The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

**1952-13**

# School on Stochastic Geometry, the Stochastic Lowener Evolution, and Non-Equilibrium Growth Processes

*7 - 18 July 2008*

## Background information on Conformal invariance in the 2D Ising model (further notes)

Stanislav SMIRNOV

*Universite' de Geneve*
*Section de Mathematiques*
*CH-1211 Geneve*
*Switzerland*

We saw how to show using discrete holomorphicity that in the FK Ising del at criticality, some

observables have conformally invariant scaling limits. $\star$

In the last two lectures (largely independent of the first three) we will discuss how to use $\star$ to construct scaling limits.

Possible objects

- single interface $\leftarrow$ SLE
- full interface collection
- height functions, fields

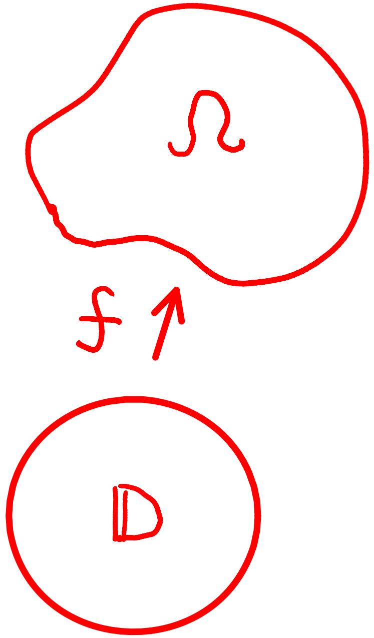
# Löwner evolution

- a tool to study variation of domains & maps in $\mathbb{C}$.
- introduced to attack Bieberbach's conjecture

**K. Löwner, Untersuchungen über schlichte konfornie Abbildung des Einheitskreise, I.** ***Math. Ann.*** **89, 103-121 (1923).**

- was instrumental in its proof

**L. de Branges, A proof of the Bieberbach conjecture.** ***Acta. Math.*** **154, 137-152 (1985).**




## Bieberbach's conjecture

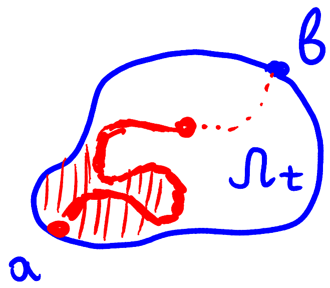
## de Branges' theorem!!!!!

$f : \mathbb{D} \to \Omega$ a conformal map

$f(z) = \sum a_n z^n$.

Then $|a_n| \leq n|a_1|$, attained for $\Omega = \mathbb{C} \setminus \mathbb{R}_-$

Deform domain by growing a slit from $a \in \partial\Omega$ to $b \in \Omega$ (or $b \in \partial\Omega$)

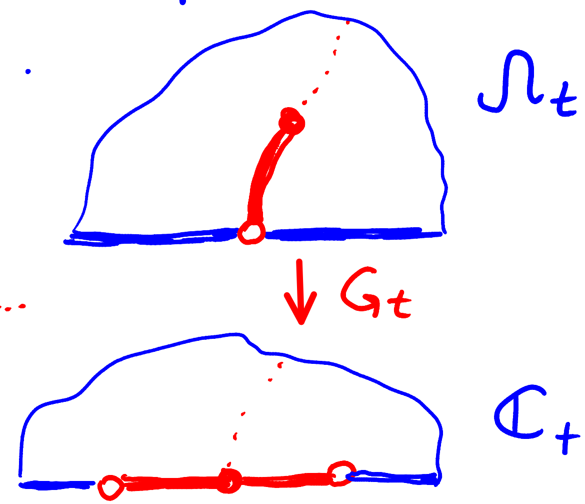



Map $\Omega$ to $\mathbb{C}_+$, so that $a \mapsto 0$, $b \mapsto \infty$.

Parametrize slit $\gamma$ by time $t$.

Set $\Omega_t = \mathbb{C}_+ \setminus \gamma[0,t]$, component at $\infty$

$G_t : \Omega_t \to \mathbb{C}_+$ a conformal map with $\infty \mapsto \infty$, $G_t'(\infty) = 1$, $\gamma(t) \to 0$.



