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From Critical to Off-Critical Interfaces
(SLE/CFT and application)

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FROM CRITICAL TO OFF-CRITICAL INTERFACES (07/2008)

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with M. Bauer

I/ statistical interfaces and SLE

1. Interfaces
2. statistical mechanics and curves
3. statistical martingales
4. SLE/CFT correspondance

II/ SLE/CFT and applications

1. Loewner eq, (algebraic version)
2. CFT martingales
3. Application 1: coupling to Gaussian field.
4. Application 2: SLE(κ,ρ) and generalisation.

(5. : N-SLE

III/ Off-critical SLE

1. (Warm-up) random walk / Girsanov theorem.
2. S.A.W as an example (of off-critical curve)
3. Off-critical drift and partition f^{ct}.
   " SLE and field theory
4. Example 1: $\left[\begin{array}{l} c=1+\text{mass} \\ SLE_4 + \text{"} \end{array}\right.$ (massive free field)
5. Example 2: off-critical LERW

II/ SLE/CFT and applications

First, we shall make contact between some algebraic aspect of CFT and probabilistic nature of SLE.

Recall (expected) CFT martingales (via statistical martingales)

~~1/ basics on CFT and Virasoro algebra~~

1. Loewner eq. (algebraic version of) (in $\mathbb{H}$ = upper half plane)

For chordal SLE, Loewner eq. is $dg_t = \frac{2dt}{g_t - \xi_t}$; $\xi_t = \sqrt{\kappa} B_t$

Convenient to consider $h_t = g_t - \xi_t$ (map the tip of the curve back to the origine ξ_0)

$$dh_t = \frac{2}{h_t} dt - d\xi_t$$

Consider now an arbitrary function of h_t, say $F(h_t(z))$

By Ito calculus

$$dF(h_t) = \left[\frac{2}{h_t} F'(h_t) + \frac{\kappa}{2} F''(h_t)\right] dt - d\xi_t F'(h_t) \quad , \quad d\xi_t^2 = \kappa dt$$

$$= \left(\frac{\kappa}{2} \ell_{-1}^2 - 2\ell_{-2}\right) F(h_t)\, dt - d\xi_t (\ell_{-1} F)(h_t)$$

where ℓ_n are the vector field

$$\ell_n = -z^{n+1}\partial_z \quad , \quad [\ell_n, \ell_m] = (n-m)\,\ell_{n+m}$$

Rk:

Recall (as M. Bauer explained last week), we ~~can~~ view the Loewner map as a group element (group of germs of holo. map at ∞, with product equals to the composition

Action on function is $\hat{g}_t \cdot F = F \circ g_t$

Stochastic eq. for $\hat{g}_t$ is

$$\hat{g}_t^{-1} d\hat{g}_t = \underbrace{\left(\frac{\kappa}{2} \ell_{-1}^2 - 2\ell_{-2}\right) dt}_{\text{This plays an important role in the CFT/SLE correspondence}} - \ell_{-1} d\xi_t$$

2. Basics on CFT and Virasoro algebra

Hilbert spaces of CFT are modules of Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}$$

In the CFT, the Loewner map is promoted to an operator G_t which intertwines the CFT Hilbert space in $\mathbb{H} \setminus \gamma_{[0,t]}$ and $\mathbb{H}$.
It satisfies $G_t^{-1} dG_t = (-2L_{-2} + \frac{\kappa}{2} L_{-1}^2)dt - d\xi_t L_{-1}$

We are interested in under which condition the drift term can vanish i.e. $(-2L_{-2} + \frac{\kappa}{2} L_{-1}^2) \underset{\sim}{\sim} 0$.

Consider h.w.v $|\psi\rangle$. $\begin{cases} L_n|\psi\rangle = 0 \\ L_0|\psi\rangle = h|\psi\rangle \end{cases}$ $\rightsquigarrow$ $\mathcal{U}(Vir_-)|\psi\rangle$ form a Vir-module

Question? when $(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2)|\psi\rangle \simeq 0$?
The meaning is when is this vector an h.w.v. in $\mathcal{U}(Vir_-)|\psi\rangle$

Compute $L_n|\Theta\rangle$, $(|\Theta\rangle \equiv (-2L_{-2} + \frac{\kappa}{2}L_{-1}^2)|\psi\rangle)$.
It is enough to compute $L_1|\Theta\rangle$ and $L_2|\Theta\rangle$.

