for the scalar with a source, one obtains the Itô equation

$$\mathrm{d}_t\theta(t,\boldsymbol{r}) \;=\; -\big(\boldsymbol{v}(t,\boldsymbol{r})\mathrm{d}t\big)\cdot\boldsymbol{\nabla}\theta(t,\boldsymbol{r}) \;+\; \tfrac{1}{2}D_0\,\boldsymbol{\nabla}^2\theta(t,\boldsymbol{r})\,\mathrm{d}t \;+\; g(t,\boldsymbol{r})\,\mathrm{d}t \tag{4.5}$$

Let us suppose that $g(t,\boldsymbol{r})$ is also a Gaussian process, independent from velocities, with mean zero and covariance

$$\big\langle g(t,\boldsymbol{r})\,g(t',\boldsymbol{r}')\big\rangle \;=\; \delta(t-t')\,\chi(\boldsymbol{r}-\boldsymbol{r}')\,.$$

For the product $\prod_{n=1}^{N}\theta(t,\boldsymbol{r}_n)$, one obtains then from Eq. (4.5) the Itô SDE

$$\begin{aligned}\mathrm{d}_t\prod_{n=1}^{N}\theta(t,\boldsymbol{r}_n) \;\;=&\;\; -\sum_{n=1}^{N}\big(\boldsymbol{v}(t,\boldsymbol{r}_n)\mathrm{d}t\big)\cdot\boldsymbol{\nabla}\theta(t,\boldsymbol{r}_n)\prod_{n'\neq n}\theta(t,\boldsymbol{r}_{n'}) \;+\; \sum_{n=1}^{N}\tfrac{1}{2}D_0\boldsymbol{\nabla}^2\theta(t,\boldsymbol{r}_n)\prod_{n'\neq n}\theta(t,\boldsymbol{r}_{n'})\,\mathrm{d}t\\ &+\;\sum_{m<n}D^{ij}(\boldsymbol{r}_m-\boldsymbol{r}_n)\,\nabla_i\theta(t,\boldsymbol{r}_m)\nabla_j\theta(t,\boldsymbol{r}_n)\prod_{n'\neq n,m}\theta(t,\boldsymbol{r}_{n'})\,\mathrm{d}t\\ &+\;\sum_{n=1}^{N}\big(g(t,\boldsymbol{r}_n)\mathrm{d}t\big)\prod_{n'\neq n}\theta(t,\boldsymbol{r}_{n'}) \;+\; \sum_{n<n'}\chi(\boldsymbol{r}_n-\boldsymbol{r}'_{n'})\prod_{n''\neq n,n'}\theta(t,\boldsymbol{r}_{n'})\,\mathrm{d}t\,.\end{aligned}$$

For the expectation values, this gives the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle\prod_{n=1}^{N}\theta(t,\boldsymbol{r}_n)\Big\rangle \;=\; \mathcal{M}_N\,\Big\langle\prod_{n=1}^{N}\theta(t,\boldsymbol{r}_n)\Big\rangle \;+\; \sum_{n<n'}\chi(\boldsymbol{r}_n-\boldsymbol{r}'_{n'})\,\Big\langle\prod_{n''\neq n,n'}\theta(t,\boldsymbol{r}_{n'})\Big\rangle\,,$$

where the operator $\mathcal{M}_N$ is the same as in Eq. (4.2) for the evolution of expectations of functions of positions of N Lagrangian particles.