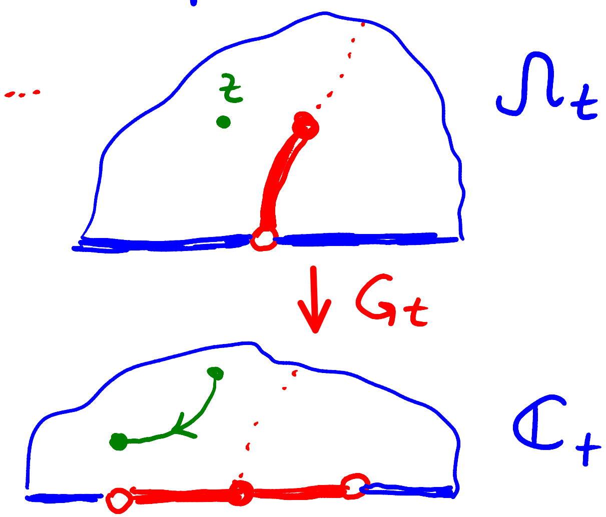

Expand at $\infty$:

$$G_t(z) = z + a_0(t) + \frac{a_{-1}(t)}{z} + \frac{a_{-2}(t)}{z^2} + \dots$$

Note: $G_t : \mathbb{R} \circlearrowleft \Rightarrow a_k \in \mathbb{R}$

$G_t : \Omega_t \to \mathbb{C}_+$ a conformal map

$$G_t(z) = z + a_0(t) + \frac{a_{-1}(t)}{z} + \frac{a_{-2}(t)}{z^2} + \dots$$




Note: $a_{-1}(t) = \mathrm{cap}_{\mathbb{C}_+}(\gamma[0,t])$

$\Rightarrow$ continuously increases $\Rightarrow$

can change time $a_{-1}(t) = 2t$

Denote $w(t) := -a_0(t)$.

Löwner equation

$$\boxed{\partial_t \left( G_t(z) + w(t) \right) = \frac{2}{G_t(z)}}$$

B.C. $G_0(z) = z$, $G_t(z) = z - w(t) + \frac{2t}{z} + \dots$ at $\infty$

gives a bijection $\{\text{nice slits } \gamma\} \leftrightarrow \{\text{continuous } w\}$

- ODE for $G_t(z)$ involves $w(t)$ only!
- $\partial_t w(t)$ = "the turning speed"

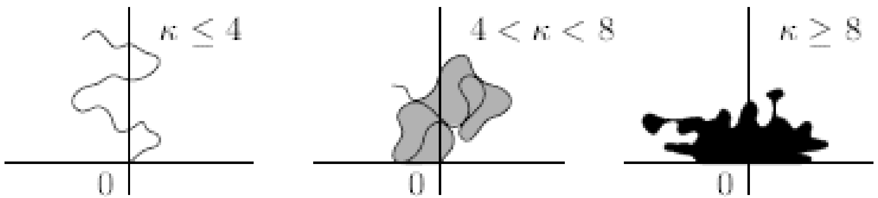
# Schramm-Löwner Evolution

deterministic $w$ $\leftrightarrow$ deterministic $\gamma$

random $W$ $\leftrightarrow$ random $\gamma$ ($\to \mu \in \text{Prob}\{\text{curves}\}$)

