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School on Stochastic Geometry, the Stochastic Lowener Evolution, and Non-Equilibrium Growth Processes

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Background information on Conformal invariance in the 2D Ising model (further notes)

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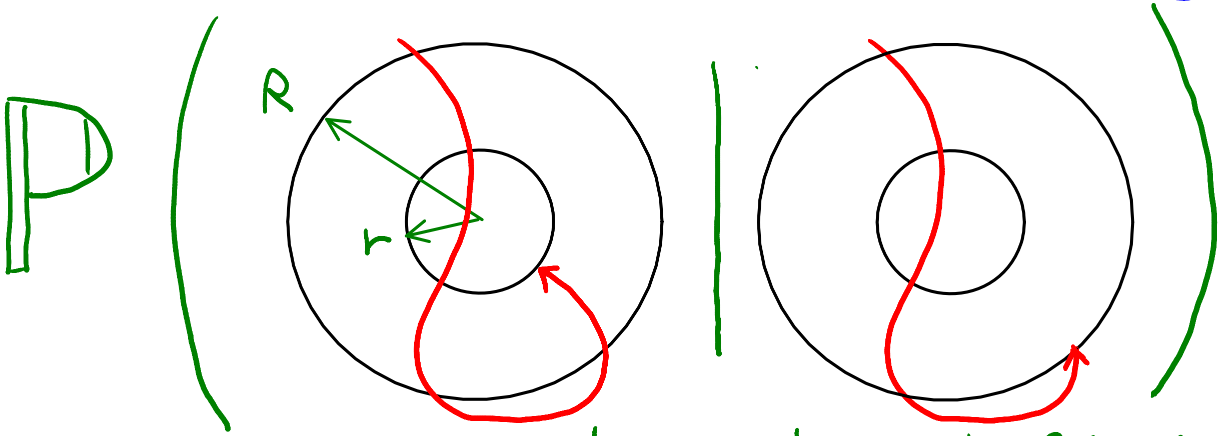
We want to use martingale observable to deduce A, B, C

A $\{\mu^\varepsilon\}$ is precompact
B γ a.s. is a Löwner slit
C $w(t)$ is a.s. continuous with bounded moments

Proposition Let $\{\gamma^\varepsilon(t)\}_\varepsilon$ be a family of random curves with no transversal self intersections. The following estimate implies A B C: for $\frac{R}{r} > M$

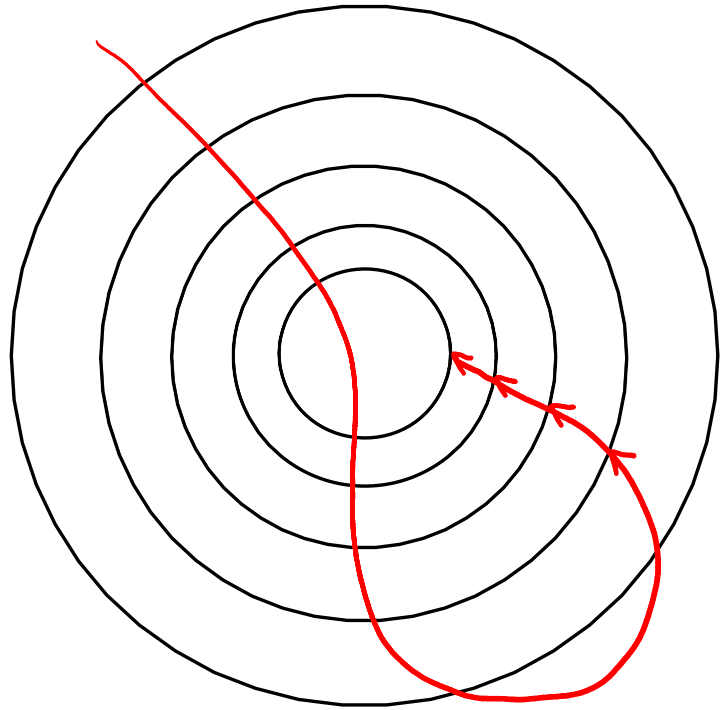


$< \frac{1}{2}$ $(\star)$

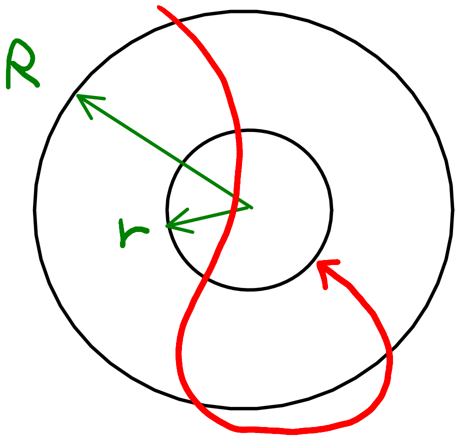
any fixed annulus with $R/r > M$

(A) Assume that (*) holds if $R/r > K$.

