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From Critical to Off-Critical Interfaces (Off-critical SLE)

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FROM CRITICAL TO OFF-CRITICAL INTERFACES (07/2008)

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with M. Bauer

I/ statistical interfaces and SLE

1. Interfaces
2. statistical mechanics and curves
3. statistical martingales
4. SLE/CFT correspondance

II/ SLE/CFT and applications

1. Loewner eq, (algebraic version)
2. CFT martingales
3. Application 1: coupling to Gaussian field.
4. Application 2: SLE(κ, ρ) and generalisation.

(5. : N-SLE

III/ Off-critical SLE

1. (Warm-up) random walk / Girsanov theorem.
2. S.A.W as an example (of off-critical curve)
3. Off-critical drift and partition f^{ct}
   " SLE and field theory
4. Example 1: $\left[\begin{array}{l} c=1+\text{mass} \\ SLE_4 + \text{"} \end{array}\right.$ (massive free field)
5. Example 2: off-critical LERW

III/ Off-critical SLE [redacted]

ex of problem
- percolation : $p \neq p_c$ but $p \to p_c$
- Ising : $T \neq T_c$ but $T \to T_c$
- SAW : different fugacity
- then
  - LERW.
  - $c=1$ + mass.

Try to present a possible approach (via field theory technique + probability) to description of off-critical SLE.

We want to described interfaces in the scaling regime near a critical point. Examples (are give above)

Percolation : .....
Ising : .....
SAW : .....

We shall discuss more in details the case of LERW and $c=1$ + mass

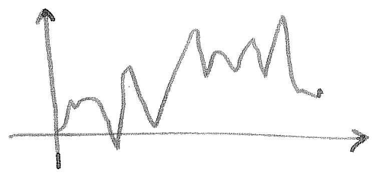
Plan :

a

c/ Warm-up : random walk or [off-critical model / Girsanov th.]

• biaised random walk of step a

$$X_N = a \sum_{j=1}^{N} \epsilon_j \quad , \quad \epsilon_j = \pm \quad \text{prob } p/1-p$$



The sym. walk is for $p=1/2$. For $p \gtrless 1/2$, $X_N \gtrless 0$ almost surely.
and there is a phase transition at $p=1/2$
We want to look at the 'near-critical' theory $p \to 1/2$ $\begin{matrix} N \to \infty \\ a \to 0 \end{matrix}$

Let $N_\pm$ be the number of positive/negative step.

$$\begin{cases} X_t = a(N_+ - N_-) \\ t = a^2 N = a^2(N_+ + N_-) \end{cases}$$

The proba of the walk is $p^{N_+}(1-p)^{N_-}$, so relative to the critical theory we get the weight

$$\mathbb{E}_p[\ldots] = \mathbb{E}_{p=1/2}[M \ldots] \quad \text{with} \quad M = (2p)^{N_+}(2(1-p))^{N_-}$$

We get weight $M \equiv e^{S}$, S the action, (M = martingale)

• We now take the scaling limit $a \to 0$, $\begin{pmatrix} N \to \infty \\ t \text{ fixed} \end{pmatrix}$

At the critical theory ($p=1/2$),
$X_t \equiv$ Brownian motion $(dX_t)^2 = dt$.

To get a finite contribution (for S) we have to scale p with a

$$p = \frac{1}{2}(1+\mu a) \quad , \quad a \to 0$$

then $S_t = \mu X_t - \frac{\mu^2}{2} t$.

It is easy to check that $M_t = e^{S_t} = e^{\mu X_t - \frac{\mu^2}{2} t}$
is a martingale for X_t a Brownian motion: $M_t^{-1} dM_t = \mu dB_t = \mu dX_t$
The off-critical measure is

$$\mathbb{E}_\mu[\dots] = \mathbb{E}_{Br}[M_t \dots]$$

With respect to that measure X_t is no longer a Brownian motion. It has a drift: it is easy to check that $E_\mu[X_t] = \mu t$. One can directly check (eg by looking at the Fokker–Planck operator) that w.r.to $\mathbb{E}_\mu[\dots]$, X_t satisfy the SDE

$$dX_t = dB_t + \mu dt \qquad \text{with } B_t \equiv \mathbb{E}_\mu\text{- Brownian}$$

The drift coincide with $M_t^{-1} dM_t = \mu dt$

- This is the essence of the Girsanov theorem.