**SLE($\kappa$)** is LE with $w(t) = \sqrt{\kappa}\, B_t$, $\kappa \in \mathbb{R}_+$

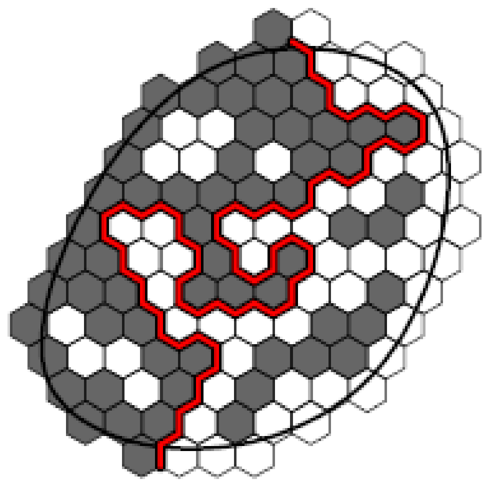
- $\gamma$ a.s. a simple curve $0 \le \kappa \le 4$ [Rohde-Schramm]
  a self-touching curve $4 < \kappa < 8$
  a random Peano curve $8 \le \kappa$
- $\text{HDim}(\gamma) = \min(1 + \frac{\kappa}{8}, 2)$ a.s. [Beffara]
- $\partial(SLE(\kappa)) = SLE(\frac{16}{\kappa})$, $\kappa > 4$ [Zhan], [Dubedat]




SLE computations = Itô calculus

# Schramm's principle

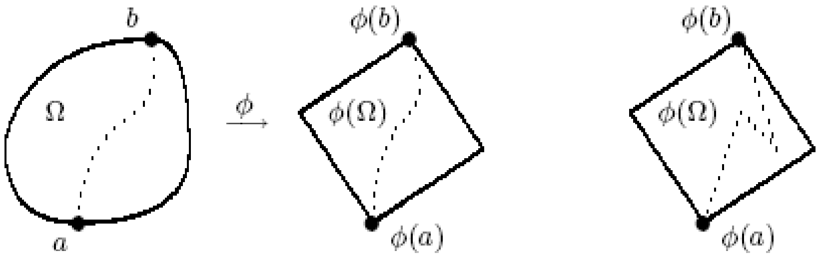
Assume that an interface has a conformally invariant scaling limit.
Then it is SLE($\kappa$) for some $\kappa$.



## Conformal invariance

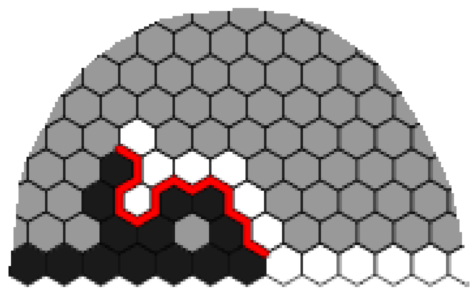
same law

$\mu(\Omega,a,b) \quad \overset{\phi}{\longmapsto} \quad \phi\left(\mu(\Omega,a,b)\right) \qquad \mu\left(\phi(\Omega),\phi(a),\phi(b)\right)$




← holds in the limit

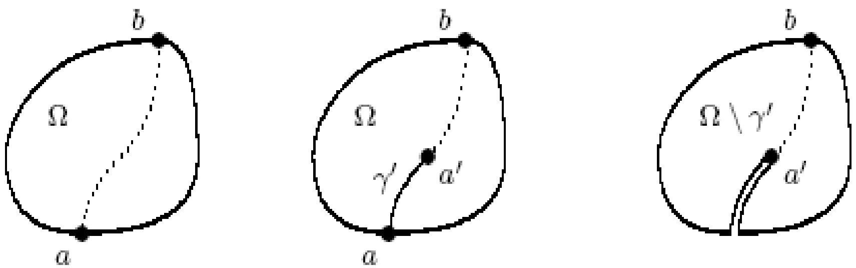
## Domain Markov



same law

condition

$\mu(\Omega,a,b) \quad \longmapsto \quad \mu(\Omega,a,b)\,|\,\gamma' \qquad \mu\left(\Omega\setminus\gamma',a',b\right)$



