The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

1952-10

School on Stochastic Geometry, the Stochastic Lowener Evolution, and Non-Equilibrium Growth Processes

7 - 18 July 2008

Background information ON Conformal invariance in the 2D Ising model (Discrete Harmonic and Discrete Holomorphic functions)

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DISCRETE HARMONIC AND DISCRETE HOLOMORPHIC FUNCTIONS [La Pietra '2008]

- DISCRETE HARMONIC FUNCTIONS [ON ISORADIAL GRAPHS]

  CONVERGENCE AS MESH $\longrightarrow 0$

- DISCRETE HOLOMORPHIC AND "STRONG - HOLOMORPHIC" FUNCTIONS

SQUARE LATTICE

La Pietra 2

[HAVING ISORADIAL GRAPHS IN MIND]

Harmonic measure

$$\omega^\delta(u, b^\delta a^\delta, \Omega^\delta)$$
$$\parallel$$
$$\mathbb{P}^u\{RW^\delta_I \in b^\delta a^\delta\}$$

I first hit to $\partial\Omega^\delta$

Continuous setup:

$$\omega(u, ba, \Omega)$$
$$\parallel$$
$$\mathbb{P}^u\{B_I \in ba\}$$

If $(\Omega^\delta, a^\delta, b^\delta) \to (\Omega, a, b)$

then $\omega^\delta(u, b^\delta a^\delta, \Omega^\delta)$

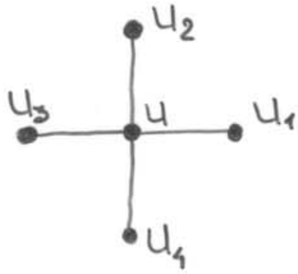
$\to \omega(u, ba, \Omega)$

- **WHY ?**

[• universal proof for different "lattices"]

[• what does "$\to$" mean ?]

Definition: [square lattice]



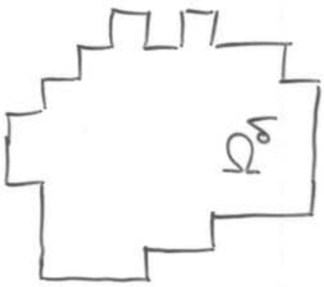
H defined on $\Gamma = \Gamma^\delta$

- $(\Delta^\delta H)(u) := \frac{1}{\delta^2} \sum_{k=1}^{4} (H(u_k) - H(u))$

  and $m^\delta(u) := \delta^2$

- H is **discrete harmonic** iff $\Delta^\delta H \equiv 0$

EXERCISE: (Dirichlet boundary value problem)



$? H : \begin{cases} \Delta^\delta H \equiv 0 & \text{inside } \Omega^\delta \\ H \equiv F & \text{on the boundary } \Omega^\delta \end{cases}$

Prove that

there exists unique solution

Hint:

Minimize discrete Dirichlet energy: $\sum_{\langle u_j u_k \rangle} (H(u_j) - H(u_k))^2$

Note:

$\omega^\delta(\cdot, b^\delta a^\delta, \Omega^\delta)$ is the solution of Dirichlet boundary value problem:

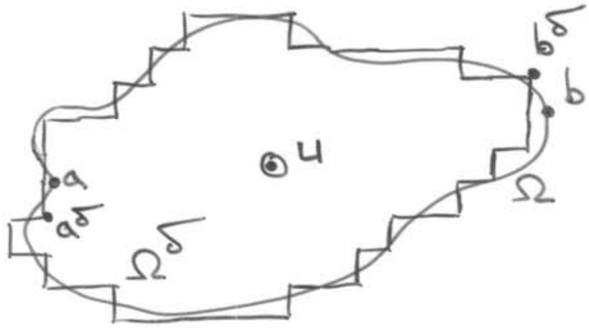
$$\begin{cases} \omega^\delta(\cdot, b^\delta a^\delta, \Omega^\delta) \equiv 0 & \text{on } a^\delta b^\delta \\ \omega^\delta(\cdot, b^\delta a^\delta, \Omega^\delta) \equiv 1 & \text{on } b^\delta a^\delta \end{cases}$$