$$L_1|\Theta\rangle = L_1(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2)|\psi\rangle = [-6L_{-1} + \kappa L_{-1} + 2\kappa h L_{-1}]|\psi\rangle$$

$$\begin{cases} [L_1, L_{-2}] = 3L_{-1} \\ [L_1, L_{-1}] = 2L_0 \\ [L_1, L_{-1}^2] = 2L_0L_{-1} + 2L_{-1}L_0 = 4L_{-1}L_0 + 2L_{-1} \end{cases}$$

$\rightsquigarrow$ need $2\kappa h + \kappa - 6 = 0$

$$L_2|\Theta\rangle = L_2(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2)|\psi\rangle = (-8h - c + 3\kappa h)|\psi\rangle$$

$$[L_2, L_{-2}] = 4L_0 + c/2$$
$$[L_2, L_{-1}] = 3L_1$$
$$[L_2, L_{-1}^2] = 3L_1L_{-1} + 3L_{-1}L_1 = 6L_0 + 6L_{-1}L_1$$

$\rightsquigarrow$ need $8h + c - 3\kappa h = 0$

i.e. $\begin{cases} h = \dfrac{6-\kappa}{2\kappa} \quad ; \quad c = \dfrac{(3\kappa-8)(6-\kappa)}{2\kappa} \end{cases}$

These are the relations which fixes the SLE/CFT dictionnary
The state $|\psi\rangle$ is the state created by the bdry ~~[illegible]~~ changing op. $\psi(0)$.

3. CFT martingales

We can now easily prove that CFT correlation f^{ct} are SLE martingales

chordal SLE in $\mathbb{H}$

$$\text{eg}\quad \mathcal{O} = \prod_\alpha \psi_\alpha(y_\alpha) \quad \rightsquigarrow \quad {}^{g_t}\mathcal{O} = \prod_\alpha |g_t'(y_\alpha)|^{\Delta_\alpha}\ \psi_\alpha(g_t(y_\alpha))$$

CFT martingales are

$$t \to \langle \psi(x_\infty)\ {}^{g_t}\mathcal{O}\ \psi(\xi_t)\rangle_{\mathbb{H}} \equiv C(\underset{\substack{\parallel \\ Y_\alpha^t}}{g_t(y_\alpha)}\ ;\ \xi_t)$$

provided that $\psi(\xi_t)$ is the operator whith null vector at level two

ie $\left(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2\right)\psi = 0$

Indeed, Ito calculus on $dC \Rightarrow$

$$dC = \underbrace{\left[\frac{\kappa}{2}\partial_\xi^2 + \sum_\alpha\left(-\frac{\Delta_\alpha}{(Y_\alpha^t - \xi_t)^2} + \frac{1}{(Y_\alpha - \xi_t)}\partial_{Y_\alpha}\right)\right]}_{=0}C\ dt + \# d\xi_t$$

$= 0$ as a consequence of $\left(-2L_{-2} + \frac{\kappa}{2}L_{-1}^2\right)\psi = 0$

(trop technique)

Rk: 2 ways to prove

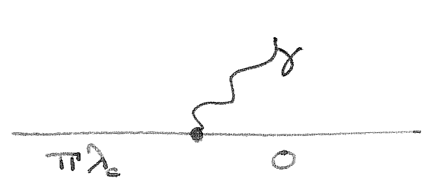
(1) as above

(2) or via operator formalis $\langle \psi\ {}^{g}\mathcal{O}\ \psi\rangle = \langle\psi|G_t^{-1}\mathcal{O}\ G_t|\psi\rangle$

Application: coupling to Gaussian field (un peu long)

i) GFF, $c=1$. ($\kappa=4$)

We consider Gaussian free field (GFF) on the upper-half plane $\mathbb{H}$, with Dirichlet boundary conditions. It is fixed by its one- and two-point functions



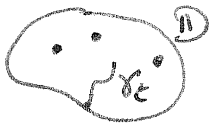
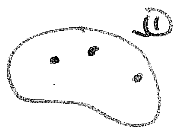
$$\langle \phi(z) \rangle_{\mathbb{H}} = \lambda_c \, \mathrm{Im}(\log z) \equiv C_z \quad \text{(harmonic)}$$

$$\langle \phi(z)\phi(w) \rangle^c_{\mathbb{H}} = -\log \left| \frac{z-w}{z-\bar{w}} \right|^2 \equiv G(z,w)$$

(choice of normalization)

discontinuity will be fixed by demanding that there is an exact coupling between $GFF_{c=1}$ and $SLE_{\kappa=4}$

ie

$$\mathbb{E}_{\kappa=4} \int \mathcal{D}\phi|_{\gamma_t}$$  $$=\int \mathcal{D}\phi$$ 

ie as usual summing of field configuration in $\mathbb{D}\setminus\gamma_t$ and summing over the curve via SLE should reproduce the sum over field configuration in $\mathbb{D}$.