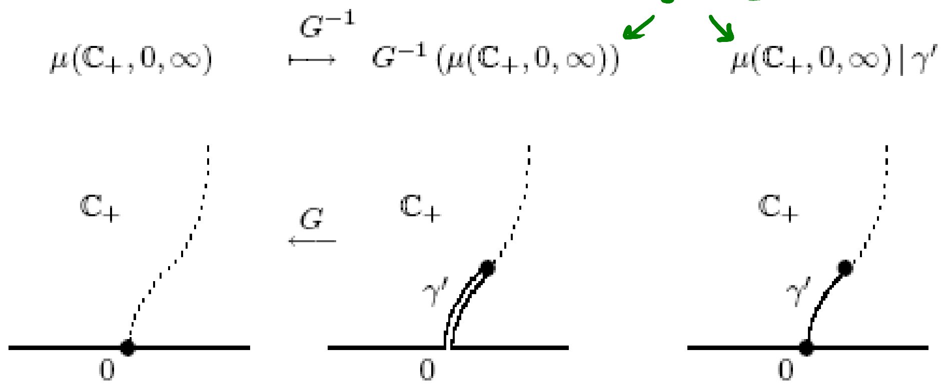

← holds on the lattice (nearest neighbour interaction)

Conformal invariance + domain Markov $\Rightarrow$

**Conformal Markov Property**

$$\boxed{\begin{aligned} &G_{t+s} \mid G_t \\ &= G_t(G_s) \end{aligned}}$$

same law



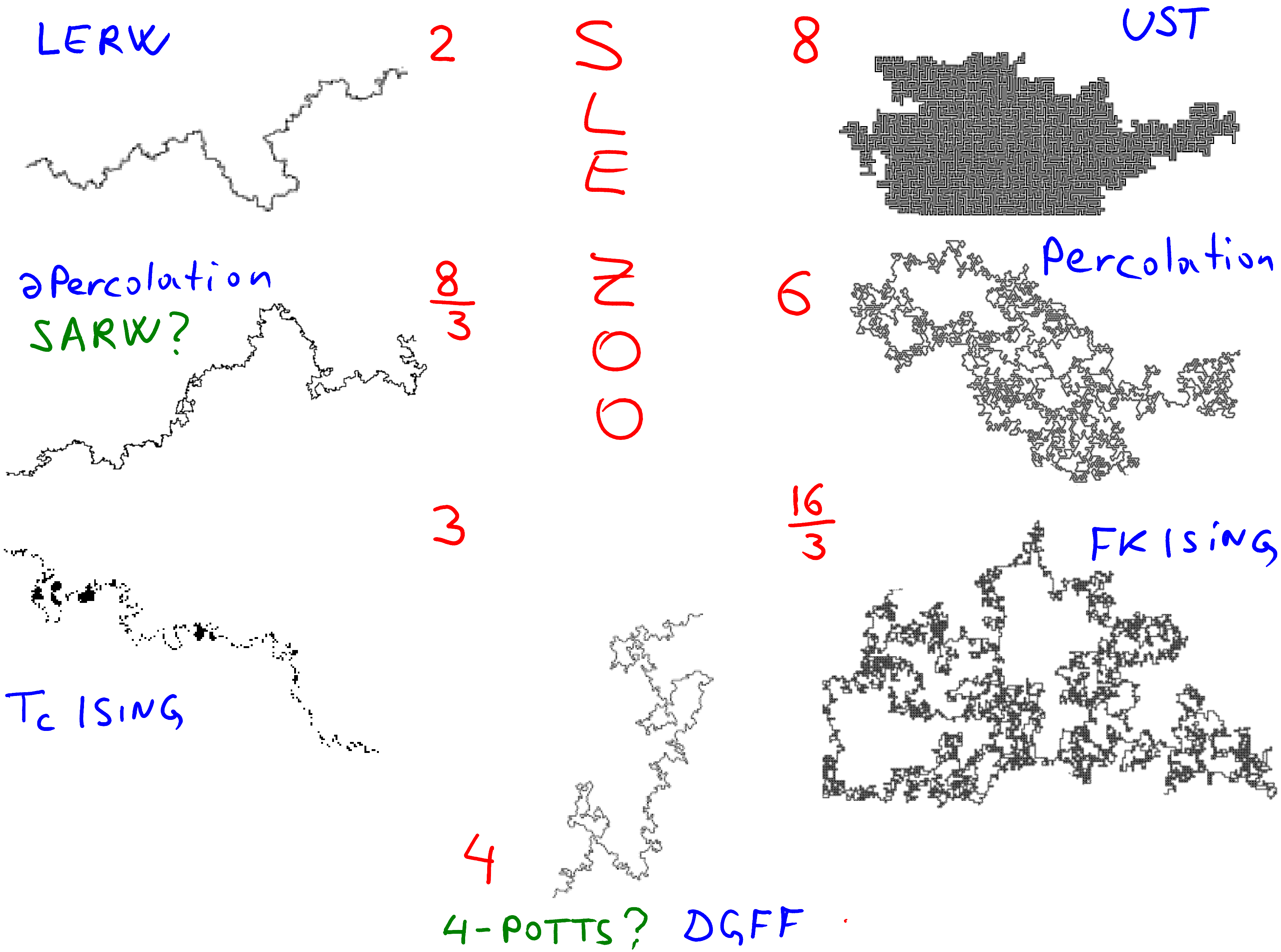

Expanding at $\infty$: $z - w(t+s) + \ldots \mid G_t =$