La Pietra 4

Start with some "nice" (Hausdorf) convergence $(\Omega^\delta, a^\delta, b^\delta) \to (\Omega, a, b)$

Let

$h^\delta(\cdot) := \omega^\delta(\cdot, b^\delta a^\delta, \Omega^\delta)$



① HARNACK'S ESTIMATE

$$\left|\frac{H(u_k)-H(u)}{\delta}\right| \leq \frac{\text{const}}{\text{dist}(u, \partial\Omega^\delta)} \cdot \underbrace{\max_{\Omega^\delta} |H|}_{\leq 1}$$

$\Rightarrow \{h^\delta\}$ are uniform Lip's inside

$\Rightarrow$ PRECOMPACTNESS IN $C(K)$, $K \subset \Omega$

$\Rightarrow$ $\exists$ subsequential limit

$$h^{\delta_k} \overset{K}{\rightrightarrows} h \quad , \quad h : \Omega \to [0,1]$$

② APPROXIMATION PROPERTY

$\forall \varphi : \mathbb{R}^2 \to \mathbb{R}$ - smooth

$$\Delta^\delta(\varphi|_{\Gamma^\delta}) = (\Delta\varphi)|_{\Gamma^\delta} + O(\delta)$$

Then :

$\forall \varphi : \ \operatorname{supp} \varphi \subset \Omega$

$$\iint_\Omega h(u)(\Delta\varphi)(u)\, dm(u) = \lim_{\delta \downarrow 0} \sum_{\Omega^\delta} h(u)(\Delta\varphi)(u)\, \overbrace{m^\delta(u)}^{1/\delta^2}$$

$$= \lim_{\delta\downarrow 0} \underbrace{\sum_{\Omega^\delta} h^\delta(u)(\Delta^\delta\varphi)(u)\, m^\delta(u)}$$

$$\| \quad \text{[discrete integration by parts]}$$

$$\sum_{\Omega^\delta} (\Delta^\delta h^\delta)(u)\, \varphi(u) m^\delta(u)$$

$$\|$$

$$0$$

$\Rightarrow$ h IS HARMONIC INSIDE Ω

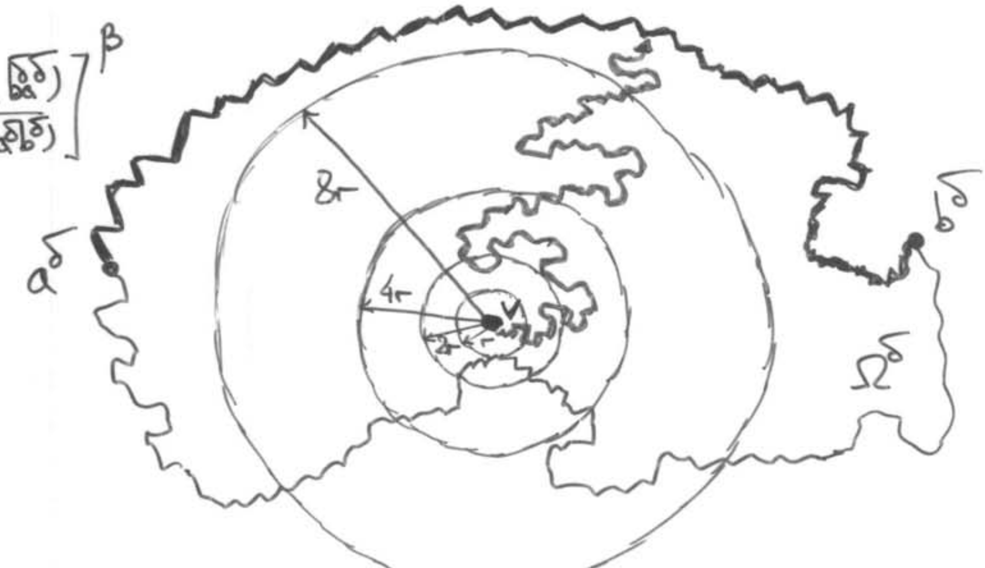
(3) WEEK BEUERLING - TYPE ESTIMATE

$$h^\delta(v) \leq \text{const.} \left[\frac{\operatorname{dist}(v, a^\delta b^\delta)}{\operatorname{dist}(v, a^\delta b^\delta)}\right]^\beta$$