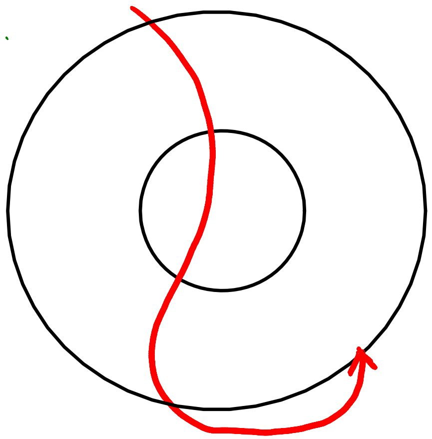
Splitting an annulus of modulus R/r into $\log_K R/r$ annuli of modulus K we obtain:



$\mathbb{P}\Big($



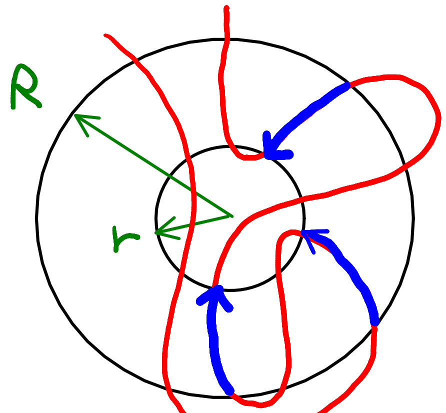
$\Big|$



$\Big) \lesssim \left(\frac{r}{R}\right)^{\delta}$

Hence

$$\mathbb{P}\Big($$



$$\Big) \lesssim \left(\frac{r}{R}\right)^{\lambda_k}, \quad \lambda_k = \frac{k-2}{2}\,\delta$$

Draw γ, prove by induction.

Note: $\lambda_k \nearrow \infty$ as $k \to \infty$.

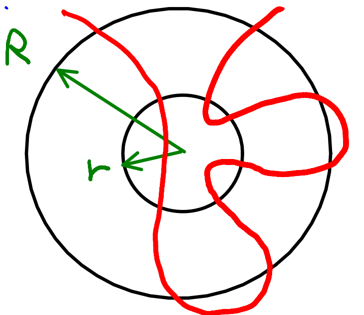
Thm [Aizenman-Burchard] if $\lambda_k > 2$ for some k, then γ has stochastically bounded α-Hölder norm for $\alpha < \frac{1}{2} - \frac{1}{\lambda_k}$.

Hence $\{\mu_\varepsilon\}$ is precompact in $M(\text{Höl}(\alpha))$ with $w*$ topology (Prokhorov's theorem) $\Rightarrow$ Ⓐ

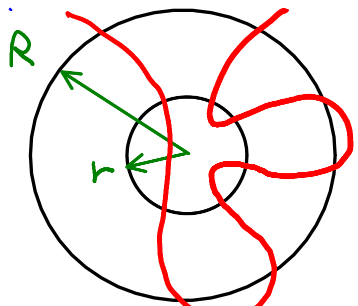
$\uparrow$ Borel measures.

Aizenman-Burchard: if $\lambda_\kappa > 2$

$\mathbb{P}\Bigg($



some annulus $\Bigg) \lesssim \mathbb{P}\Bigg($



fixed annulus $\Bigg) \cdot \#\left\{\begin{array}{c}\text{distinct}\\ r\text{-}R\\ \text{annuli}\end{array}\right\}$

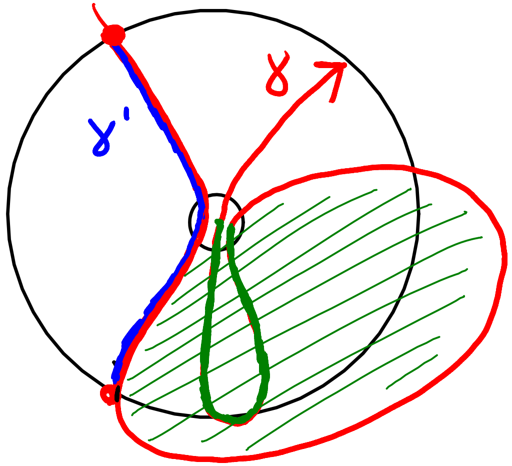
$$\underbrace{\left(\frac{r}{R}\right)^{\lambda_\kappa} \qquad\qquad \left(\frac{1}{r}\right)^2}_{r^{\lambda_\kappa - 2}\,\cdot\,\mathrm{Const}(R,\Omega)}$$

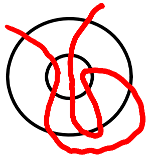
So, probability to see κ crossings somewhere tends to zero as $r \to 0$,

In particular, γ will have stochastically bounded number of disjoint pieces of diameter R

Hölder continuity follows.

(B) Obstruction to being a Löwner slit:
once green region is surrounded, its interior is invisible for conformal observer. Must show "$\lambda_6 > 2$" "No 6 arms"

