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Conformally invariant random simple curves on Riemann surfaces.

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CONFORMALLY INVARIANT RESTRICTION MEASURES ON RIEMANN SURFACES II: WERNER'S MEASURE

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In this lecture we will give a direct construction of *Werner's measure*, the unique conformally invariant measure on self-avoiding loops in Riemann surfaces.

1. From Cross-cuts to boundary bubbles

Let $\mu^{\mathbb{H}}_{z,-1/z}(x,\infty)$ be as defined in the first lecture, i.e. the law of chordal $\mathrm{SLE}_{8/3}$ in the upper half-plane from x to ∞. By a formula of Schramm,

$$\mu^{\mathbb{H}}_{z,-1/z}(x,\infty)[\gamma \text{ passes to the right of } i] = \frac{1}{2}\left(1+\frac{x}{\sqrt{1+x^2}}\right)$$

$$= \frac{1}{2}(1-\cos\theta), \tag{1}$$

where $x = -\cot\theta$ is the angle of the straight line segment from x to i with the real axis.

If $\Phi(z) = -1/z$, then Φ maps the upper half-plane conformally onto itself, sending i to i, 0 to ∞, and ϵ to $-1/\epsilon$. Then, by definition,

$$\begin{aligned}
&\mu^{\mathbb{H}}_{z,z}(\epsilon,0)[\gamma \text{ disconnects } i \text{ from } \infty] \\
&\quad = (\Phi'(\epsilon)\Phi'(0))^{5/8}\,\mu^{\mathbb{H}}_{z,-1/z}(-1/\epsilon,\infty)[\gamma \text{ passes to right of } i] \\
&\quad = \epsilon^{-5/4}\frac{1}{2}(1-(1+\epsilon^2)^{-1/2}) = \epsilon^{-5/4}(\epsilon^2/4 + o(\epsilon^2)),
\end{aligned}$$

since $\Phi'(\epsilon) = -\epsilon^2$, and $\Phi'(0) = 1$ with respect to the chosen uniformizers. This asymptotic behavior suggests the following ansatz for the definition of a measure on boundary bubbles:

$$\mu^{\mathbb{H},i}_{z}(0) = \lim_{\epsilon\to 0} 4\epsilon^{-3/4}\mu^{\mathbb{H}}_{z,z}(0,\epsilon)\restriction_{[\gamma \text{ disconnects } i \text{ from } \infty]},$$

where $\mu \restriction_A$ stands for the restriction of the measure μ to the set A. More generally, for a simply connected domain D, a point $r \in D$, $q \in \partial D$, u a boundary uniformizer at q, and s another boundary point, set

$$\mu^{D,r}_{u}(q) = \lim_{p\to q} 4|u(p)-u(q)|^{-3/4}\mu^{D}_{u,u}(p,q)\restriction_{[\gamma \text{ disconnects } r \text{ from } s]}\,.$$

Then

$$\mu_u^{D,r}(q)$$
$$= \lim_{p\to q} 4|u(p)-u(q)|^{-3/4}\mu^{\mathbb{H}}_{z,z}(\epsilon,0) \restriction_{[\gamma \text{ disconnects } i \text{ from } \infty]} \circ\Phi \ (\Phi'(p)\Phi'(q))^{5/8}$$
$$= \Phi'(q)^{5/4} \lim_{p\to q} \left|\frac{\Phi(p)-\Phi(q)}{u(p)-u(q)}\right|^{3/4} \lim_{\epsilon\to 0} 4\epsilon^{-3/4}\mu^{\mathbb{H}}_{z,z}(\epsilon,0) \restriction_{[\gamma \text{ disconnects } i \text{ from } \infty]} \circ\Phi$$

(2)
$$= \Phi'(q)^2\mu_z^{\mathbb{H},i}(0)\circ\Phi,$$

and, if v is another uniformizer at q,

$$\mu_v^{D,r}(q)$$
$$= \lim_{p\to q}\left|\frac{v(p)-v(q)}{u(p)-u(q)}\right|^{-3/4} 4|u(p)-u(q)|^{-3/4}\mu^D_{u,u}(p,q)\left(\frac{du}{dv}(p)\frac{du}{dv}(q)\right)^{5/8}$$
$$= \left(\frac{du}{dv}(q)\right)^2 \mu_u^{D,r}(q).$$

Thus bubble measures transform as *quadratic differentials.*

2. Existence of the limit

The existence of the limit, namely as a weak limit of measures is a subtle problem. Our normalization with the factor $\epsilon^{-3/4}$ only guaranteed that the total mass of the measure converges. However, it is possible that, in the end, some or all of the mass escapes. For example, the cross-cut measures are supported on curves of Haussdorf dimension 4/3 and we would like the bubble measures to also be supported on curves of Haussdorf dimension 4/3. For weak convergence one usually has to establish that the sequence of measures is tight, i.e. that all the measures are nearly supported on one common compact set of curves. However, using Schramm's result, we will go a different route.