$$= (z - w(t) + \ldots) \circ (z - w(s) + \ldots) = z - (w(t) + w(s)) + \ldots$$

So $\left.\begin{array}{l} w(t+s) - w(t) \mid G_t = w(s) \\ w(t) \text{ a.s. continuous} \end{array}\right\} \Rightarrow$ $w(t) = \sqrt{\varkappa}\, B_t + \alpha t$

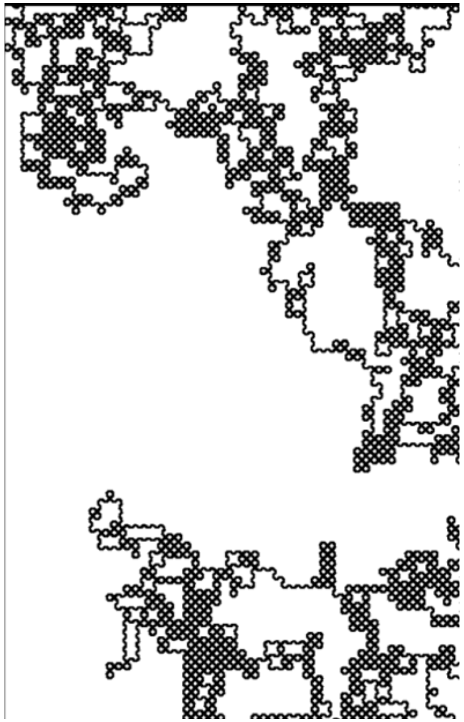
$\varkappa \in \mathbb{R}_+ ,\ \alpha \in \mathbb{R}$