4.3.2 Zero modes and anomalous scaling

Under steady forcing, the scalar statistics reaches a stationary state in the weakly compressible phase with direct scalar energy cascade. In this state,

$$\mathcal{M}_N\,\Big\langle\prod_{n=1}^{N}\theta(t,\boldsymbol{r}_n)\Big\rangle \;=\; -\sum_{n,n'}\chi(\boldsymbol{r}_n-\boldsymbol{r}_{n'}))\,\Big\langle\prod_{n''\neq n,n'}\theta(t,\boldsymbol{r}_{n'})\Big\rangle\,.$$

The above equations determine the scalar correlations $\big\langle\prod\theta(t,\boldsymbol{r}_n)\big\rangle$ inductively in N, up to zero modes of the operators $\mathcal{M}_N$. It appears that if the source g is concentrated mostly at long distances then the short distance scaling of the scalar "structure functions" $S_N(\boldsymbol{r})=\big\langle(\theta(t,\boldsymbol{r})-\theta(t,\boldsymbol{0}))^N\big\rangle$,

$$S_N(\boldsymbol{r})\;\propto\;|\boldsymbol{r}|^{\zeta_N}\qquad\text{for small }|\boldsymbol{r}|\,, \tag{4.6}$$

is determined by the contribution of the scaling zero-modes f_N of $\mathcal{M}_N$:

$$\mathcal{M}_N f_N\;=\;0\,,\qquad f_N(\lambda\underline{\boldsymbol{R}})\;=\;\lambda^{\zeta_N}f_N(\underline{\boldsymbol{R}})\,.$$

Recall that the functions of the N Lagrangian trajectories evolve by Eq. (4.2). The zero-modes of $\mathcal{M}_N$ correspond to functions $f(\underline{\boldsymbol{R}}(t))$ of the N-particle process that are conserved in mean:

$$\big\langle f(\underline{\boldsymbol{R}}(t;\underline{\boldsymbol{r}}_0))\big\rangle\;=\;f(\underline{\boldsymbol{r}}_0)\,.$$

They are martingales of the effective multi-particle diffusions.

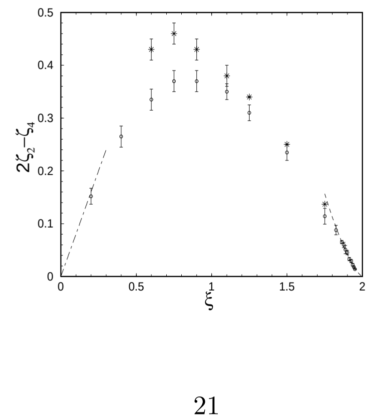
The scaling zero-modes of $\mathcal{M}_N$ giving the leading contributions to the structure function $S_N(\boldsymbol{r})$ have been found for homogeneous and isotropic flows in the leading order of the perturbation expansion in powers of ξ [47, 18] and of $\frac{1}{d}$ [28, 27] in the years 1995-6. The result was:

$$\zeta_N=\begin{cases}\frac{N}{2}(2-\xi)-\frac{N(N-2)(1+2\wp)\,\xi}{2(d+2)}+O(\xi^2)\\ \underbrace{\frac{N}{2}(2-\xi)}_{\substack{\text{dimensional}\\ \text{scaling}}}-\underbrace{\frac{N(N-2)(1+2\wp)\,\xi}{2d}+O(\frac{1}{d^2})}_{\text{anomaly}}\end{cases} \tag{4.7}$$

Terms to the 3$^{\rm rd}$ order are known [1]. The fact that $\zeta_N<\frac{N}{2}\zeta_2$ means that

$$\frac{S_N(\boldsymbol{r})}{S_2(\boldsymbol{r})^{N/2}}$$

becomes large for small $|\boldsymbol{r}|$. This signals the presence of long tails in the stationary distribution of the scalar increments $\big(\theta(t,\boldsymbol{r})-\theta(t,\boldsymbol{0})\big)$, getting more and more pronounced at short distances. This is a signal of small scale intermittency of the scalar statistics observed also in other turbulent states. The perturbative results (4.7) were confirmed by numerical simulations, see the figure from [43]



representing the results for $2\zeta_2 - \zeta_4$ as a function of ξ in two- (upper points) and three-dimensional (lower points) incompressible Kraichnan model, with the broken line in the lower left corner representing the perturbative result (4.7) for $d = 3$. Zero-modes were also shown numerically to be also responsible for the intermittency of scalar transport by the inverse cascade of 2D turbulence [25].