To establish convergence it is enough to do it in one geometric set-up, i.e. a simply connected domain, an interior point, a boundary point, and a uniformizer. It then follows for all other set-ups. So consider

$$\mu^{\mathbb{H},i}_{-1/z}(\infty) = \lim_{x\to-\infty} 4|x|^{3/4}\mu^{\mathbb{H}}_{-1/z,-1/z}(x,\infty) \restriction_{[\gamma \text{ passes to right of } i]}$$

(3)
$$= \lim_{x\to-\infty} 4|x|^2\mu^{\mathbb{H}}_{z,-1/z}(x,\infty) \restriction_{[\gamma \text{ passes to right of } i]} .$$

We will now reparametrize time for the SLE-curve, namely by *conformal radius from i*. The SLE maps g_t are given by solving

$$\partial_t g_t = \frac{3/4}{g_t(z)-U_t}, \quad g_0(z) = z,$$

where $U_t = -B_t$ for a standard Brownian motion with $B_0 = x$. Here we follow Lawler's notation. Let $\gamma[0,t]$ be the SLE-curve up to time t. Define the conformal radius r of a point z in a simply connected planar domain as follows: If $D = \mathbb{H}$, then $r = 1/\Im z$, while otherwise $r = |\Phi'(z)|/\Im\Phi(z)$, where Φ maps D onto $\mathbb{H}$. Set $Z_t = X_t + iY_t = g_t(i) - U_t$ and $\theta_t = \arg(Z_t)$. Then the conformal radius r_t of $\mathbb{H}\backslash\gamma[0,t]$ (as seen) from i is

$$r_t = \frac{|g_t'(i)|}{\Im g_t(i)},$$

and by Itô's formula

$$\partial_t r_t = r_t \frac{(3/2)Y_t^2}{(X_t^2+Y_t^2)^2}$$
$$d\theta_t = -(1/2)\frac{X_tY_t}{(X_t^2+Y_t^2)^2}\,dt - \frac{Y_t}{X_t^2+Y_t^2}\,dB_t.$$

In particular, r_t is strictly increasing. Change time so that $\hat{r}_t = r_{\sigma(t)} = e^{3t/2}$, if $r_\infty = \lim_{s\to\infty} r_s \geq e^{3t/2}$. Then $\hat{\theta}_t = \theta_{\sigma(t)}$ satisfies

$$d\hat{\theta}_t = -\frac{1}{2}\cot\hat{\theta}_t\,dt + d\hat{B}_t,$$

where $\hat{B}_t$ is also a standard Brownian motion. The solution to this stochastic differentia equation is known as a Legendre process on $[0,\pi]$ of index -1. It is absorbed when it reaches the boundary. Its behavior near the boundary is that of a 0-dimensional Bessel process. In particular, the event $\hat{\theta}$ is absorbed before time t corresponds to the event $r_\infty < e^{3t/2}$.

By Schramm's formula

$$\mu^{\mathbb{H},i}_{-1/z}(\infty) = \lim_{x\to-\infty} \frac{\mu^{\mathbb{H}}_{z,-1/z}(x,\infty)\restriction_{[\gamma \text{ passes to right of } i]}}{\mu^{\mathbb{H}}_{z,-1/z}(x,\infty)[\gamma \text{ passes to right of } i]}$$
$$= \lim_{x\to-\infty} \mu^{\mathbb{H}}_{z,-1/z}(x,\infty)[\,\cdot\,|\gamma \text{ passes to right of } i], \tag{4}$$

where the first line is to be read as "the left-hand side, which is a limit, exists, if and only if the limit on the right exists, in which case they are equal." The second equality is the definition of conditioning. The conditioning can be done by an h-transform, using the probability from Schramm's formula, which is "harmonic" for the generator of the Legendre process of index -1. The additional drift is the logarithmic derivative of this probability

$$\frac{\partial}{\partial\theta}\log[\frac{1}{2}(1-\cos\theta)] = \cot\frac{\theta}{2}.$$