for some $\beta > 0$.

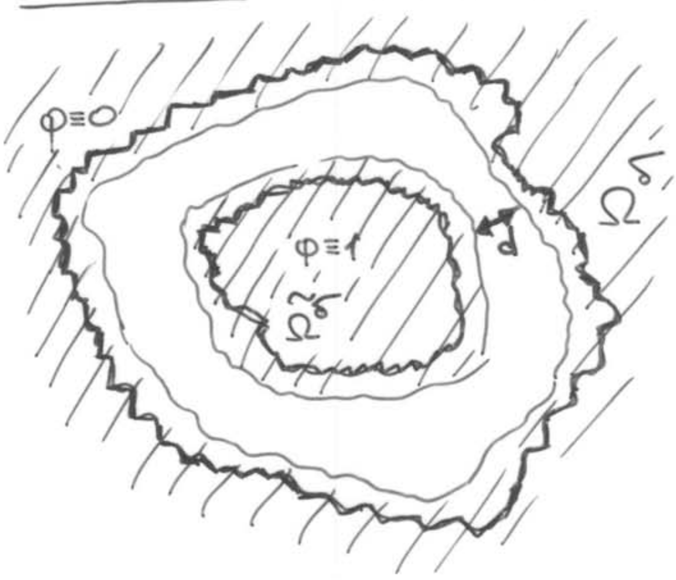
$\Uparrow$

probability to cross the annulus without touching the boundary $(a^\delta b^\delta)$ is bounded away from 1. $\Leftarrow$ APPROXIMATION PROPERTY



THUS : $$h(v) \leq \text{const.} \left[\frac{\operatorname{dist}(v, ab)}{\operatorname{dist}(v, ba)}\right]^\beta \xrightarrow[v \to ab]{} 0$$

HARNACK'S ESTIMATE:



Continuous case:

$$0 = \iint_\Omega \varphi h \underbrace{\Delta h}_{\equiv 0}$$

$$= - \iint_\Omega \nabla(\varphi h) \nabla h =$$

$$= - \iint_\Omega \varphi \|\nabla h\|^2 - \iint_\Omega \nabla\varphi \cdot \underbrace{h \nabla h}_{= \frac{1}{2}\nabla(h^2)}$$

$$\Rightarrow \iint_\Omega \varphi \|\nabla h\|^2 = \frac{1}{2} \iint_\Omega \underbrace{\Delta\varphi}_{\asymp 1/d^2} \cdot h^2$$

$$\Rightarrow \|\nabla h\|_{L^2(\tilde\Omega)} \leq \frac{\text{const}}{d} \|h\|_{L^2(\Omega)}$$

EXERCISE:

Do DISCRETIZATION of this proof

[Subtle point:
φ and ∇h are defined on different lattices]

MEAN VALUE PROPERTY FOR DISCRETE HOLOMORPHIC (GRADIENTS OF DISCRETE HARMONIC) FUNCTIONS

$$|F(v)| \leq \frac{\text{const}}{r^2} \sum_{B^\delta(v,r)} |F|$$

[Cauchy's formula together with asymptotics of Cauchy's kernel]

KENYON '03 ON ISORADIAL GRAPHS

So, if $(\Omega^\delta, a^\delta, b^\delta) \to (\Omega, a, b)$

[I say, in Hausdorf sense]

then $\omega^\delta(u, b^\delta a^\delta, \Omega^\delta) \to \omega(u, ba, \Omega)$

[or, equivalently,

$$|\omega^\delta(u, b^\delta a^\delta, \Omega^\delta) - \omega(u, b^\delta a^\delta, \Omega^\delta)| \to 0,$$

since the continuous harmonic measure is stable under $\Omega^\delta \to \Omega$]

WE NEED SOME UNIFORM ESTIMATE HERE!