Conclusion. *In the non-smooth Kraichnan velocities modeling the inertial range turbulence, Lagrangian particles exhibit non-standard behaviors impossible in differentiable dynamical systems but expected to be common in non-differentiable ones with Hölder-continuous vector fields. These behaviors include explosive separation of trajectories generating their spontaneous randomness and an implosive collapse of deterministic trajectories, with possible stickiness. Such unusual behaviors of trajectories are source of non-equilibrium transport phenomena involving cascades with non-zero fluxes of conserved quantities. They condition the appearance of short-scale intermittency in the direct cascade of scalars advected by fully turbulent flows. The Kraichnan model allowed to associate the scalar intermittency to hidden statistical conservation laws of multi-particle evolution: the "zero modes". The zero-mode scenario permitted perturbative calculations of the anomalous scaling exponents of the scalar structure functions in this model. Similar mechanism is believed to be responsible for intermittency is other problems with dynamics governed by linear evolution equations [4].*

5 Final remarks

The passive transport in turbulent flows is related to the behavior of Lagrangian or inertial particles carried by the fluid. In the regime where velocities are smooth in space, i.e. for intermediate Reynolds numbers, the particle dynamics provides examples of random differential dynamical systems and may be studied with tools of the chaotic dynamical systems theory. It requires, nevertheless, subtle information about rare dynamical events (multiplicative large deviations). The statistics of rare fluctuations obeys fluctuation relations, sharing this property with other non-equilibrium systems [30].

In the regime of large Reynolds numbers, i.e. in the inertial range, the motion of particles may be viewed as providing an example of random non-differentiable dynamical system. As the work on the Kraichnan model has shown, such systems lead to unconventional flows with explosive trajectory separation or implosive trajectory trapping. The exotic behavior of particle trajectories has dramatic effect on the transport properties and conditions the appearance of cascades of conserved quantities and of intermittency related to hidden conservation laws.

Some of the lessons from studying passive turbulent transport remain valid for reactive particles like water droplets or chemical or biological agents [38, 76]. How many of them may be be transformed to insights about turbulence itself remains to be seen [2].

References

[1] L. Ts. Adzhemyan, N. V. Antonov, V. A. Barinov, Yu. S. Kabrits, A. N. Vasil'ev, "Calculation of the anomalous exponents in the rapid-change model of passive scalar advection to order ε^3", Phys. Rev. **E 64** (2001), 056306/1-28

[2] L. Angheluta, R. Benzi, L. Biferale, I. Procaccia, F. Toschi, "Anomalous scaling exponents in nonlinear models of turbulence", Phys. Rev. Lett. **97** (2006), 160601/1-4

[3] R. A. Antonia, E. Hopfinger, Y. Gagne, F. Anselmet, "Temperature structure functions in turbulent shear flows", Phys. Rev. **A 30** (1984), 2704-2707

[4] I. Arad, L. Biferale, A. Celani, I. Procaccia and M. Vergassola, "Statistical conservation laws in turbulent transport", Phys. Rev. Lett. **87** (2001), 164502/1-4

[5] L. Arnold, *Random Dynamical Systems*, Springer, Berlin 2003

[6] E. Balkovsky, G. Falkovich, A. Fouxon, "Clustering of inertial particles in turbulent flows", chao-dyn/9912027 and "Intermittent distribution of inertial particles in turbulent flows", Phys. Rev. Lett. **86** (2001), 2790-2793

[7] E. Balkovsky, A. Fouxon, "Universal long-time properties of Lagrangian statistics in the Batchelor regime and their application to the passive scalar problem", Phys.Rev. E, **60**, (1999) 4164-4174

[8] E. Balkovsky and A. Fouxon and V. Lebedev, "On the Turbulent Dynamics of Polymer Solutions", Phys. Rev. Lett. **84** (2000), 4765-4768

[9] M. M. Bandi, J. R. Cressman Jr., W. I. Goldburg, "Test of the Fluctuation Relation in compressible turbulence on a free surface", J. Stat. Phys. **130** (2008), 27-38