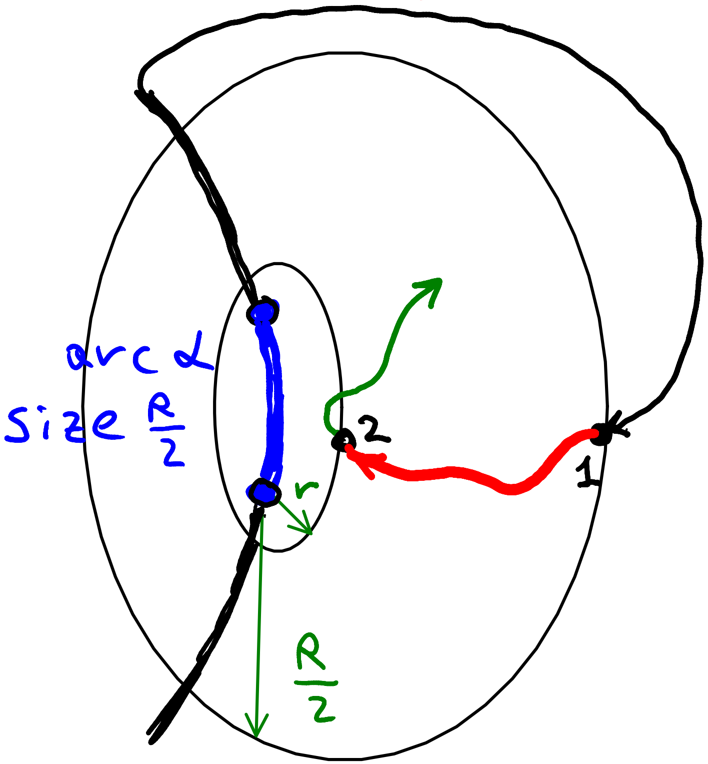


e.g. $\mathbb{P}\Big($  $\Big) \lesssim \left(\frac{r}{R}\right)^{2+\Delta}$, so never happens.

Also must exclude similar pictures on $\partial\Omega$
(but in our method no difference)

Fix R. $\circledast \Rightarrow$ Aizenman-Burchard $\Rightarrow$ curve γ can be cut into stochastically bounded number of pieces of diameter $\frac{R}{2}$. $\Rightarrow$ can assume γ' is one of those (i.e. fixed)

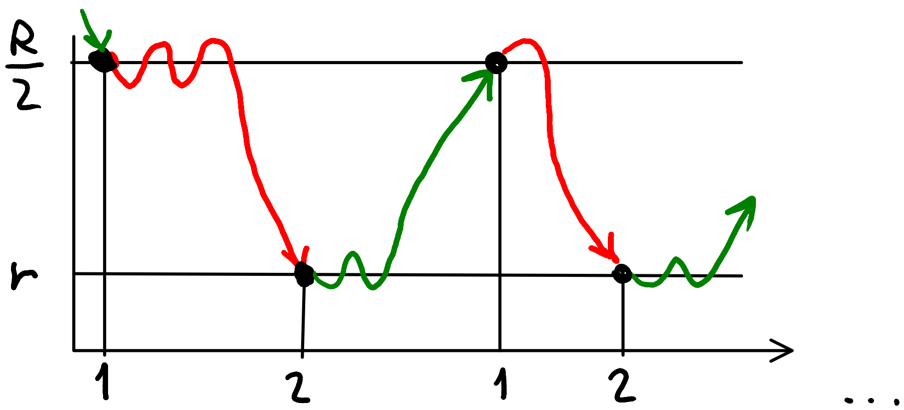
Excluding 6 arms against a fixed curve I



To create setup for 6 arms (in red) we need to

① go $\frac{R}{2}$-away from α

② come r-close to α

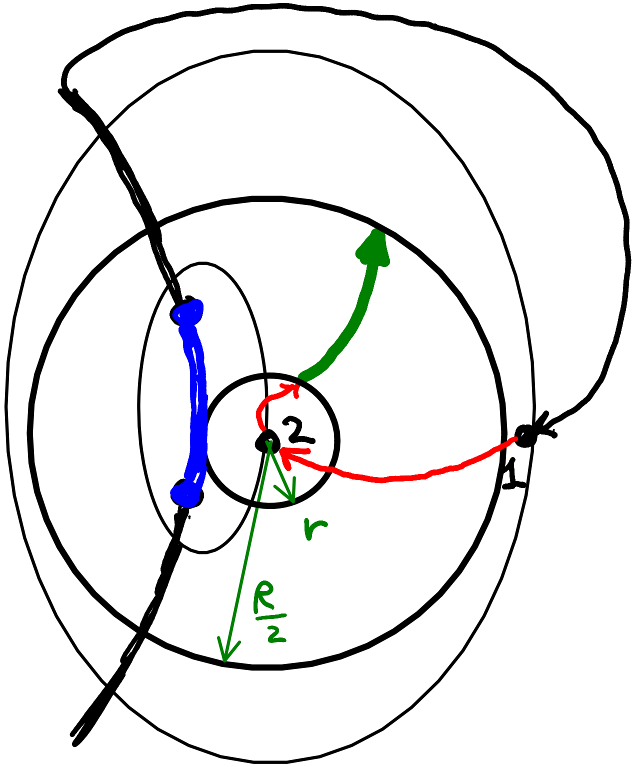


Fix R, pick $r \ll R$ later

Number of setups is stochastically bounded by [Aizenman - Burchard]

they contain disjoint red arcs of size $\approx \frac{R}{2}$

Excluding 6 arms against a fixed curve II



Number of arcs α and points 2 is stochastically bounded in terms of R.