Thus, $\hat{\theta}$ for the conditional SLE satisfies the equation

$$d\hat{\theta}_t = (\cot\frac{\hat{\theta}}{2} - \frac{1}{2}\cot\hat{\theta})\ dt + d\hat{B}_t \tag{5}$$

The boundary behavior of this process at the point 0 is that of a 4-dimensional Bessel process, i.e 0 is an entrance boundary never to be reached again, while at π the boundary behavior is unchanged, i.e that of a 0-dim. Bessel process, and the process is absorbed once it reaches π. The solution of (5) is known to be a Feller process. In particular, if P_θ is the law of the solution to (5) with $\hat{\theta}_0 = \theta$, then $\theta \to \theta'$ implies the weak convergence of P_θ to $P_{\theta'}$. Since the weak convergence of the driving function $\hat{\theta}$ for the SLE-curve γ implies the weak convergence of the SLE-curve itself (in an appropriate topology on the space of curves), we get the existence of the desired limit.

3. Werner's measure: The conformally invariant measure on self-avoiding loops

A loop is a homeomorphism ϕ from S^1 into $\mathbb{C}$. We consider two such homeomorphisms ϕ_1, ϕ_2 equivalent if there exists a homeomorphism $h : S^1 \to S^1$ such that $\phi_1 = \phi_2 \circ h$. An annular region U is a doubly connected domain whose boundary components are both non-degenerate, i.e. contain more than one point. In fact, usually we will assume that both boundaries are loops. An annular region U is conformally equivalent to an annulus $\{1 < |z| < \rho\}$ for a unique $\rho \in (1, \infty)$. The annulus $\{1 < |z| < \rho\}$ is conformally equivalent to

$$A_\rho \equiv \mathbb{H}\backslash\{|z - i\frac{\rho^2+1}{\rho^2-1}| \le \frac{2\rho}{\rho^2-1}\}$$

under the map $z \to i(w+1)/(w-1)$.

Suppose $\mu = \mu^{\mathbb{C}}$ is a measure on loops $\ell \in \mathbb{C}$. For any planar domain D, set $\mu^D = \mu \restriction_{[\ell\subset D]}$. μ is said to satisfy *conformal restriction* if whenever $\Phi : D \to D'$ is conformal and onto, then $\mu^D = \Phi^*\mu^{D'}$.

Proposition 1 (Werner). *If μ exists, it is unique (up to a multiplicative constant). In fact, if $z \in D' \subset D$, where D and D' are simply connected domains, then*

$$\mu[\ell \subset D, \ell \not\subseteq D', \ell \textit{ disconnects } z \textit{ from } \partial D] = c\log\Phi'(z),$$

where $\Phi : D' \to D$ is conformal and onto with $\Phi(z) = z, \Phi'(z) > 0$.

Proof. Let $D = \{|z| < 1\}$ and suppose $0 \in D'' \subset D' \subset D$. Then

$$\begin{aligned} F(\Phi_{D''}) &\equiv \mu^D[\ell \nsubseteq D'', \ell \text{ disconnects } 0 \text{ from } \partial D] \\ &= \mu^{D'}[\ell \nsubseteq D'', \ell \text{ disconnects } 0 \text{ from } \partial D'] \\ &\quad + \mu^D[\ell \nsubseteq D', \ell \text{ disconnects } 0 \text{ from } \partial D] \\ &= \mu^D[\ell \nsubseteq \Phi_{D'}(D''), \ell \text{ disconnects } 0 \text{ from } \partial D] \\ &\quad + \mu^D[\ell \nsubseteq D', \ell \text{ disconnects } 0 \text{ from } \partial D], \\ &= F(\Phi_{D''} \circ \Phi_{D'}^{-1}) + F(\Phi_{D'}), \end{aligned}$$

where $\Phi_{D'}, \Phi_{D''}$ are conformal from D', resp. D'' onto D, which fix 0 and have positive derivative there. Thus the map $\Phi_{D'} \mapsto F(\Phi_{D'})$ is a homomorphism from the semigroup of conformal maps $\Phi_{D'}$ under composition into the additive semigroup $\mathbb{R}^+$.