[10] G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press 1967

[11] P. H. Baxendale, D. W. Stroock, "Large deviations and stochastic flows of diffeomorphisms", Prob. Theor.& Rel. Fields **80** (1988), 169-215

[12] J. Bec, "Multifractal concentrations of inertial particles in smooth random flows", J. Fluid Mech. **528** (2005), 255-277

[13] J. Bec, L. Biferale, M. Cencini, A. Lanotte, S. Musacchio, F. Toschi, "Heavy particle concentration in turbulence at dissipative and inertial scales", Phys. Rev. Lett. **98** (2007), 084502/1-4

[14] J. Bec, M. Cencini, R. Hillerbrand, "Heavy particles in incompressible flows: the large Stokes number asymptotics", Physica **D 226** (2007), 11-22

[15] J. Bec, M. Cencini, R. Hillerbrand, "Clustering of heavy particles in random self-similar flow" Phys. Rev. **E 75** (2007), 025301(R)/1-4

[16] J. Bec, K. Gawędzki, P. Horvai, "Multifractal clustering in compressible flows", Phys. Rev. Lett. **92** (2004), 224501-2240504

[17] R. Benzi, B. Levant, I. Procaccia, E. S. Titi, "Statistical properties of nonlinear shell models of turbulence from linear advection models: rigorous results", arXiv:nlin.CD/0612033

[18] D. Bernard, K. Gawędzki, A. Kupiainen, "Anomalous scaling in the N-point functions of a passive scalar", Phys. Rev. **E 54** (1996), 2564-2572

[19] D. Bernard, K. Gawędzki, A. Kupiainen, "Slow modes in passive advection", J. Stat. Phys. **90** (1998), 519-569

[20] R. B. Bird, C. F. Curtiss, R. C. Armstrong, O. Hassager, *Dynamics of Polymeric Liquids, Vol. 2, Kinetic Theory*, Wiley, New York 1987

[21] G. Boffetta, J. Davoudi, B. Eckhardt, J. Schumacher, "Lagrangian tracers on a surface flow: the role of time correlations", Phys. Rev. Lett. **93** (2004), 134501/1-4

[22] T. Bohr, M. H. Jensen, G. Paladin, A. Vulpiani, *Dynamical Systems Approach to Turbulence*, Cambridge University Press 1998

[23] G. Boffetta, J. Davoudi, F. De Lillo, "Multifractal clustering of passive tracers on a surface flow", Europhys. Lett., **74** (2006), 62-68

[24] F. Bonetto, G. Gallavotti, G. Gentile, "A fluctuation theorem in a random environment", mp_arc/06-139

[25] A. Celani, M. Vergassola, "Statistical Geometry in Scalar Turbulence", Phys. Rev. Lett. **86** (2001), 424-427

[26] M. Chertkov, "Polymer Stretching by Turbulence", Phys. Rev. Lett. **84** (2000), 4761-4764

[27] M. Chertkov, G. Falkovich, "Anomalous scaling exponents of a white-advected passive scalar", Phys. Rev. Lett. **76** (1996), 2706-2709

[28] M. Chertkov, G. Falkovich, I. Kolokolov, V. Lebedev, "Normal and anomalous scaling of the fourth-order correlation function of a randomly advected scalar", Phys. Rev. **E 52** (1995), 4924-4941

[29] M. Chertkov, I. Kolokolov, M. Vergassola, "Inverse versus direct cascades in turbulent advection", Phys. Rev. Lett. **80** (1998), 512-515

[30] R. Chetrite, K. Gawędzki, "Fluctuation relations for diffusion processes", Commun. Math. Phys., in print, available online

[31] R. Chetrite, J.-Y. Delannoy, K. Gawędzki, "Kraichnan flow in a square: an example of integrable chaos", arXiv:nlin.CD/0606015, J. Stat. Phys. in press

[32] L. Chevillard, C. Meneveau, "Intermittency and universality in a Lagrangian model of velocity gradients in three-dimensional turbulence", C. R. Mechanique **335** (2007), 187-193