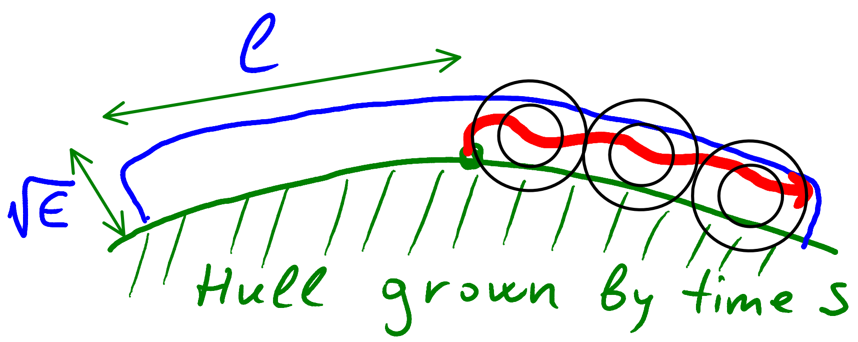
So it is enough to show that going $\frac{R}{2}$-away from point 2 into a fjord has small probability.

But the latter requires crossing an r - $\frac{R}{2}$ annulus.

so $\mathbb{P} \lesssim \left(\frac{2r}{R}\right)^{\delta} \xrightarrow[r\to 0]{} 0 \qquad \Longrightarrow$ B

© $w(t)$ is continuous,
increments have bounded moments.

Enemy: γ which follows closely $\partial\Omega$ (or its own past)



Fix ϵ, ℓ, take an ϵ-neighborhood of diameter ℓ piece of $\partial\Omega_s$

Let t_0 be the exit time, $t_0 - t \lesssim \epsilon$

$|w(t_0) - w(t)| > \ell$ iff we exit through the side.

For this we traverse $\approx \frac{\ell}{\sqrt{\epsilon}}$ annuli of const moduli

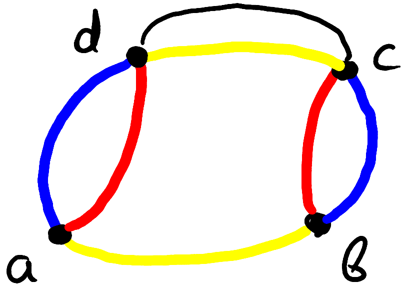
So $\mathbb{P}\left(\sup_{[t_0, t_0+\epsilon]} |w(t) - w(t_0)| > \ell\right) \lesssim \exp\left(-C\frac{\ell}{\sqrt{\epsilon}}\right)$

$\Rightarrow$ ©

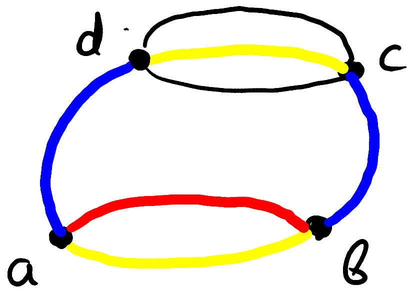
4-point function

Domain Ω with 4 sources

Join 2 by an imaginary edge, 2 possibile configurations.



$\mathbb{E}\, X_{z \in \gamma_{ad} \text{ or } \gamma_{bc}}\, W$

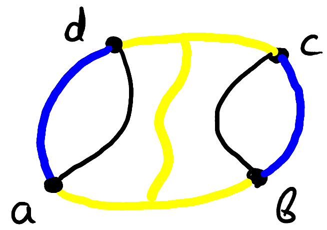


$\mathbb{E}\, \sqrt{2}\, X_{z \in \gamma_{ab}}\, W$

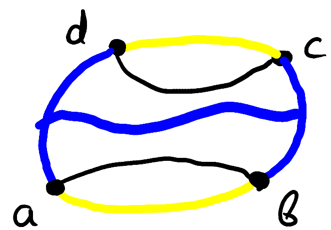
Contributions to F_{abcd} given in green.

$F_{abcd} \in Hol$, $H_{abcd} = \mathrm{Im} \int F^2_{abcd} \in Harm$

H_{abcd} has jumps at a, b, c, d which can be written in terms of crossing probabilities.



p_1

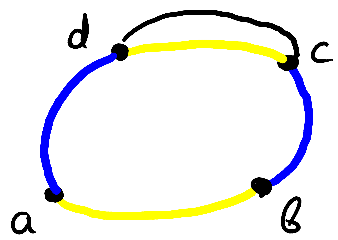


p_2

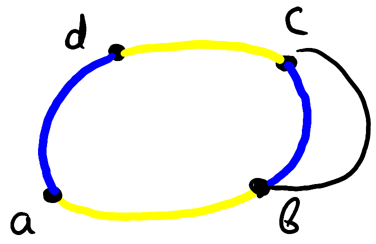
$p_1 + p_2 = 1$

The same unknown p_1 appears in a similar function F_{dabc}, and can be

F_{abcd}



F_{dabc}



eliminated by considering $F := p_1 F_{abcd} + p_2 F_{dabc}$.

One obtains F and uses it to obtain p_1, F_{abcd}, F_{dabc}.