This is OK if GFF correlation in $\mathbb{H}_t$ are martingales.

1 pt
$$\langle \phi(z) \rangle_{\mathbb{H}_t} = C[h_t(z)] \qquad \text{ok martingale for } \kappa=4$$
$$d\,C[h_t] = \lambda_c \, \mathrm{Im}\left(\frac{2}{h_t(z)}\right) dB_t$$

2 pt
$$\langle \phi(z)\phi(w) \rangle_{\mathbb{H}_t} = C(h_t(z))\,C(h_t(w)) + G(h_t(z), h_t(w))$$

is a martingale if $\lambda_c = \sqrt{2}$

because $d\,G(h_t(z), h_t(w)) = -2\left(\mathrm{Im}\,\frac{2}{h_z}\right)\left(\mathrm{Im}\,\frac{2}{h_w}\right) dt$.

Rk1. if one and 2 point are ok all N point f^{ts} are ok

because
$$\langle e^{i(\jmath,\varphi)} \rangle_{\mathbb{H}} = e^{i(C_{\mathbb{H}},\jmath) - \frac{1}{2}(\jmath\, G_{\mathbb{H}}\, \jmath)}$$

Rk2 In $\mathbb{H}_t$, we ~~ss~~ have Dirichlet boundary condition, so that there is a discontinuity of the field (of all its N-p^{ts} f^{ct}) along γ and the discontinuity jump is $\pi\lambda_c = \pi\sqrt{2}$. (cf Schramm-Sheffield)

ii) GFF and coupling with $SLE_{\kappa<4}$

Since $SLE_{\kappa<4}$ corresponds to central charge $c = \frac{(6-\kappa)(3\kappa-8)}{2\kappa} < 1$ we have to add a "background charge" to the GFF, so that now the stress tensor is $T(z) = -\frac{1}{2}(\partial\phi)^2 + i\alpha_0(\partial^2\phi)$

$$c = 1 - 12\,\alpha_0^2$$

In presence of background charge the field ϕ is not scalar, (the current $J = i\partial\phi$ is not a primary field of dim 1) there is an anomaly in its conformal transformation.
This reflects in the coupling SLE_κ / GFF_{α_0}.

1^{pt} in $\mathbb{H}\setminus\gamma_t \equiv \mathbb{H}_t$ is a martingale $\Rightarrow$

$$C_t(z) = \lambda\left[\operatorname{Im}\log h_t(z) - \left(\frac{\kappa-4}{4}\right)\operatorname{Im}\log h_t'(z)\right]$$

reflect the anomalous transformation with

$$dC_t(z) = \lambda\,\operatorname{Im}\left(\frac{\sqrt{\kappa}}{h_t}\right) dB_t$$

2 pt f^{ct} in $\mathbb{H}_t$

the connected 2 pt f^{ct} is still the Green function

$$\langle \phi(z)\phi(w)\rangle^c_{\mathbb{H}_t} = G(h_t(z), h_t(w))$$

so that the full 2 pt function is a martingale iff

$$\lambda = \sqrt{8/\kappa}$$

Pf As before, if the one and two point f^{ct} are martingale, the same is true for the N. pt f^{ct} $\Rightarrow$ exact coupling.

From the value of λ and from the anomalous transf of ϕ, we read the value of α_0

$$\alpha_0 = \sqrt{\frac{2}{\kappa}}\left(\frac{\kappa-4}{4}\right) \to c = 1 - 12\alpha_0^2 = 1 - 6\,\frac{(\kappa-4)^2}{4\kappa}.$$

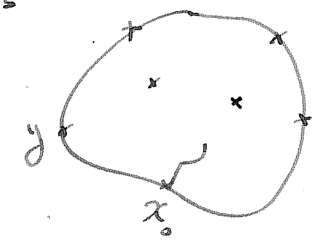
Rk: In presence of background charge, transformation law are

$$\begin{cases} \langle \phi(z)\rangle_U = \langle \phi(f(z))\rangle_V + 2 i\alpha_0 \log f'(z) \\ \langle J(z)\rangle_U = f'(z)\langle J(f(z))\rangle + 2 i\alpha_0 \frac{f''}{f'} \end{cases}$$

Application: SLE(κ, ρ) and généralisation (un peu long)

Q? how to describ SLE with more marked points

e.g.



$\leadsto$ motivation for introducing Z_0 via statistical/off martingale

We start from chordal SLE from 0 to ∞ in the upper half plane (with the hydrodynamic normalization : $dg_t = \dfrac{2dt}{g_t - \xi_t}$

We shall use a tiny lemma (which is a simple consequence of Ito calculus, link to what is called h-transform in probability theory.