CARATHEODORY CONVERGENCE:

$\mathbb{D} = \{z : |z| < 1\}$

Normalization:

$\varphi^\delta(0) = u$

$(\varphi^\delta)'(0) > 0$

[fiords beyond thin straits are OK]

- Set of all domains Ω:

  $$B(u,r) \subset \Omega \subset B(u,R) \quad [0<r<R \text{ are fixed}]$$

  is compact in the Carathéodory topology

  [so, the set of all (Ω, a, b) is compact too]

- $\omega(u, ba, \Omega)$ is continuous in the Carathéodory topology (Ω, a, b)

- Still,

  $$(\Omega^\delta, a^\delta, b^\delta) \xrightarrow{\text{Cara}} (\Omega, a, b)$$
  $$\Rightarrow \omega^\delta(u, b^\delta a^\delta, \Omega^\delta) \longrightarrow \omega(u, ba, \Omega)$$

THUS, WE ARRIVE AT

UNIFORM ESTIMATE

$$\boxed{\left|\omega^\delta(u, b^\delta a^\delta, \Omega^\delta) - \omega(u, b^\delta a^\delta, \Omega^\delta)\right| \leq \varepsilon(\delta) \xrightarrow[\delta \downarrow 0]{} 0}$$

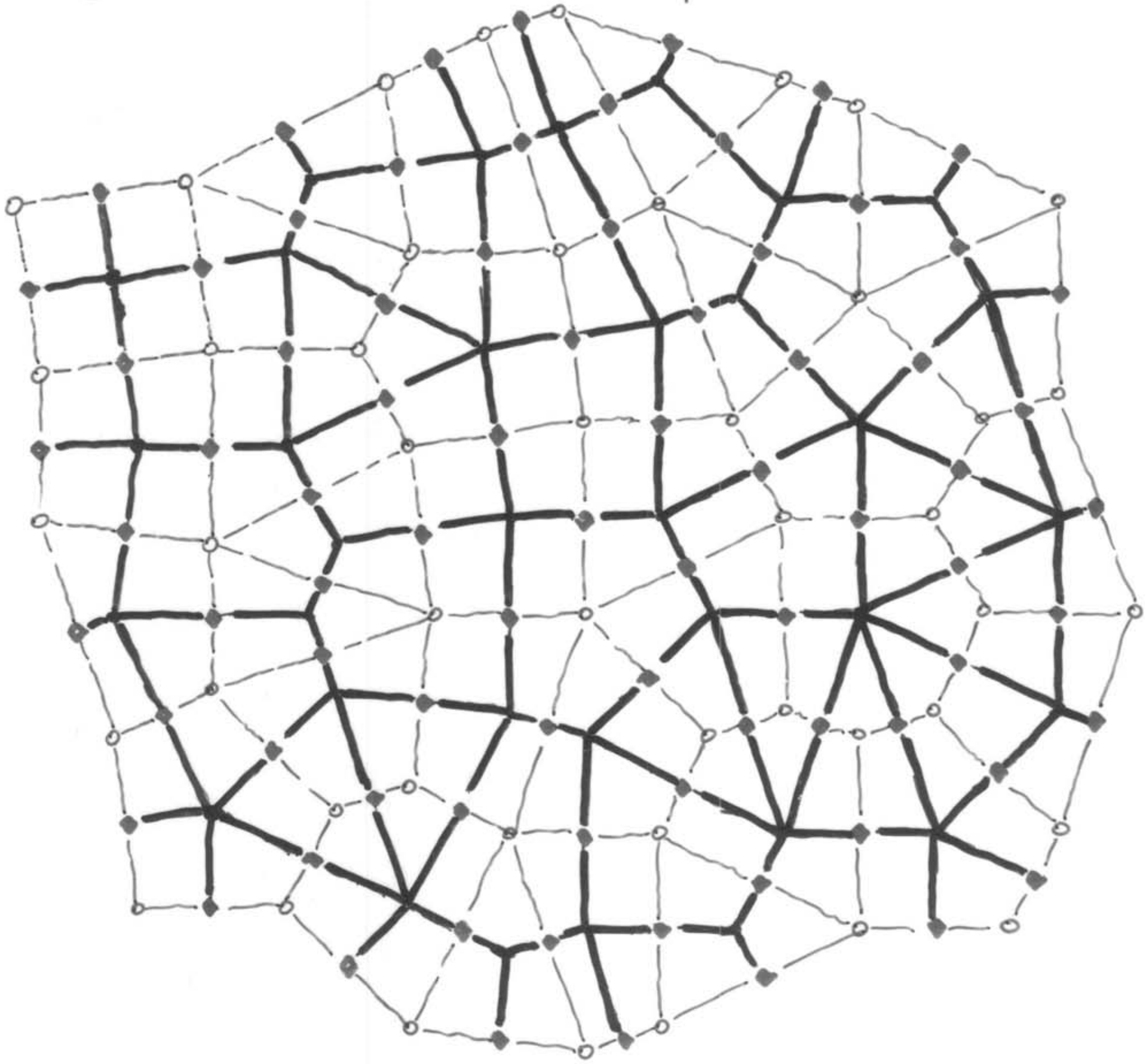
ON ARBITRARY ISORADIAL GRAPHS

[δ = common radius of circles]

ISORADIAL GRAPH:

La Pietra Ga

[dual vertices = centers of circles]