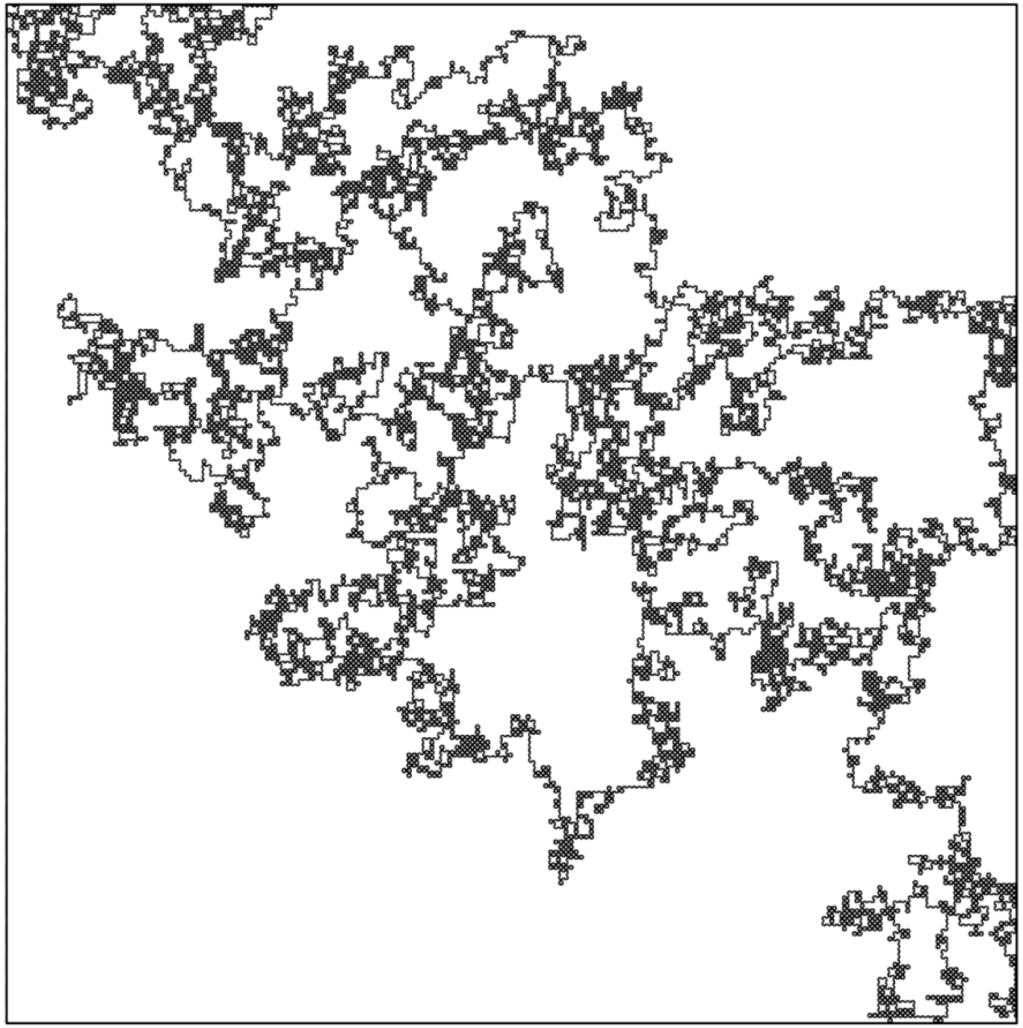
- $\alpha = 0$ by symmetry or scaling
- The principle is very nice, but hard to apply: need to establish conformal invariance first, then evaluate some observable to find $\varkappa$...


SLE ZOO
LERW
2
8
UST
∂Percolation
SARW?
8/3
6
Percolation
3
16/3
FK ISING
Tc ISING
4
4-POTTS? DGFF

**Theorem** if a sequence of discrete domains converges: $\Omega^\varepsilon, a^\varepsilon, b^\varepsilon \xrightarrow{\text{Cara}} \Omega, a, b$ then interface in the critical FK Ising converges to $SLE(16/3)$

(weak * convergence of Borel measures on Hölder curves)





**Martingale principle**: if some "martingale" observable $F$ has a conformally covariant limit, then interface converges to $SLE(\varkappa_F)$

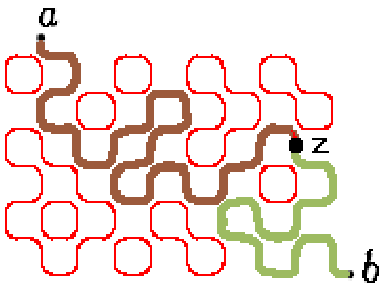
**Martingale property** of $F$:

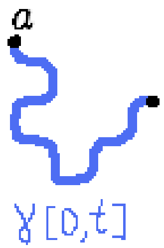
$$F(z,\Omega^\epsilon,a,b) = \mathbb{E}_{\gamma^\epsilon[0,t]} F(z,\Omega^\epsilon\setminus\gamma^\epsilon[0,t],\gamma^\epsilon(t),b)$$

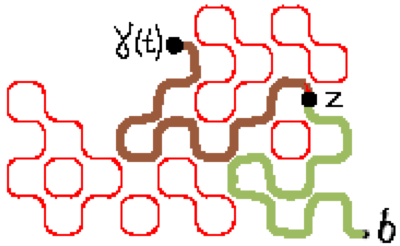
$$\| \qquad\qquad\qquad \|\,(\star)$$

$$\frac{1}{\sqrt{2\epsilon}}\,\mathbb{E}\,X_{z\in\gamma}\cdot W(b\to z) = \frac{1}{\sqrt{2\epsilon}}\,\mathbb{E}_{\gamma[0,t]}\,\mathbb{E}\left(X_{z\in\gamma}\,W(b\to z)\,\middle|\,\gamma[0,t]\right)$$