[33] W. E, E. Vanden Eijnden, "Generalized flows, intrinsic stochasticity, and turbulent transport", Proc. Nat. Acad. Sci. **97** (2000), 8200-8205

[34] W. E, E. Vanden Eijnden, "Turbulent Prandtl number effect on passive scalar advection", Physica D **152-153** (2001), 636-645

[35] J.-P. Eckmann, "Roads to turbulence in dissipative dynamical systems" Rev. Mod. Phys. **53** (1981), 643 - 654

[36] D. J. Evans, D. J. Searles, "Equilibrium microstates which generate the second law violating steady states" Phys. Rev. **E 50** (1994), 1645-1648

[37] Evans, D. J., Searles, D. J.: *The fluctuation theorem.* Adv. in Phys., **51** (2002), 1529-1585

[38] G. Falkovich, A. Fouxon, M. G. Stepanov, "Acceleration of rain initiation by cloud turbulence", Nature **419** (2002), 151-154

[39] G. Falkovich, K. Gawedzki, M. Vergassola, "Particles and fields in fluid turbulence", Rev. Mod. Phys. **73** (2001), 913-975

[40] I. Fouxon, P. Horvai, "Fluctuation relation and pairing rule for Lyapunov exponents of inertial particles in turbulence", J. Stat. Mech.: Theor. Experim. **8** (2007), L08002/1-9

[41] I. Fouxon and P. Horvai, "Separation of heavy particles in turbulence", Phys. Rev. Lett. **100** (2008), 040601/1-4

[42] U. Frisch: *Turbulence: the Legacy of A. N. Kolmogorov*, Cambridge University Press, Cambridge 1995

[43] U. Frisch, A. Mazzino, A. Noullez, M. Vergassola, "Lagrangian method for multiple correlations in passive scalar advection", Phys. Fluids **11** (1999), 2178-2186

[44] G. Gallavotti, E. D. G. Cohen, "Dynamical ensembles in non-equilibrium statistical mechanics", Phys. Rev. Lett. **74** (1995), 2694-2697

[45] K. Gawędzki, "Soluble models of turbulent transport", in: Non-equilibrium Statistical Mechanics and Turbulence, Eds. S. Nazarenko, O. V. Zaboronski, Cambridge University Press 2008

[46] K. Gawędzki, P. Horvai, "Sticky behavior of fluid particles in the compressible Kraichnan model", J. Stat. Phys. **116** (2004), 1247-1300

[47] K. Gawędzki, A. Kupiainen, "Anomalous scaling of the passive scalar", Phys. Rev. Lett. **75** (1995), 3834-3837

[48] K. Gawędzki, M. Vergassola, "Phase transition in the passive scalar advection", Physica **D 138** (2000), 63-90

[49] I. S. Gradshteyn, I. M. Ryzhik, Table of Integrals, Series and Products. 7^{th} Edition. Eds. Jeffrey, A., Zwillinger, D., Academic Press, New York 2007

[50] P. Grassberger, R. Baddi, A. Politi, "Scaling laws for invariant measures on hyperbolic and nonhyperbolic attractors", J. Stat. Phys. **51** (1988), 135-178

[51] B. I. Halperin, "Green's functions for a particle in a one-dimensional random potential", Phys. Rev. **139** (1965), A104-A117

[52] P. Horvai, "Lyapunov exponent for inertial particles in the 2D Kraichnan model as a problem of Anderson localization with complex valued potential", arXiv:nlin.CD/0511023

[53] Jarzynski, C.: *A nonequilibrium equality for free energy differences.* Phys. Rev. Lett. **78** (1997), 2690-2693

[54] A. N. Kolmogorov, "The local structure of turbulence in incompressible viscous fluid for very large Reynolds' numbers", C. R. Acad. Sci. URSS **30** (1941), 301-305

[55] R. H. Kraichnan, "Small-scale structure of a scalar field convected by turbulence", Phys. Fluids **11** (1968), 945-963