Let $Z_0(\xi_t, g_t(w_1)\ldots)$ be a martingale for SLE chordal (from 0 to ∞ in the hydro. normalisation

$Z(\xi_t, g_t(w_1)\ldots)$ " " "

then the ratio $\dfrac{Z(\xi_t, g_t(w_1)\ldots)}{Z_0(\xi_t, g_t(w_1)\ldots)}$ is a martingale

for the SLE process $dg_t = 2dt/_{g_t - \xi_t}$ with source (with drift)

$$\boxed{d\xi_t = \sqrt{\kappa}\, dB_t + \kappa\,(\partial_\xi \log Z_0)(\xi_t, g_t(w_1)\ldots)\, dt}$$

(dem, see below)

We apply this to different exemples. The point is that we know (from stat. mech previous discussion) that the martingales of process associated to some stat. mech. models should be ratio of partition functions.

Soit $Z_0(\xi_t, g_t(z))$ une martingale pour SLE chordale

Soit $d\xi_t = \sqrt{\kappa}\, dB_t + \kappa (\log Z_0)'(\xi_t, g_t(z)).dt$ où $(\log Z_0)'$ denote dérivée / à ξ.

Les martingales du processus avec drift sont:

$$\frac{Z(\xi_t, g_t(z))}{Z_0(\xi_t, g_t(z))}$$

où Z est aussi martingale pour SLE chordale.

Appl. construction des processus SLE(κ, ρ) et généralisation

dem) On calcul drift(Z/Z_0).

On a $(Z/Z_0)' = \frac{Z}{Z_0}(\log Z)' - \left(\frac{Z}{Z_0}\right)(\log Z_0)'$

$$(Z/Z_0)'' = \frac{Z}{Z_0}\left[\frac{Z''}{Z} - 2(\log Z)'(\log Z_0)' + 2(\log Z_0)'^2 - \frac{Z_0''}{Z_0}\right]$$

d'où, via calcul à la Itô,

$$\frac{Z_0}{Z}\cdot \operatorname{drift}\left(\frac{Z}{Z_0}\right) = \frac{2}{g(z)-\xi}(\partial_z \log Z) - \frac{2}{g(w)-\xi}(\partial_w \log Z_0) \quad \text{(venant de Loewner)}$$

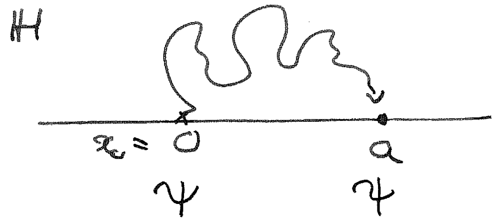
$$+ \kappa\, \cancel{(\log Z_0)'\left[(\log Z)' - (\log Z_0)'\right]} \quad \text{(venant de drift } d\xi)$$

$$+ \frac{\kappa}{2}\left[\frac{Z''}{Z} - \cancel{2(\log Z)'(\log Z_0)'} + \cancel{2(\log Z_0)'^2} - \frac{Z_0''}{Z_0}\right] \quad \text{(venant de Itô)}$$

$$= \underbrace{\frac{1}{Z}\left[\frac{2}{g(z)-\xi}\partial_z Z + \frac{\kappa}{2} Z''\right]}_{} - \underbrace{\frac{1}{Z_0}\left[\frac{2}{g(w)-\xi}\partial_w Z_0 + \frac{\kappa}{2} Z_0''\right]}_{}$$

Ts les deux, ce sont conditions pour que Z et Z_0 sont martingales pour SLE-chordale.

i) Chordal SLE from 0 to a in $\mathbb{H}$ (with hydrodynamic)



there are two boundary condition changing operator at 0 and at α.

In the continuum model, the partition function should be (proportional to) the 2-point function

$$Z_0(x_0, a) = \langle \psi(x_0)\, \psi(x_0) \rangle \quad \text{in } \mathbb{H}$$

In the cut-domain $\Rightarrow$

$$Z_0(\xi_t, a_t) = \langle \psi(a_t)\, \psi(\xi_t) \rangle \qquad \text{(without the Jacobian)}$$
$$= (\xi_t - a_t)^{2h_{12}} \qquad h_{12} = \frac{6-\kappa}{2\kappa}$$