Note: $\gamma$ does not distinguish $\partial\Omega$ from its past $\Rightarrow$

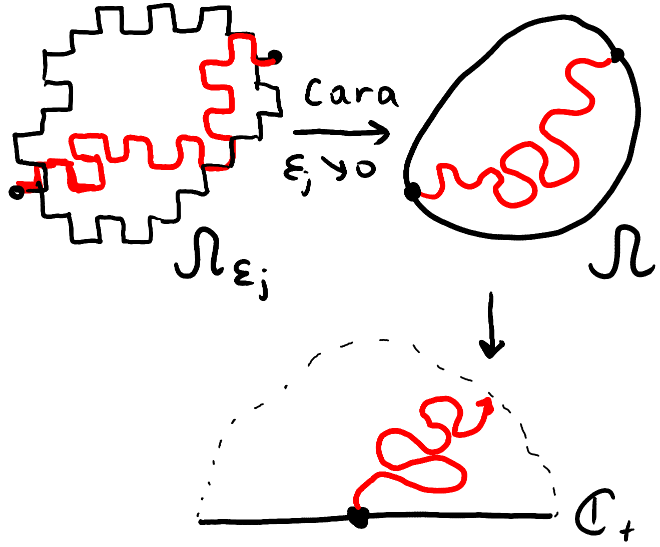
$\mathbb{E}\Bigg($ 

$\Bigg|$ 

$\Bigg) \overset{(\star)}{=} \mathbb{E}\Bigg($ 

$\Bigg)$

Take our random curve $\gamma^\varepsilon$ in $\Omega^\varepsilon \approx \Omega$.




1. Choose converging subsequence
   $\gamma^{\varepsilon_i} \to \gamma'$ (i.e. $\mu^{\varepsilon_j} \xrightarrow{w*} \mu$)
2. Map $\Omega$ to $\mathbb{C}^+$, get random $\gamma$
3. Describe $\gamma$ by LE with
   random driving force $w(t)$
4. Use martingale to determine $w(t)$

Assume for now

(A) $\{\mu^\varepsilon\}$ is precompact (for 1)

(B) $\gamma$ a.s. is a Löwner slit (for 3)

(C) $w(t)$ is a.s. continuous with bounded moments (for 4)

# Martingale property of F:

$$F(z,\Omega^\epsilon,a,b) = \mathbb{E}_{\gamma^\epsilon[0,t]} F(z,\Omega^\epsilon\setminus\gamma^\epsilon[0,t],\gamma^\epsilon(t),b)$$

$$\downarrow \epsilon_j \searrow 0 \qquad\qquad\qquad \downarrow \epsilon_j \searrow 0$$

$$\sqrt{\Phi'(z,\Omega)} \qquad\qquad \mathbb{E}_{\gamma[0,t]} \sqrt{\Phi'(z,\Omega\setminus\gamma[0,t])}$$

Passing to a limit is legal: $\Omega^{\epsilon_j} \xrightarrow{\text{Cara}} \Omega$ and $\Omega \mapsto \Phi(\cdot,\Omega)$ is continuous in Carathéodory sense.

Observe that $\Phi(z,\mathbb{C}_+) = \log z \Rightarrow \Phi' = 1/z$

$\Phi(z,\mathbb{C}_+\setminus\gamma[0,t]) = \log G_t(z) \Rightarrow \Phi' = G_t'(z)/G_t(z)$

Conclusion:

$$\sqrt{1/z} = \mathbb{E}_{G_t} \sqrt{G_t'(z)/G_t(z)}$$

Problems: no info about $w(t)$ yet (Markov? ...)