If M_t martingale for Brownian motion X, $M_t^{-1} dM_t = V_t \, dX_t$
Then, with respect to the measure $\mathbb{E}_\mu[\dots] \equiv \mathbb{E}_X[M_t \dots]$,
the X_t satisfies

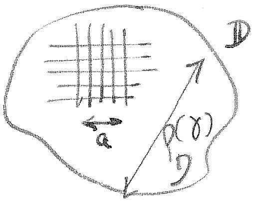
$$dX_t = d\hat{B}_t + V_t dt$$

with $\hat{B}_t$ - Brownian w.r.t. $\mathbb{E}_\mu[\dots]$

1/ SAW as an example (used to see what kind of questions we may ask, what are the problems...)

i) scaling limit

on a lattice embeded in a domain $\mathbb{D}$



SAW are simplest nearest neighbouring walk never visiting twice any lattice site

weight $w_\gamma = x^{|\gamma|}$, $|\gamma|$ = "length" of γ = # of steps

Proba is $w_\gamma / Z_\mathbb{D}$, $Z_\mathbb{D} = \sum_\gamma' x^{|\gamma|}$ partition function.

For $x = x_c$ the typical walks have macroscopique size. scaling limit as $a \to 0$. with x near x_c

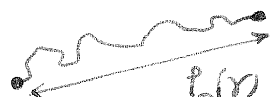
At x_c, SAW $\simeq$ SLE $8/3$ (conjecturally)

How to take the scaling limite?

Let us rewrite $Z = \sum_\gamma' x^{|\gamma|} = \sum_\gamma' x_c^{|\gamma|} (x/x_c)^{|\gamma|}$

so that $Z/Z_\mathbb{D} = \mathbb{E}[(x/x_c)^{|\gamma|}]$ with $\mathbb{E}$ the critical measure

At criticality, $|\gamma|$ is related to macroscopic size via the fractal dimension $|\gamma| \simeq [\ell_\mathbb{D}(\gamma)/a]^{d_f}$



So the weights will be finite as $a \to 0$ if $a^{-d_f} \log(x/x_c)$ finite

$\simeq \left[\left(\frac{x - x_c}{x_c}\right) \simeq -\mu\, a^{d_f} \right.$ $d_f = 1 + \kappa/8 = 4/3$ for SAW.

If we set $L_\mathbb{D}(\gamma) \simeq \ell_\mathbb{D}(\gamma)^{d_f}$, the off critical weight becom

$$\left[\; Z_\mathbb{D} = \mathbb{E}\left[e^{-\mu L_\mathbb{D}(\gamma)} \ldots \right] \right.$$

Natural parametrisation.

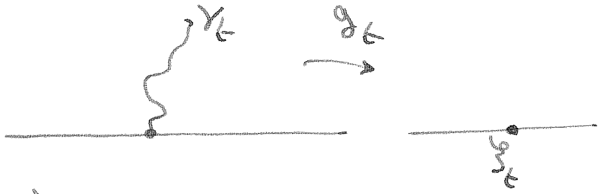
From this we can learn about the dim. of the perturbing operator

$[\mu] = d_f \quad \Rightarrow \quad [\Phi_{pert}] = 2 - d_f \quad \to \phi_{0;1}$

"Modulo" that the natural parametrisation has really been defined by mathematician, this gives a description of off-critical SAW (or SLE_κ, one of the possible perturbation)

ii) <u>how to describe it via Loewner</u> (may be not the best idea but let us try)

Eg $\mathbb{H}$ (now the domain matters)



; $\dfrac{dg_t(z)}{dt} = \dfrac{2}{g_t(z) - \xi_t}$

The source $t \to \xi_t$ parametrize the curve.
It depends on the perturbing parameter μ i.e. $\xi_t^{(\mu)}$
what are the expected properties?