Theorem If $\Omega^\epsilon, a^\epsilon, b^\epsilon, c^\epsilon, d^\epsilon \xrightarrow[\epsilon \searrow 0]{\text{Cara}} \Omega, a, b, c, d$,

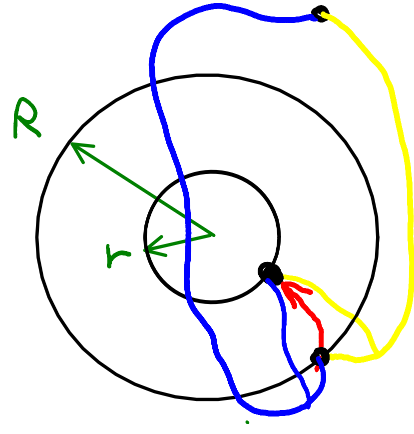
then $$p_1 \to \frac{\sqrt{1+u}+1-\sqrt{2}-\sqrt{u}}{(2-\sqrt{2})(1-\sqrt{u})}$$

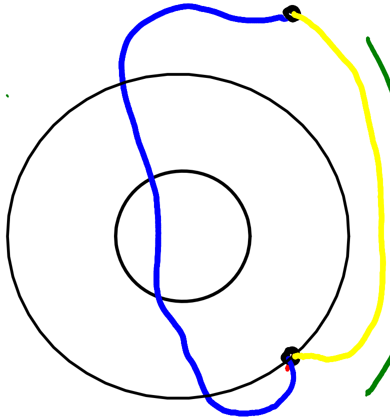
$$F_{abcd} \underset{\mathrm{Int}\,\Omega}{\rightrightarrows} \sqrt{\Phi'_{abcd}}\,,$$ Φ_{abcd} a 2-to-1 slitmap.

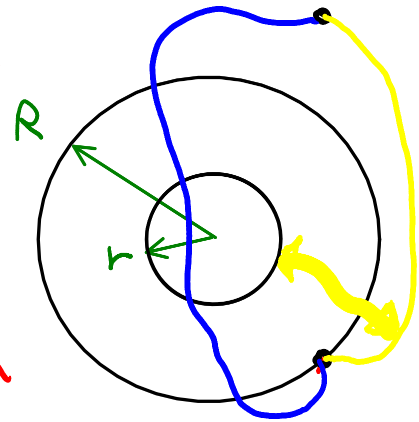
So crossing probabilities have conformally invariant limits, and we calculated their values.

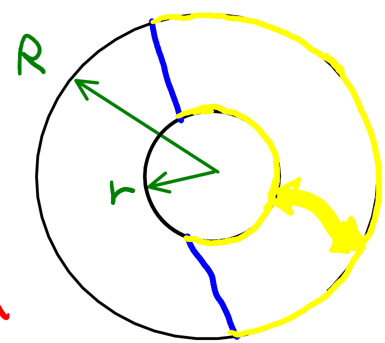
CROSSING PROBABILITIES $\Rightarrow$ ★

Recall that interface separates cluster from dual cluster

$\mathbb{P}\Big($ 

$\Big|$  $\Big) \le$

$\le \mathbb{P}\Big($ 

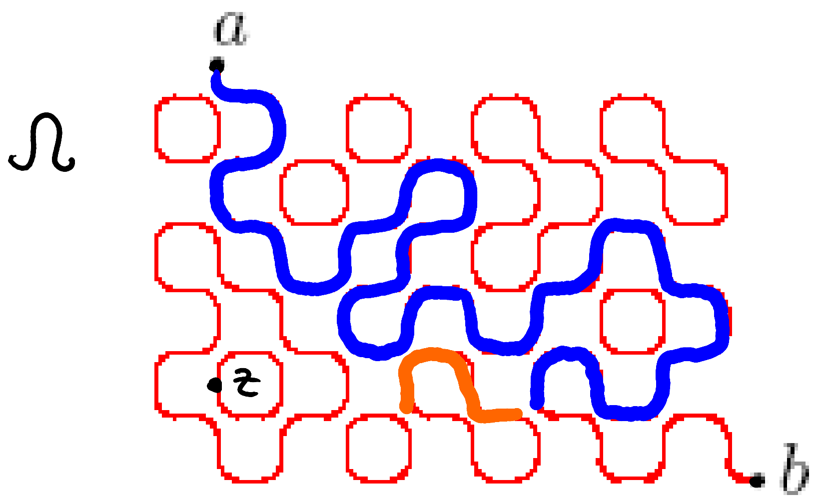
$\Big) \le \mathbb{P}\Big($ 

$\Big)$

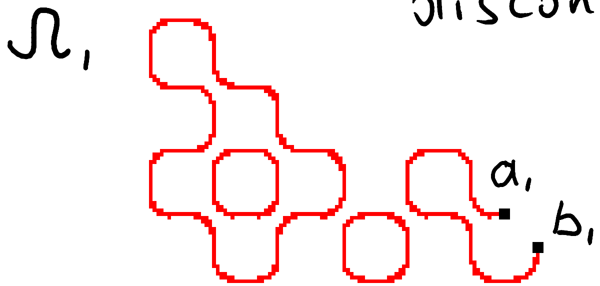
Here u is the modulus of a 2π-sector between r, R.

$$\le p_1(u) \xrightarrow[\frac{r}{R} \to 0]{} 0$$

Interface tree for FK model.