Consider the family of domains $D_t \equiv D \backslash [r_t, 1)$, where r_t is such that $\Phi'_{D_t}(0) = e^t$. Then, by symmetry,

$$\Phi_{D_t}(D_{t+s}) = D \backslash [r_{t+s,t}, 1) \equiv D_{t+s,t}$$

for some $r_{t+s,t}$. But $\Phi_{D_{t+s}} = \Phi_{D_{t+s,t}} \circ \Phi_{D_t}$, so that

$$e^{t+s} = \Phi'_{D_{t+s}}(0) = \Phi'_{D_{t+s,t}}(0) e^t.$$

This implies $\Phi'_{D_{t+s,t}}(0) = e^s$ and hence $D_{t+s,t} = D_s$ and $\Phi_{D_{t+s}} = \Phi_{D_t} \circ \Phi_{D_s}$ so that $t \to \Phi_{D_t}$ is a semigroup. Since $t \mapsto F(\Phi_{D_t})$ is a non-decreasing function from $(0, \infty)$ into $(0, \infty)$ such that

$$F(\Phi_{D_{t+s}}) = F(\Phi_{D_t}) + F(\Phi_{D_s}),$$

it follows that

$$F(\Phi_{D_t}) = ct = c \log \Phi'_{D_t}(0)$$

for some constant $c > 0$.

For each $\theta \in [0, 2\pi)$, $t > 0$, set $D_{t,\theta} = D \backslash [r_t e^{i\theta}, e^{i\theta})$, with the r_t from above. Then, by rotational invariance of D, $\Phi'_{D_{t,\theta}}(0) = e^t$, and, by rotational invariance of μ^D,

$$F(\Phi_{D_{t,\theta}}) = F(\Phi_{D_t}).$$

Next, consider the semigroup $\mathcal{U}$ of conformal maps generated by the family $\{\Phi_{D_{t,\theta}} : t > 0, \theta \in [0, 2\pi)\}$. If $\Phi \in \mathcal{U}$, then

$$\Phi = \Phi_{D_{t_1,\theta_1}} \circ \cdots \circ \Phi_{D_{t_n,\theta_n}},$$

and, by the semigroup property of F,

$$\begin{aligned} F(\Phi) &= F(\Phi_{D_{t_1,\theta_1}}) + \cdots + F(\Phi_{D_{t_n,\theta_n}}) \\ &= c\left(\log \Phi'_{D_{t_1,\theta_1}}(0) + \cdots + \log \Phi'_{D_{t_n,\theta_n}}(0)\right) \\ &= c \log \Phi'(0). \qquad (6) \end{aligned}$$

Finally, the family $\mathcal{U}$ is "dense" in the class of conformal maps $\Phi_{D'}$ for arbitrary simply connected subdomains $0 \in D' \subset D$. This means there exists a sequence of domains D'_n belonging to maps from $\mathcal{U}$ such that $D'_n \searrow D'$. This implies

$$\Phi'_{D'_n}(0) \nearrow \Phi'_{D'}(0),$$

and also $\ell \nsubseteq D'$ if and only if $\ell \nsubseteq D'_n$ from some n on. Hence

$$F(\Phi_{D'}) = \lim F(\Phi_{D'_n}) = c \lim \Phi'_{D'_n}(0) = c\Phi_{D'}(0).$$

□

4. Existence of μ

For an annular domain U, denote $\langle U \rangle$ the set of non-contractible loops in U. Let $\sigma(A_\rho)$ be the σ-algebra on the space $\langle A_\rho \rangle$ generated by events $\langle U \rangle$, where U is an annular region in A_ρ containing at least one loop which is non-contractible in A_ρ. We say U is an annular region *properly contained* in A_ρ. If U is conformally equivalent to the annulus $\{1 < |z| < \rho'\}$, then we say the modulus of U is

$$\text{mod } (U) = -\ln \rho'$$

Note that $1 < \rho' < \rho$. Denote γ an oriented simple curve which begins on the real line and ends at i. Then

- form $\gamma \cap U$, which consists of at most countably many oriented and ordered components,
- remove all components after the first component that ends on "inner" boundary of U,
- add left endpoints of all remaining components.