- <u>short distance</u>: $\frac{1}{\lambda} g_{\lambda^2 t}(\lambda z)$ satisfies Loewner with source $\frac{1}{\lambda} \xi^{(\mu)}_{\lambda^2 t}$

  so we expect $\frac{1}{\lambda} \xi^{(\mu)}_{\lambda^2 t} \xrightarrow[\lambda \to 0]{} \sqrt{\kappa} B_t$

  We expect a stronger statement, as the curve is expected to look locally like a SLE curve (around any point along the curve) we expect
$$\frac{1}{\lambda}\left(\xi^{(\mu)}_{s+\lambda^2 t} - \xi^{(\mu)}_s\right) \xrightarrow[\lambda \to 0]{} \sqrt{\kappa}\, B_t \quad \text{for any } s.$$

- <u>decomposition</u>: from this, we expect to be able to decompose
$$\xi^{(\mu)}_t = \sqrt{\kappa}\, B_t + A_t$$
with A_t no contributing to the quadratic variation

and similarly we expect that $\xi^{(\mu)}_t$ will satisfy a stochastic eq
$$d\xi^{(\mu)}_t = \sqrt{\kappa}\, dB_t + \text{drift}$$
← coding for the off-critical perturbation.

ii) off-critical drift and partition f^{ct}

describe how drift / weighting of SLE measure related to partition f^{ct}

take again SAW. → off critical correspond to weight by the natural parametrization.

ie if O is an observable $E^{\mu}[O] = \frac{1}{Z_D} \mathbb{E}[e^{-\mu L_D} O]$

→ Imagine that O only depends on the curve up to time t, then

$$E^{\mu}[O] = \mathbb{E}[M_t O] \quad \text{with} \quad M_t = \frac{1}{Z_D} \mathbb{E}[e^{-\mu L_D} \,|\, \mathcal{F}_t]$$

ie M_t is a martingale and we are (exactly) in the framework of the Girsanov theorem.

↳ the drift term can be computed from M_t

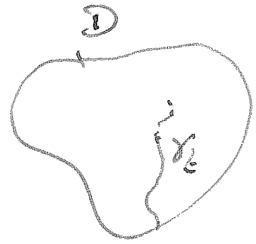
since M_t is martingale $dM_t = M_t V_t \, dB_t$

then drift (of $d\xi_t$) $= \kappa V_t \, dt = M_t^{-1} dM_t$

ie "logarithmic" derivative of M_t

→ how to compute M_t / via partition f^{ct}

M_t is link to partition f^{ct}



take again SAW: since L_D counts the number of steps we expect

$$L_D(\gamma_{[0,s]}) = L_D(\gamma_{[0,t]}) + L_{D\setminus\gamma_{[0,t]}}(\gamma_{[t,t+s]})$$

thus

$$M_t = e^{-\mu L_D(\gamma_{[0,t]})} \frac{Z_{D\setminus\gamma_t}}{Z_D}$$

"energy term"

ratio of partition functions

⇝ Generalisation eg. for Ising

3/ Off-critical SLE and field theory

from the previous discussion, the drift (à la Girsanov) is given by the logarithmic derivative of the condition partition f^{ct}. This, in general, is proportional to the partition f^{ct} in the cut-domain

$$M_t = e^{\mathcal{E}_t} \cdot Z_t = e^{\mathcal{E}_t} \left\langle \exp\left[- \int d^2z \, \mu . \phi(z,\bar{z}) \right] \right\rangle_{\mathbb{D}_t ;\, b.c}$$

to compensate for M_t to be a martingale (see eg the SAW case)

$\mathcal{E}_t$ = "surface energy term"

perturbing operator.