After the first disconnection



Fix $z \in \Omega$

Follow $\gamma : a \to b$ until hit z or disconnect it at time t_0

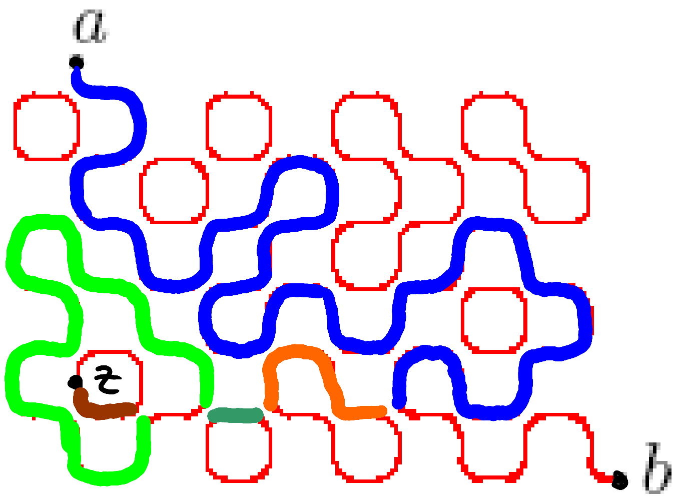
Set $\Omega_1 := \Omega \setminus \gamma[0, t_0]$ (connectivity component at z)

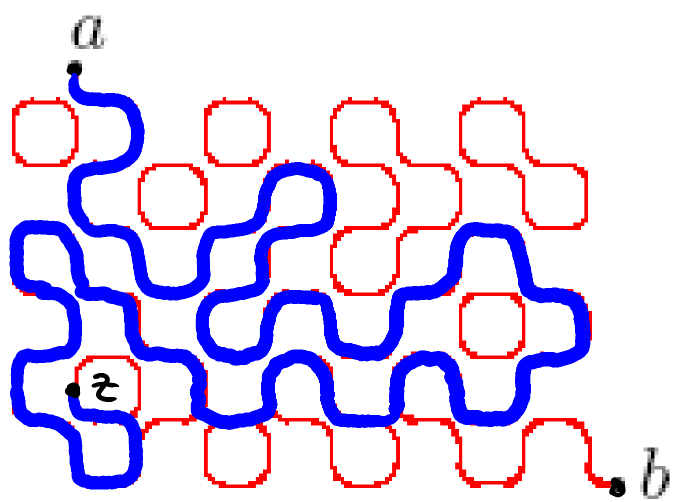
$a_1, b_1 := \gamma(t_0)$,

$\gamma_1 :=$ loop from a_1 to b_1.

Repeat the procedure ...

Interface tree for FK model.





Fix $z \in \Omega$

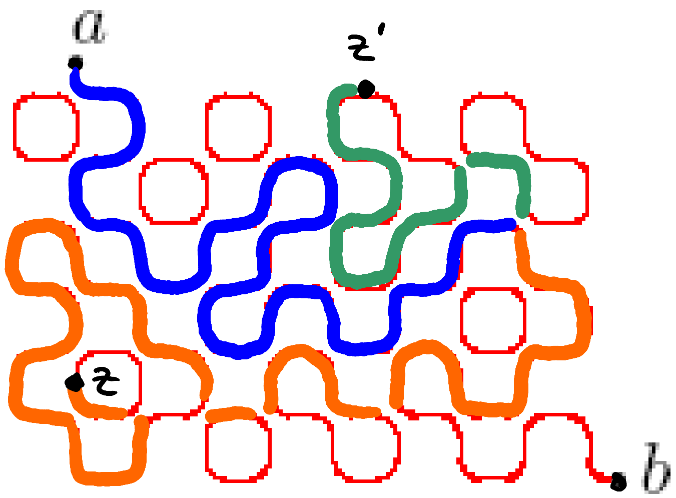
Follow $\gamma : a \to b$
until hit z or
disconnect it at time t_0

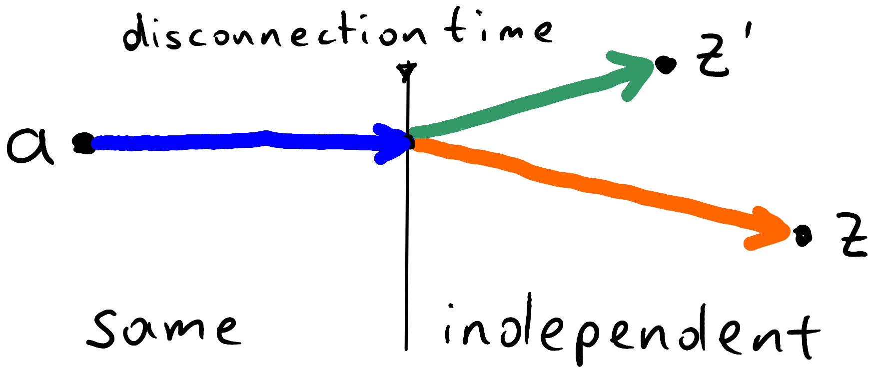
Set $a_1, b_1 := \gamma(t_0)$, follow
the loop $\gamma_1 : a_1 \to b_1$
until hit z or
disconnect it at time t_1

Set $a_2, b_2 := \gamma_2(t_2), \ldots$

Eventually arrive
at z, call the
resulting curve
Branch$(a \to z, b)$

Coupling of branches in the tree





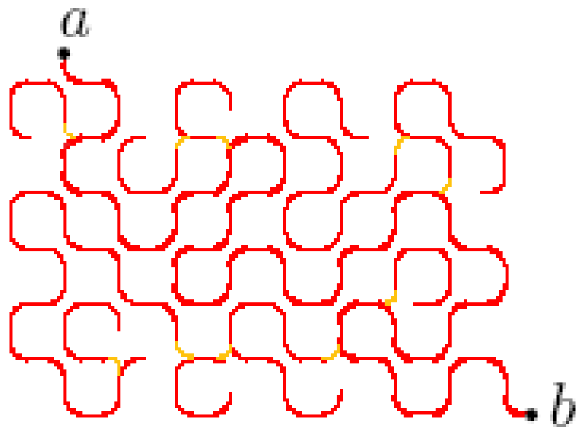
Run the procedure for two points, z and z'.