$G_t$ solves non-linear ODE

Rewrite: $(1/z)^{\sigma} = \mathbb{E}_{G_t}\left(G_t'(z)/G_t(z)\right)^{\sigma}$, $\sigma = \frac{1}{2}$ a parameter

Recall $G_t(z) = z - w(t) + \frac{2t}{z} + \ldots$ $G_t'(z) = 1 - \frac{2t}{z^2} + \ldots$

Plug in, expand

Note: absolute bounds for "…" moments of $w(t)$.

$$\frac{1}{z^{\sigma}} = \mathbb{E}_{G_t}\left(\frac{1 - 2t/z^2 + \ldots}{z - w(t) + 2t/z + \ldots}\right)^{\sigma}$$

$$= \mathbb{E}_{G_t} \frac{1}{z^{\sigma}}\left(1 + \frac{\sigma}{z} w(t) + \frac{1}{z^2}\frac{\sigma(\sigma+1)}{2}\left(w(t)^2 - \frac{8t}{\sigma+1}\right) + \ldots\right)$$

Equate coefficients: $0 = \mathbb{E}_{G_t} w(t)$, $0 = \mathbb{E}_{G_t} w(t)^2 - \frac{8t}{\sigma+1}$ $\circledast$

Do the same for domains $\mathbb{C}_+ \setminus \gamma[0,s]$, $\mathbb{C}_+ \setminus \gamma[0,t]$,

get $\circledast$ for increments, so $w(t)$ is a.s. continuous,

$w(t)$ and $w(t)^2 - \frac{8t}{\sigma+1}$ are martingales

$\xRightarrow{\text{Lèvy}}$ $w(t) = \sqrt{\frac{8}{\sigma+1}}\, B_t$ for FK Ising $\sqrt{\frac{16}{3}}\, B_t$ $\blacksquare$

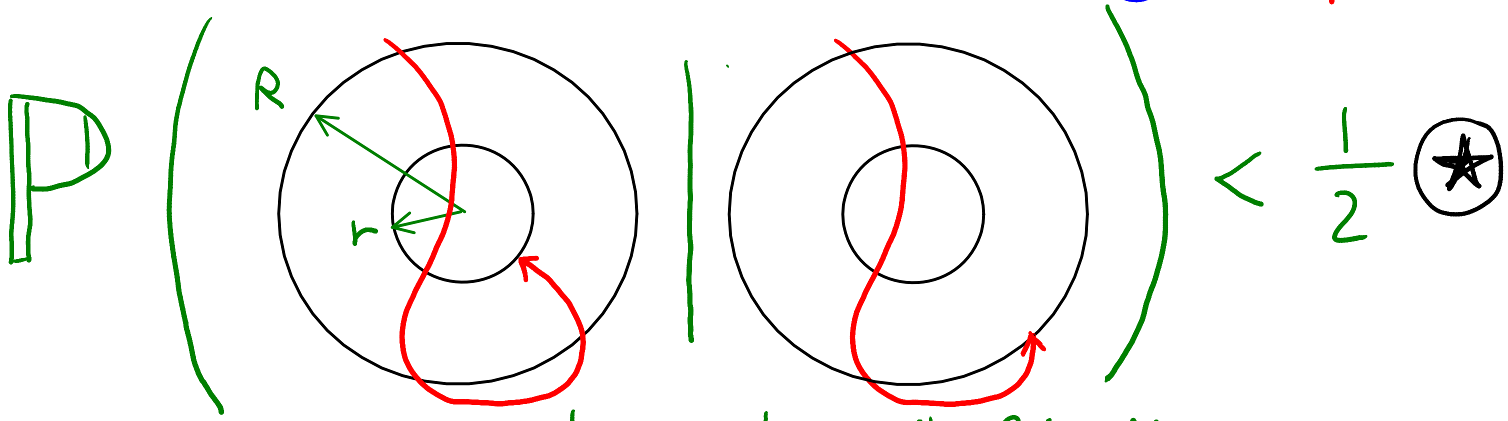
We want to use martingale observable to deduce (A)(B)(C)

(A) $\{\mu^\varepsilon\}$ is precompact

(B) $\gamma$ a.s. is a Löwner slit

(C) $w(t)$ is a.s. continuous with bounded moments

**Proposition** Let $\{\gamma^\varepsilon(t)\}_\varepsilon$ be a family of random curves with no transversal self intersections. The following estimate implies (A)(B)(C): for $\frac{R}{r} > M$




any fixed annulus with $R/r > M$