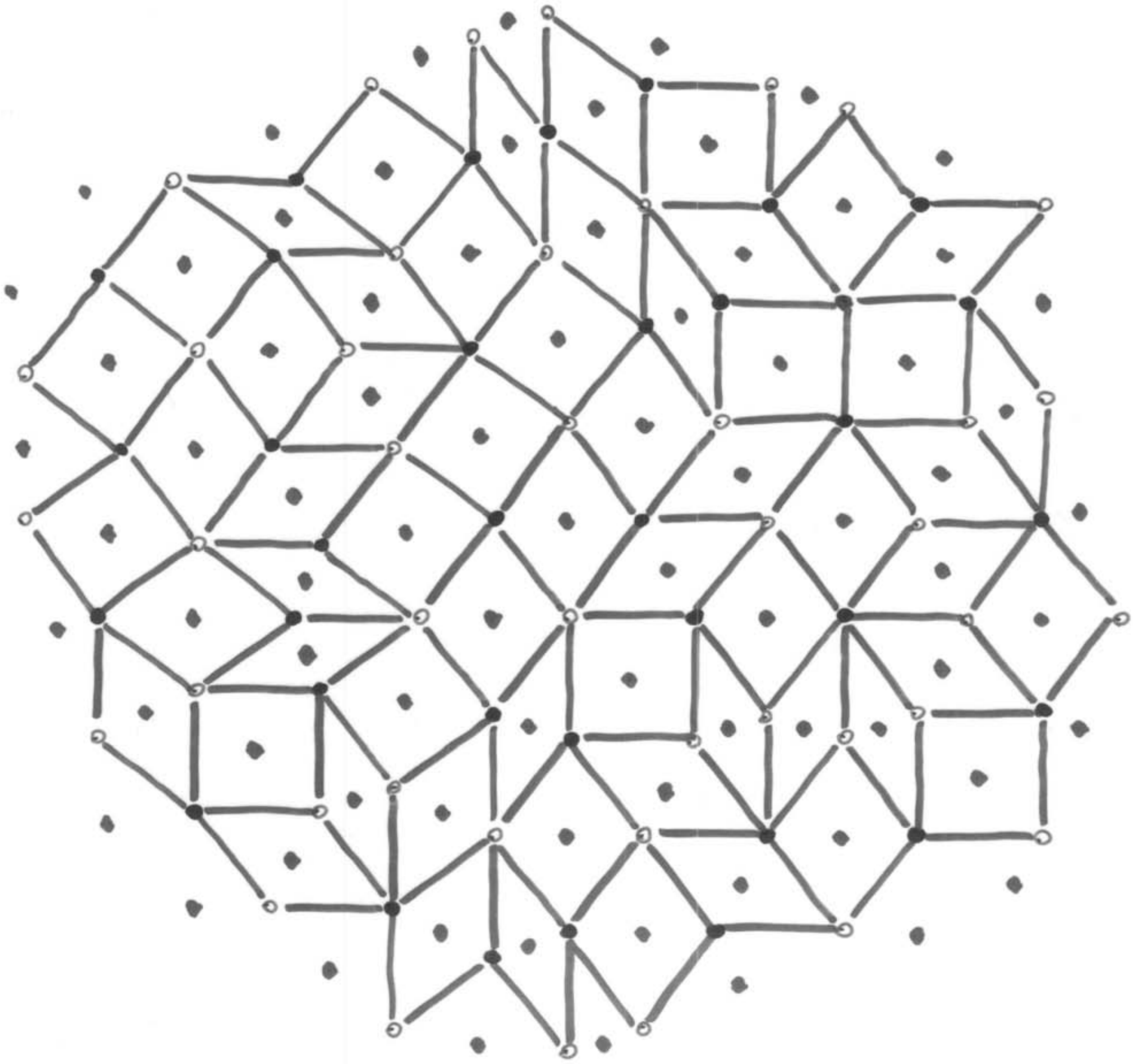


G (black) is the original (spin) lattice

G^* (white) is the dual lattice

$\Lambda = G \cup G^*$

La Pietra 9b

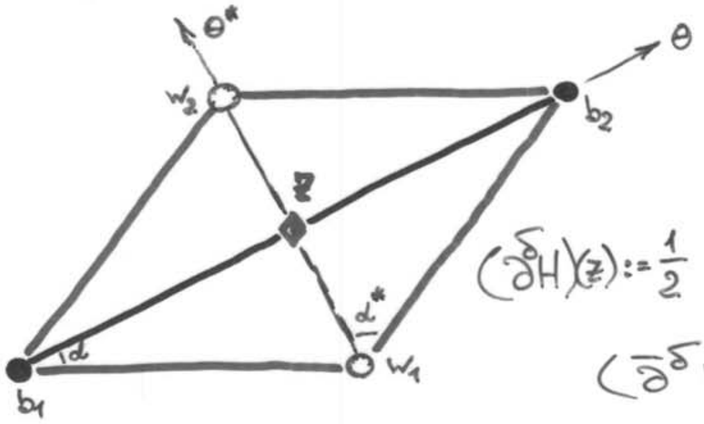


◊ = set of • = "diamond" lattice

ISORADIAL GRAPHS

La Pietra '06

Diff. operators, weights:

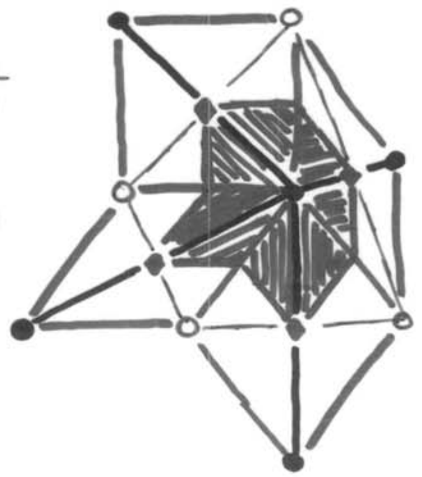
① 

$\partial^\delta, \bar\partial^\delta : \mathcal{F}(\Lambda) \to \mathcal{F}(\diamond)$

$$(\partial^\delta H)(z) := \frac{1}{2}\left[\frac{H(b_2)-H(b_1)}{b_2-b_1} + \frac{H(w_2)-H(w_1)}{w_2-w_1}\right]$$