Until we disconnect them, we follow the same curve.

Afterwards we follow independent curves in two disjoint domains

Interface tree

- a, b for boundary conditions ($a = b$ ok)
- $\forall z$ get BRANCH$(a \to z, b)$
- BRANCH$(a \to z)$, BRANCH$(a \to z')$ coincide till disconnection, then are independent.



Conclusion: converges to a tree of coupled $SLE(\varkappa, \varkappa - 6)$. [Kemppainen, S]

Proof F_{zzab} has a martingale property for the BRANCH$(a \to z\ b)$ use the same trick.

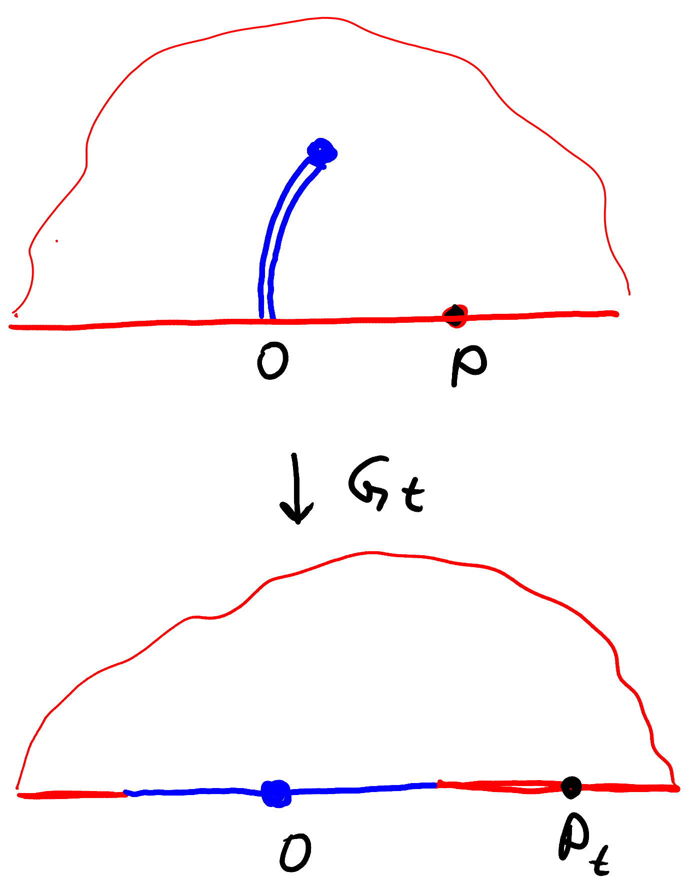
SLE(κ, ρ) with drift

Map Ω to $\mathbb{C}_+$

$a \mapsto 0$, $z \mapsto \infty$, $b \mapsto p$

Now we have an SLE where driving term has a drift depending on p_t

$$dw(t) = \sqrt{\kappa}\, dB_t + \frac{\rho}{p_t}\, dt$$



Note $d(G_t(z) + w(t)) = 2/G_t(z)\, dt$

For $z = p$ $\quad d(p_t + w(t)) = 2/p_t\, dt$

Hence

$$\boxed{dp_t = -\sqrt{\kappa}\, dB_t + \frac{2-\rho}{p_t}\, dt,}$$

a Bessel process

There is a general theory of SLE$(\varkappa, \rho)$:

- why drift $\approx \frac{\rho}{P_t}$
- why independence up to disconnection time implies $\rho = \varkappa - 6$

But we do not need it.

Our martingale observable $F_{zz\,ab}$ implies that $\sigma\sqrt{P_t}$, $P_t^2 - 8t$ are martingales.

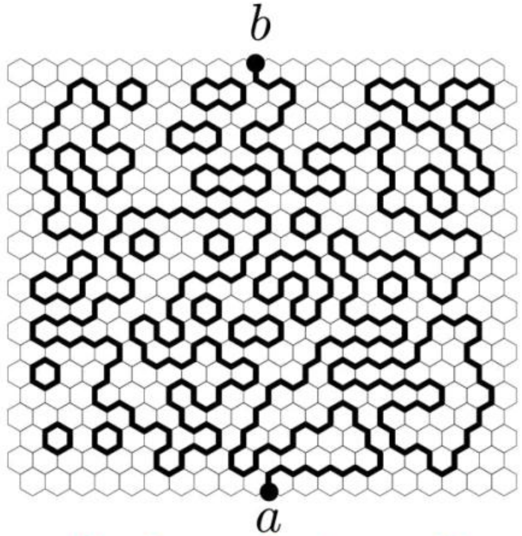
$\sigma = \pm$ independently on every γ_k

Lèvy-type characterization [e.g. Arbib 1965] implies

$$dP_t = -\sqrt{\varkappa}\, dB_t + \frac{2-\rho}{P_t}\, dt \qquad \begin{array}{l}\varkappa = 16/3\\ \rho = \varkappa - 6\end{array}$$

so we get an SLE$(\frac{16}{3}, \frac{16}{3} - 6)$ tree. $\blacksquare$

$\boldsymbol{O(n)}$ **loop gas.** Configurations of disjoint simple loops on hexagonal lattice. Loop-weight $n \in [0, 2]$, edge-weight $x > 0$.



$$\boldsymbol{Z = \sum_{\text{configs}} n^{\#\text{ loops}} \, x^{\#\text{ edges}}}$$

Dobrushin boundary conditions:

besides loops, an interface $\gamma : a \leftrightarrow b$.