We call this the crossing of U induced by γ. Denote crossing also by γ. Then the crossing can be parameterized on $[-\ln \rho', 0]$ so that

$$U \backslash \gamma[-\ln \rho', t] \text{ has modulus } t.$$

Define a measure $\mu^{\rho,\gamma}$ on $(\langle A_\rho \rangle, \sigma(A_\rho))$ by

$$\mu^{\rho,\gamma} = \int_{-\ln \rho}^{0} dt (\varphi_t^{\rho,\gamma})^* \mu_z^{\mathbb{H},i}(0) \restriction_{\langle A_{e^{-t}} \rangle},$$

where $A_\rho\backslash\gamma[-\ln\rho,t]$ has modulus t and $\varphi_t^{\rho,\gamma}$ maps $A_\rho\backslash\gamma[-\ln\rho,t]$ conformally onto $A_{e^{-t}}$, sending the tip γ_t to 0. Then

$$\begin{aligned}\mu^{\rho,\gamma}[\langle U\rangle] &= \int_{-\ln\rho}^{0} dt\ \mu_z^{\mathbb{H},i}(0)[\langle\varphi_t^{\rho,\gamma}(U)\rangle] \\ &= \int_{-\ln\rho}^{0} dt\ \mu_z^{\mathbb{H},i}(0)[\langle\Phi_t\circ\varphi_t^{\rho,\gamma}(U)\rangle]\Phi_t'(0)^2 \\ &= \int_{-\ln\rho'}^{0} \mu_z^{\mathbb{H},i}(0)[\langle A_{e^{-s}}\rangle]. \qquad (7)\end{aligned}$$

The first equality in (7) is merely the definition of the pull-back of the measure $\mu_z^{\mathbb{H},i}(0)$ under φ, in the second equality Φ_t is the conformal map from $\varphi_t^{\rho,\gamma}(U)$ onto $A_{e^{-s}}$, where s is the modulus of $\varphi_t^{\rho,\gamma}(U)$, and the appearance of the factor $\Phi_t'(0)^2$ is due to a well-known argument of Beffara, and the third equality stems from the fact that under the transformation Φ_t the resulting crossing is no longer parameterized by modulus, the relation between the parameterizations being given by $ds=\Phi_t'(0)^2dt$. But the last expression in (7) shows that $\mu^{\rho,\gamma}[\langle U\rangle]$ and hence the measure $\mu^{\rho,\gamma}$ are independent of the crossing γ. Call these measures

$$\mu^\rho, \quad \rho>1.$$

If U is an annular region (not necessarily planar) with $\mod(U) = -\ln\rho$, define μ^U by

$$\mu^U=\Phi^*\mu^\rho, \quad \text{where } \Phi: U\to A_\rho.$$

If U is properly contained in $A_{\rho'}$ and V is an annular region properly contained in U with $\mod(V)=\rho''$, then

$$\begin{aligned}\mu^U[\langle V\rangle] &= \mu^\rho[\langle\Phi(V)\rangle] \\ &= \int_{-\ln\rho''}^{0} ds\ \mu_z^{\mathbb{H},i}(0)[\langle A_{e^{-s}}\rangle] \\ &= \mu^{\rho'}[\langle V\rangle]. \qquad (8)\end{aligned}$$

So $\mu^{\rho'}\restriction_{\langle U\rangle}=\mu^U$. Similarly, if U,V are two annular regions and $E\in\sigma(U)\cap\sigma(V)$, then

$$\mu^U[E]=\mu^V[E],$$

i.e., the measures on annuli are compatible. By general measure theory, this implies that

Theorem 2 (Werner's measure). *Given any Riemann surface S there exists a unique measure μ^S on loops in S such that if U is an annular* *region in S of modulus $-\ln\rho$, then*

$$\mu^S[\langle U\rangle] = \int_{-\ln\rho}^{0} dt\ \mu_z^{\mathbb{H},i}(0)[\langle A_{e^{-t}}\rangle].$$

The measures μ^S satisfy conformal restriction, i.e.

- *if $\Phi : R \to S$ is conformal, then $\mu^R = \Phi^* \mu^S$;*
- *if $T \subset S$, then $\mu^T = \mu^S \restriction_{[\ell\subset T]}$.*

Up to a multiplicative constant, the the family $\{\mu^S\}$ is the only family satisfying conformal restriction.

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