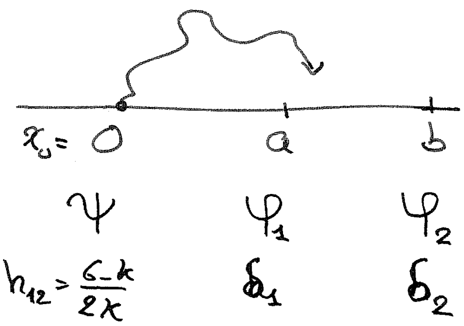
From the previous we know that the drift is $\kappa\, \partial_\xi \log Z_0$.
So this SLE process is described by

$$dg_t(z) = \frac{2dt}{g_t(z) - \xi_t} \quad ; \quad da_t = \frac{2dt}{a_t - \xi_t}$$

$$d\xi_t = \sqrt{\kappa}\, dB_t + (6-\kappa)\frac{dt}{\xi_t - a_t}$$

Rk: $\kappa = 6 \rightarrow$ locality

ii) SLE(κ, ρ) with marked point a, b in $\mathbb{H}$



- operator ψ (with null vector at level two) at starting position of the curve
- operators φ_1 and φ_2 at the two marked points (with dim δ_1 and δ_2)

The partition function will be (proportional to) the 3-pt f^{ct}:

$$Z_0(x_0,a,b) = \langle \varphi_2(b)\,\varphi_1(a)\,\Psi(x_0)\rangle \qquad \text{on } \mathbb{H}$$

$$= \text{cst.}\ (x_0-a)^{\delta_2-\delta_1-h}\,(x_0-b)^{\delta_1-\delta_2-h}\,(b-a)^{h-\delta_1-\delta_2}$$

But, since Ψ satisfies null vector eq. at level two, there are fusion rules, and this 3-pt f^{ct} is non vanishing only if the dimension $\delta_{1,2}$ satisfies a quadratic eq..
They can be parametrized as

$$(\text{at } b) \rightsquigarrow \delta_2 = \rho(\rho+4-\kappa)/4\kappa \quad ; \quad \delta_1 = (\rho+2)(\rho+6-\kappa)/4\kappa \leftarrow (\text{at } a)$$

$$\delta_1-\delta_2 = \frac{2\rho+\kappa-6}{2\kappa} = \frac{\rho}{\kappa} - h_{12}$$

The drift term is given by $\kappa(\partial_\xi \log Z_0)(\xi, a_t, b_t)$ so that

$$d\xi_t = \sqrt{\kappa}\,dB_t - \frac{\rho\,dt}{\xi_t - b_t} + \frac{(\rho+\kappa-6)dt}{\xi_t - a_t}$$

<u>Rk 1</u>: The symmetric case corresponds to $\rho = \frac{6-\kappa}{2}$, ie $\delta_1=\delta_2=h_{0;1/2}$
it is SLE-dipolar

<u>Rk 2</u>: Case $\rho=0$ reproduces SLE from 0 to a.
Case $\rho = 6-\kappa$ " SLE from 0 to b

ii) <u>Generalisation : SLE(κ, Z_0)</u>
with any number of marked points (on the bdry, in the bulk)

$$Z_0(x_0; y_1, \dots, w_1, \dots) = \langle \Psi(x_0)\,\varphi_1(y_1)\dots\,\Phi_1(w_1,\bar w_1)\dots\rangle$$

and the drift will be $\kappa(\partial_{\xi_t}\log Z)(\xi_t, y_1^t, \dots, w_1^t \dots)$

<u>Rk</u>: Application to conditioning.

Application: N-SLE

N interfaces
N boundary changing operators Ψ

The partition f^ct is $\quad Z_0(x_1, \dots, x_N) = \langle \Psi(x_N) \dots \Psi(x_1) \rangle$

In ~~that~~, with hydrodynamic normalization, Loewner map satisfies

$$dg_t^{(z)} = \sum_{j=1}^{N} \frac{2\, dt}{g_t(z) - x_j^t}$$

with

$$dx_j^t = \sqrt{\kappa}\, dB_t^{(j)} + \sum_{k \neq j}' \frac{2dt}{x_j^t - x_k^t} + \kappa \left(\partial_{x_j} \log Z_0\right)(x_1^t, \dots, x_N^t)\, . dt$$

one ind^t Brownian for each curve

due to effect of the map on the curves

to ensure that ratio of partition f^ct / correlation f^ct are martingales

Rk1: Z_0 satisfies N diff. eq. (of 2^nd order), because Ψ has null vector at level two

Rk2: the # of conformal blocs (i.e. solution of diff. eq.) equals the # of inequivalent topology $\Rightarrow Z^{[\alpha]}$;

Rk3: Arch proba.: $\dfrac{Z^{[\alpha]}}{Z}$