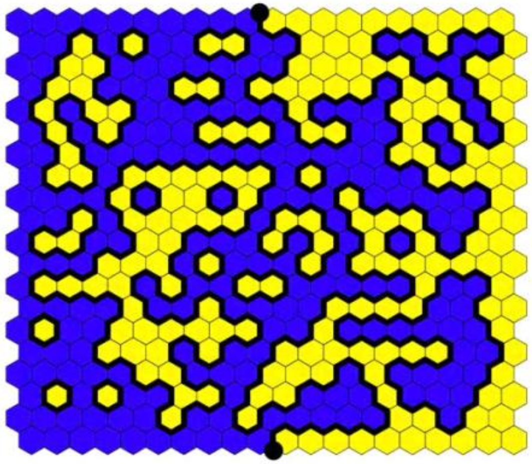
Conjecture [Kager-Nienhuis,...]. $\exists$ *conformally invariant scaling limits for* $x = x_c(n) := 1/\sqrt{2 + \sqrt{2 - n}}$ *and* $x \in (x_c(n), +\infty)$.

Two different limits correspond to dilute / dense phases

(limiting loops are simple / non-simple)

For $x \in (0, x_c(n))$ interfaces$\rightarrow$lines, no conformal invariance [Pfister-Velenik]

Hexagons of two colors (Ising spins ± 1), which change whenever a loop is crossed.



For $\boldsymbol{n = 1}$ the partition function becomes

$$Z = \sum x^{\#\ \text{edges}}$$
$$= \sum x^{\#\ \text{pairs of neighbors of opposite spins}}$$

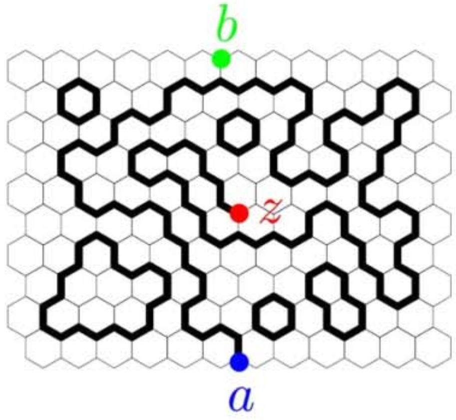
$\boldsymbol{n = 1,\ x = 1/\sqrt{3}}$: **Ising model at T_c**

Note: critical value of x is known [Wannier]

$\boldsymbol{n = 1,\ x = 1}$: **critical percolation** (on hexagons = sites of the dual triangular lattice) All configs are equally likely ($p_c = 1/2$ [Kesten, Wierman]).

$\boldsymbol{n = 0,\ x = 1/\sqrt{2+\sqrt{2}}}$: a version of **self-avoiding random walk** (no loops, only a simple curve from a to b with weight x^{length}, cf. prediction [Nienhuis] that number of length ℓ simple curves is $\approx \sqrt{2+\sqrt{2}}^{\,\ell}\,\ell^{11/32}$)

Which observable is discrete analytic for the $O(n)$ model?



Set $F(z) = Z_{+1 \text{ monodromy at z}}$
Curve stops at z before reaching b.
$2\cos(2\pi k) = n$, new spin $\sigma = 1/4 - 3k/2$
Ising: $n = 1$, $k = -1/6$, $\sigma = 1/2$, a fermion

Theorem. *For the Ising model at T_c*

$$\boldsymbol{F(z)/F(b) \rightrightarrows (\Psi'(z)/\Psi'(b))^{\sigma}} \quad \text{inside } \Omega$$

as lattice mesh $\epsilon \to 0$. Here Ψ maps Ω to a halfplane, $a, b \mapsto \infty, 0$.

Proof: Similar yet different. Partially works for all values of n.

Explains Nienhuis predictions of critical temperature x_c! (sorry, no proof yet)

Relates to Kenyon's work (think dimer models on the Fisher lattice)

Massive SLE

Do our observable for $p \neq p_{crit}$. [MAKAROV, S]

Massive CR: $\partial_{\bar{z}} F^{\epsilon} = i\, m_{\epsilon} \overline{F^{\epsilon}}$

where $m_{\epsilon} = \dfrac{p - p_{crit}}{\epsilon}$ Same RBVP

We can send $p \to p_{crit}$, $\epsilon \to 0$ so that $m_{\epsilon} \to m$.

Obtain a Massive Martingale Observable

satisfying $\partial_{\bar{z}} F_m = i\, m \overline{F_m}$

Can reason in the same way, get conformal invariance wrt metric $m|dz|$.

__Example__ mSLE(4) $dW(t) = \sqrt{4}\, dB_t + m \left(\iint_{\Omega_t} F \cdot P_m \right) dt$

Here: F non-massive observable

P_m massive Poisson kernel $\Delta P_m = m^2 P_m$

The end.

Thanks!