Naively,
this is expected a martingale because it is (naively) the ~~expec~~ CFT expectation value of an operator in $\mathbb{D}_t$

but integral, $\int d^2z$, of a z-dependant martingale is not always a martingale (see SAW and local parametrization)

To 1^{st} order, if $\mathcal{E}_t = 0$, $M_t \simeq 1 - \int d^2z \, \mu \langle \phi(z,\bar{z}) \rangle_{\mathbb{D}_t} + \dots$

The drift term (via Girsanov) is

$$F_t = M_t^{-1} dM_t = \kappa \, \partial_{\xi_t} \log\left(e^{\mathcal{E}_t} Z_t \right).$$

Rq: The drift term can also be understood as follows (cf previous lecture)

Massive expectation value have to be martingale ~~for the m~~ for the massive SLE:

$$\langle \mathcal{O} \rangle^{\mu} = \frac{\langle \mathcal{O}\, e^{-\int d^2z\, \mu \phi} \rangle_{\mathbb{D}_t; \gamma_t}}{\langle e^{-\int d^2z\, \mu \phi} \rangle_{\mathbb{D}_t; \gamma_t}} = \frac{Z_t^{\mathcal{O}} \times e^{\mathcal{E}_t}}{Z_t \times e^{\mathcal{E}_t}}$$

Namely, Z_t and $Z_t^{\mathcal{O}}$ are martingale for SLE

A (tiny) theorem in SLE shows that this ratio of SLE-martingales is a martingale for the process with the drift

$$F_t = \kappa\, \partial_{\xi_t} \log Z_t$$

which gives the same drift (assuming that there is no energy contribution) (this is called h-transformed)

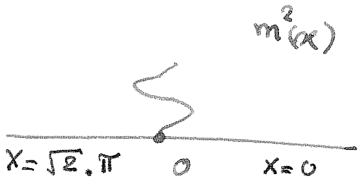
Example: $c=1$ + mass / massive free field (To appear ...)

↪ work in progress (⇒ a few détail missing)

- We perturb the $c=1$ CFT by mass term (that we assume to be point-on dependent)

$$S = \int \frac{d^2z}{8\pi}\left[(\partial X)^2 + m^2(x) X^2\right]$$

For simplicity, we consider the syst. in $\mathbb{H}$



$\simeq$ $SLE_{\kappa=4}$ but in (some kind of) perturbed environment

We choose Dirichlet boundary condition (with the critical discontinuity)
Recall that at criticality ($m^2 \equiv 0$)

$$\begin{cases} \langle X(x)\rangle_{\mathbb{H}} = \sqrt{2}\ \mathrm{Im}(\log z) & \equiv \varphi(x) \\ \langle X(x)X(y)\rangle^c_{\mathbb{H}} = G_0(z,w) = -\log\left|\frac{z-w}{z-\bar{w}}\right|^2 \end{cases}$$

The (composite) operator X^2 is defined via point-splitting (the short distance singularities are the same in the perturbed and unperturbed theories)

$$X^2(x) = \lim_{y\to x}\left(X(x)X(y) + \log|z-w|^2\right)$$

- We may described the off-critical theory à la Girsanov (as previously explained)

The drift will be described by the log. derivative of the partition f^ct

$$F_t \equiv \Pi_t^{-1} d\Pi_t = \kappa\, \partial_{g_t} \log\left(Z_t^{[m]} / Z_t^{[m=0]}\right)$$

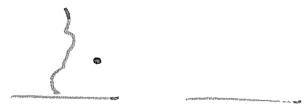
but here (for chordal) $Z_t^{[m=0]} = 1$ (with normalization choice) formally $\qquad Z_t^{[m]} = \left\langle e^{-\int \frac{d^2x}{8\pi} m^2(x) X^2(x)} \right\rangle_{\mathbb{H}_t,\, b.c.}$

- 1^{st} order computation.