[56] R. H. Kraichnan, "Anomalous scaling of a randomly advected passive scalar", Phys. Rev. Lett. **72** (1994), 1016-1019

[57] Y. Le Jan, O. Raimond, "Integration of Brownian vector fields", Ann. Probab. **30** (2002), 826-873

[58] Y. Le Jan, O. Raimond, "Flows, coalescence and noise", Ann. Probab. **32** (2004), 1247-1315

[59] I. M. Lifshitz, S. Gredeskul, L. Pastur, *Introduction to the Theory of Disordered Systems*, Wiley N.Y. 1988

[60] A. J. Majda and P. R. Kramer, "Simplified models for turbulent diffusion: Theory, numerical modelling and physical phenomena", Physics Reports **314** (1999), 237-574

[61] M. Maxey, J. Riley, "Equation of motion of a small rigid sphere in a nonuniform flow", Phys. Fluids **26** (1983), 883-889

[62] B. Mehlig, M. Wilkinson, K. Duncan, T. Weber, M. Ljunggren, "On the aggregation of inertial particles in random flows" Phys. Rev. **E 72** (2005), 051104/1-10

[63] F. Moisy, H. Willaime, J. S. Andersen, P. Tabeling, "Passive Scalar Intermittency in Low Temperature Helium Flows", Phys. Rev. Lett. **86** (2001), 4827 - 4830

[64] B. Oksendal, *Stochastic Differential Equations*, 6th ed., Universitext, Springer, Berlin 2003

[65] M. A. Olshanetsky, A. M. Perelomov: "Quantum integrable systems related to Lie algebras" Phys. Rep. **94** (1983), 313-404

[66] V. I. Oseledec, "Multiplicative ergodic theorem: characteristic Lyapunov exponents of dynamical systems", Trudy Moskov. Mat. Obšč. **19** (1968), 179-210

[67] L. Piterbarg, "The top Lyapunov exponent for a stochastic flow modelling the upper ocean turbulence", SIAM J. Appl. Math. **62** (2001), 777-800

[68] L. F. Richardson, "Atmospheric diffusion shown on a distance-neighbour graph", Proc. R. Soc. Lond. **A 110** (1926), 709-737

[69] H. Risken, *The Fokker Planck equation*, Springer, Berlin, 1989

[70] D. Ruelle, "Ergodic theory of differentiable dynamical systems", Publications Mathématiques de l'IHÉS **50** (1979), 275-320

[71] D. Ruelle, "Positivity of entropy production in nonequilibrium statistical mechanics", J. Stat. Phys. **85** (1996), 1-23

[72] D. Ruelle, "Positivity of entropy production in the presence of a random thermostat", J. Stat. Phys. **86** (1997), 935-951

[73] B. L. Sawford, "Turbulent relative dispersion", Annual Rev. Fluid Mech. **33** (2001), 289-317

[74] B. I. Shraiman, E. D. Siggia, "Anomalous scaling of a passive scalar in turbulent flow", C.R. Acad. Sci.**321** (1995), 279-284

[75] B. I. Shraiman, E. D. Siggia, “Scalar turbulence”, Nature **405** (2000), 639-646

[76] T. Tél, A. de Moura, C. Grebogi, G. Károlyi, “Chemical and biological activity in open flows: A dynamical system approach”, Phys. Rep. **413** (2005), 91-196

[77] B. Tsirelson, “Nonclassical stochastic flows and continuous products”, Probab. Surveys **1** (2004), 173-298

[78] Z. Warhaft, “Passive scalars in turbulent flows”, Annual Rev. Fluid Mech. **32** (2000), 203-240

[79] S. Wiggins, J. M. Ottino, “Foundations of chaotic mixing”, Phil. Trans. R. Soc. Lond. A **362** (2004), 937-970

[80] M. Wilkinson, B. Mehlig, “Path coalescence transition and its applications”, Phys. Rev. **E 68** (2003), 040101/1-4