$(\bar\partial^\delta H)(z)$ similarly

② $m^\delta_\diamond(z) = \delta^2 \sin 2\alpha = \delta^2 \sin 2\alpha^* =$ area of rhombus

$m^\delta_\Lambda(u) =$ area of  $= \frac{\delta^2}{4}\sum \sin 2\alpha_s$

③ $\partial^\delta, \bar\partial^\delta : \mathcal{F}(\diamond) \to \mathcal{F}(\Lambda)$

$\partial^\delta = -(\bar\partial^\delta)^*$, $\bar\partial^\delta = -(\partial^\delta)^*$

④ $\Delta^\delta : \mathcal{F}(\Lambda) \to \mathcal{F}(\Lambda)$

[in fact, $\mathcal{F}(\Gamma) \to \mathcal{F}(\Gamma)$]

$$\Delta^\delta = 4\partial^\delta\bar\partial^\delta = 4\bar\partial^\delta\partial^\delta = \frac{1}{2m^\delta_\Gamma(u)}\sum \tan\alpha_s \cdot (H(u_s)-H(u))$$

relative probabilities for RW on the isoradial graph

Remark:

$\partial^\delta\bar\partial^\delta \neq \bar\partial^\delta\partial^\delta : \mathcal{F}(\diamond) \to \mathcal{F}(\diamond)$

NO NATURAL DEFINITION OF HARMONIC FUNCTIONS ON $\diamond$

[square lattice is simpler]

SIMILARLY,

WE PROVE THE UNIFORM ESTIMATE

$$\left| P^\delta(v, a^\delta, \Omega^\delta) - P(v, a^\delta, \Omega^\delta) \right| \leq \varepsilon(\delta) \xrightarrow[\delta \downarrow 0]{} 0$$

DISCRETE POISSON KERNEL

CONTINUOUS POISSON KERNEL

(BOUNDARY VALUES: 0 on $\partial\Omega^\delta \setminus a^\delta$
>0 at a^δ

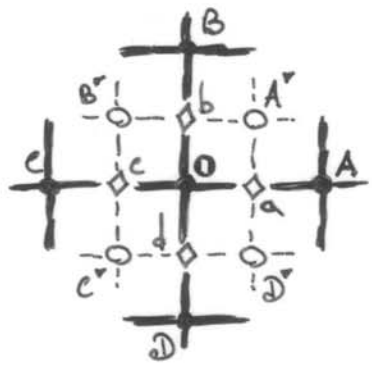
NORMALIZATION: $P^\delta(u, a^\delta, \Omega^\delta) = 1$
FOR SOME FIXED $u \in \Omega^\delta$)

IT GIVES UNIVERSALITY OF LERW ON ISORADIAL GRAPHS

[see "core argument" in Lawler - Schramm - Werner]

ALSO, UST and HE are OK.

DISCRETE HOLOMORPHIC FUNCTIONS (square lattice for simplicity)



① H is defined on •

$$F := \partial^\delta H = \tfrac{1}{2}(\partial_x - i\partial_y)^\delta H$$

$$F(a) = \frac{1}{2\delta}\left(H(A) - H(O)\right) \in \mathbb{R}$$

$$F(c) = \frac{1}{2\delta}\left(H(O) - H(C)\right) \in \mathbb{R}$$

$$F(b) = -\frac{i}{2\delta}\left(H(B) - H(O)\right) \in i\mathbb{R}$$

$$F(d) = -\frac{i}{2\delta}\left(H(O) - H(D)\right) \in i\mathbb{R}$$

Exercise:

CHECK H is harmonic at O (i.e. $(\Delta^\delta H)(O) = 0$)

if and only if

$$\boxed{F(b) - F(d) = i\left(F(a) - F(c)\right)}$$

DISCRETE CAUCHY-RIEMANN EQUATION

② $\tilde{H}$ is defined on ∘

$$F(a) = -\frac{i}{2\delta}\left(\tilde{H}(A') - \tilde{H}(D')\right) \in i\mathbb{R}$$

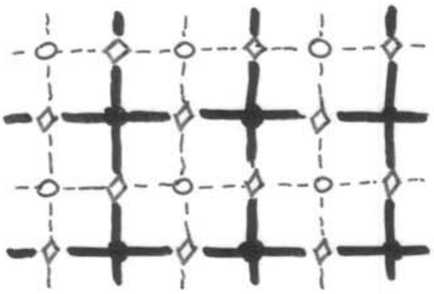
$$F(c) = -\frac{i}{2\delta}\left(\tilde{H}(B') - \tilde{H}(C')\right) \in i\mathbb{R}$$

$$F(b) = \frac{1}{2\delta}\left(\tilde{H}(A') - \tilde{H}(B')\right) \in \mathbb{R}$$

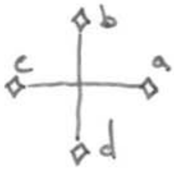
$$F(d) = \frac{1}{2\delta}\left(\tilde{H}(D') - \tilde{H}(C')\right) \in \mathbb{R}$$

$\Rightarrow$ DISCRETE C-R AT • IS ALWAYS TRUE

Definition:



Function $F : \diamond \to \mathbb{C}$ is DISCRETE HOLOMORPHIC if it satisfies



$$F(b) - F(d) = i\,(F(a) - F(c))$$

everywhere.

REMARKS:

- F is hol. $\Rightarrow$ both $\mathcal{R}F := \begin{cases} \mathrm{Re}\,F & \text{on} \\ \mathrm{Im}\,F & \text{on} \end{cases}$

  $\mathcal{I}F := \begin{cases} \mathrm{Im}\,F & \text{on} \\ \mathrm{Re}\,F & \text{on} \end{cases}$

  are discrete holomorphic

- Similar (but more complicated) ON ISO-GRAPHS

- F, G are discrete hol. $\not\Rightarrow$ FG is discrete hol.

EXERCISE:

Check that F is discrete holomorphic if and only if

DISCRETE INTEGRAL

$\int^{\delta} F(z)\, d^{\delta} z$ is WELL-DEFINED

[i.e. doesn't depend on the path of integration]

[SEPARATELY ON Γ AND Γ^*]

original lattice •

dual lattice ○

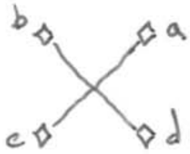
Cauchy-Riemann on Γ^*

Cauchy-Riemann on Γ

La Pietra 13

"STRONG - HOLOMORPHIC" FUNCTIONS

DISCRETE CAUCHY-RIEMANN EQUATION



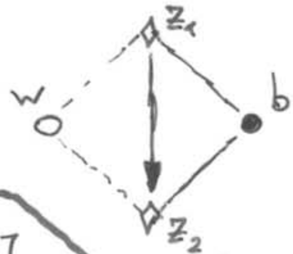
$$F(b) - F(d) = i\,(F(a) - F(c))$$

Remark: 1 complex equation per • or ○

DEFINITION:

We call F S-HOLOMORPHIC,

if $\forall\, z_1 \sim z_2$:



$\parallel e^{+i\frac{\pi}{4}}$ on the picture

$$\boxed{\Pr\left[F(z_1) ; \tfrac{1}{\sqrt{i(w-b)}}\right] = \Pr\left[F(z_2) ; \tfrac{1}{\sqrt{i(w-b)}}\right]}$$

Remark:

1 real equation per ◇—◇ :