  $X^2(x)$ is not a scalar under conformal transformation (because of the substraction). We find

$$\langle X^2(x) \rangle_{\mathbb{H}_t} = \underbrace{\left[\sqrt{2}\, \mathrm{Im} \log(h_t(x))\right]}_{\varphi(h_t(x))}^2 + \log \rho_t \qquad \rho_t = \frac{\mathrm{Im}\, h_t}{|h_t'|}$$

conformal radius at x



ie $\qquad Z_t = 1 - \int \frac{d^2x}{8\pi} m^2(x) \left[\varphi(h_t)^2 + \dots \right] + \dots$

Recall that $\varphi(h_t(x))$ is a martingale : $d\varphi(h_t(x)) = A(h_t(x))\, dB_t$

$\varphi(h_t(x))^2$ is not a martingale but $\log \rho_t$ is what is needed to make it a martingale.

Probably there is no energy contribution.

We get

$$Z_t^{-1} dZ_t = -\int_{\mathbb{H}_t} \frac{d^2x}{4\pi} A(h_t(x))\, m_x^2\, \varphi(h_t(x)) + \dots$$

Σ drift to 2^{nd} order

(solution of $-\Delta \varphi = 0$ ie harmonic)

- <u>All order</u> computation

Since the theory is gaussian, we can make the computation to all order

The (all order) result is / seems to be:

$$Z_t^{-1}\, dZ_t = - \int_{\mathbb{H}_t} \frac{d^2z}{4\pi}\, A(h_t(x))\, m^2_{(x)}\, \psi_{t}(x)$$

with ψ_t solution of $(-\Delta + m^2)\, \psi_t^{[m]}(x) = 0$ on $\mathbb{H}_t/_{\gamma}$

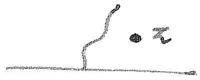
with ~~dis~~continuous Dirichlet b.c.

(There is probably no ~~other~~ surface energy contribution)

So this gives the drift to <u>all order</u> (for that perturbation)

- We are (on the way of) checking it

(1) ↳ proba for the curve to the left of a point.



From field theory argument, this is expected to be $\langle X \rangle^m$, so solution of $(-\Delta + m^2)\, P = 0$

The cond. proba should be a martingale (this has been checked, now!)

(ie $\psi_t^{[m]}$ should be martingale for that drift)

(2) ↳ this process is expected to be the scaling of variant of the (harmonic) explorer <u>with</u> killing.

LERW as an example (of off-critical SLE)

- def + weights.

$$\left\{ \begin{array}{l} \text{random walk of which you erase the loops} \\ W_\gamma = \sum_{r \downarrow \gamma} x^{|r|} \qquad \text{with } |r| = \#\text{ of step of the random walk.} \end{array} \right.$$

since $|r| \simeq a^{-2}$ for a (macroscopic) random walk, scaling limit is s.t. $a^{-2}\log(x/x_c) \simeq$ finite ie $\frac{x - x_c}{x_c} \simeq -\mu a^2$

ie perturbation by field of dimension 0 operator.

- Partition f^{ct} $Z_D = \sum_\gamma W_\gamma = \sum_r x^{|r|}$ sum of all random walks from $x_0 \to S$

In the scaling limit it becomes

$$Z_D \underset{a\to 0}{\simeq} \mathbb{E}_{\text{Brownian}}\left[e^{-\mu \tau_D} \right]$$

ie weighted by time spend the brownian in D

- Inhomogeneous generalisation and local time

change fugacity inhomogeneously in D →

$$Z_D \underset{a\to 0}{\simeq} \mathbb{E}\left[\exp\left[-\int d^2z\, \mu(z)\, \underset{\substack{\uparrow \\ \text{Brownian local time at } z}}{\ell_D(z)} \right] \right]$$

$$\ell_D(z) \stackrel{\text{def}}{=} \int^{\tau_D} ds\, \delta(z - B_s)$$

Rk It is easy to see that $\ell_D(z)$ behave like a field of dim zero

ie $\ell_D(z) \simeq \ell_{\varphi(D)}(\varphi(z))$ $\varphi : D \to \varphi(D)$

because if $\vec{B}_t$ Brownian motion in (2d),
then $\mathcal{B}_s \equiv \varphi(B_{t(s)})$ with $ds = |\varphi'(B_t)|^2 dt$ Brownian motion in 2D

- LERW, Brownian local time and symplectic fermion ($c=-2$)

↳ LERW $\equiv$ $SLE_{(\kappa=2)}$ which corresponds to $c=-2$

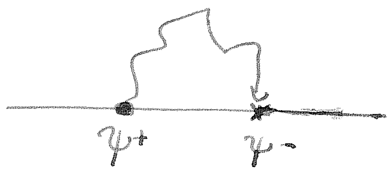
The $c=-2$ CFT is described by symplectic fermions.
with action

$$S = \int \frac{d^2x}{8\pi} \left(\partial\chi^+ \partial\chi^- \right)$$

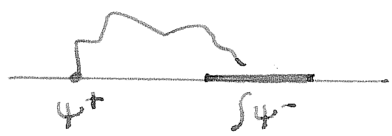
with Dirichlet b.c

$$\chi^+\big|_{\partial\mathbb{H}} = \chi^-\big|_{\partial\mathbb{H}} = 0$$

The operator which create and absorbed the curve are $\Psi^{\pm} = (\partial_y \chi^{\pm})$



If we look at dipolar geometry (curve ending on interval) we have to insert $\int_{x_-}^{x_+} dx\, \Psi^-(x)$



↳ The fugacity perturbation corresponds to massive perturbation of the symplectic fermions.

$$S = \int \frac{d^2x}{8\pi} \left(\partial\chi^+ \partial\chi^- + m^2_{(x)}\, \chi^+\chi^- \right)$$

This can be seen from a nice relation between Brownian local time and symplectic fermions:

$$\mathbb{E}_{\substack{B_0 = x_0 \\ B_{\tau_{\partial\mathbb{H}}} = x_\infty}} \left[e^{-\int d^2z\, \mu(z)\, \ell_{\mathbb{H}}(z)} \right] = \frac{\langle \Psi^+(x_0) . e^{-\int d^2z\, \mu(z) (\chi^-\chi^+)(z)} . \Psi^-(x_\infty) \rangle_{\mathbb{H}}}{\langle e^{-\int d^2z\, \mu(z) (\chi^-\chi^+)_z} \rangle_{\mathbb{H}}}$$

• Off-critical LERW drift

The off-critical drift can be computed in $\neq$ ways, (to 1st order)

i) either by computing the (logarithm derivative of the) partition f^ct

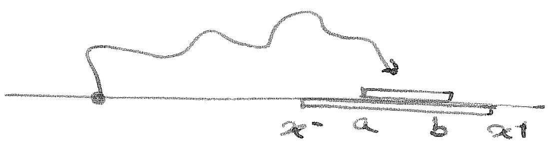
$$Z_t \simeq 1 - \int d^2z \, \mu(z) \frac{\langle \Psi^+(X_t)\, (\partial\chi^* \bar\partial\chi^+)(z) \int \Psi^- \rangle_{\mathbb{H}_t}}{\langle \Psi^+ \cdot \int \Psi^- \rangle_{\mathbb{H}_t}} + \dots$$

(There is no surface energy term.) (in the dipolar geometry)

Rk: both methods give the same result

ii) or by demanding that (some) conditioned probability are martingale. For instance

In the dipolar geometry, look for the probability that the curve hits a subinterval



From field theory, we expect this to be: equal to ratio of correlation / partition function

$$\frac{\langle \Psi^+(x_0) \int_a^b \Psi^-(y)\, dy \rangle^{[m]}}{\langle \Psi^+(x_0) \int_{x_-}^{x_+} \Psi^-(y)\, dy \rangle^{[m]}}$$

This is equal to ration of Green f^ct, $\lim_{z \to x_0} \frac{\Gamma^{[m]}_{[a,b]}(z)}{\Gamma^{[m]}_{[x_-,x_+]}(z)}$,

with $(-\Delta + m)\Gamma_{[a,b]}(z) = 0$ with Dirichlet (0/1) b.c

↳ This formula is exact as can be proved using the relation between LERW and Brownian local time

Demanding that this proba but conditioned on $\gamma_{[0,t]}$ be a martingale fixes the drift