The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

1952-12

School on Stochastic Geometry, the Stochastic Lowener Evolution, and Non-Equilibrium Growth Processes

7 - 18 July 2008

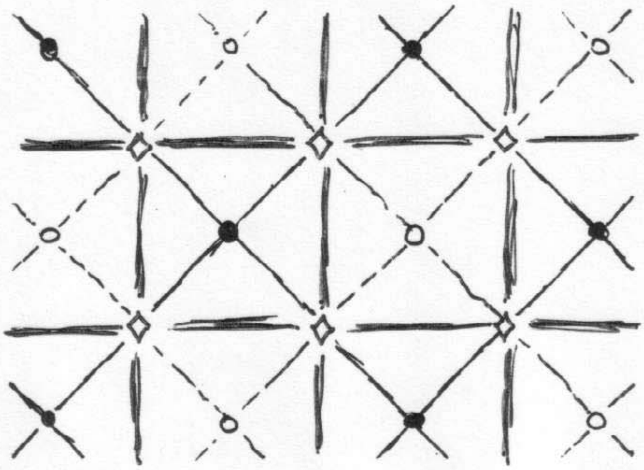
Background information ON Conformal invariance in the 2D Ising model (further notes)

Stanislav SMIRNOV

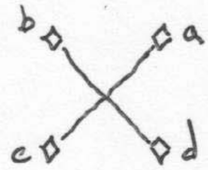
Universite' de Geneve
Section de Mathematiques
CH-1211 Geneve
Switzerland

Strada Costiera 11, 34014 Trieste, Italy - Tel. +39 040 2240 111; Fax +39 040 224 163 - sci_info@ictp.it, www.ictp.it

"STRONG - HOLOMORPHIC" FUNCTIONS



DISCRETE CAUCHY-RIEMANN EQUATION



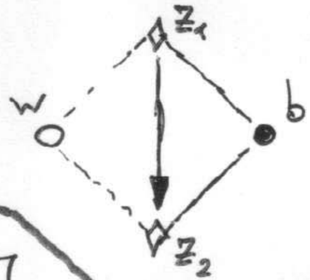
$$F(b) - F(d) = i\,(F(a) - F(c))$$

Remark: 1 complex equation per • or ○

DEFINITION:

We call F S-HOLOMORPHIC,

if $\forall\, z_1 \sim z_2$:

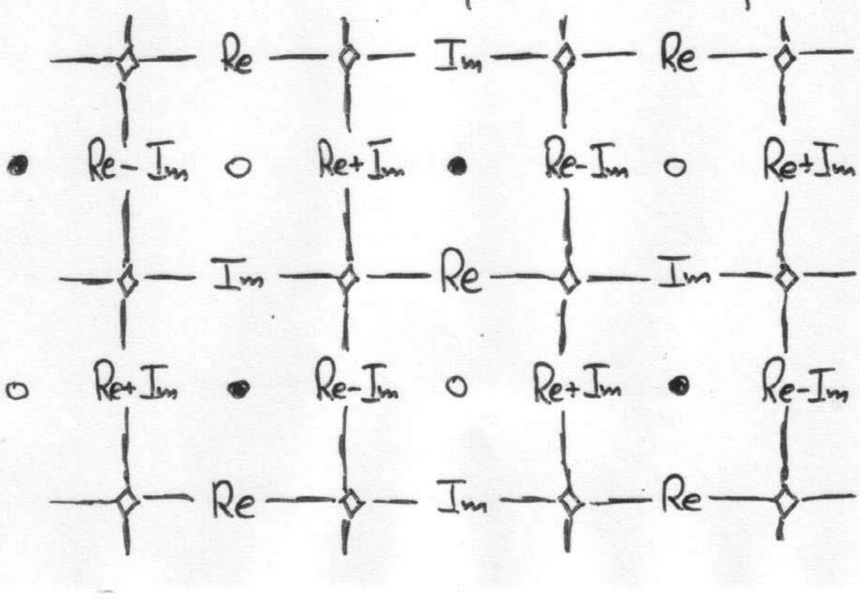


$\parallel e^{+i\frac{\pi}{4}}$ on the picture

$$\mathrm{Pr}\left[F(z_1)\,;\ 1/\sqrt{i(w-b)}\right] = \mathrm{Pr}\left[F(z_2)\,;\ 1/\sqrt{i(w-b)}\right]$$

Remark:

1 real equation per ◇—◇ :

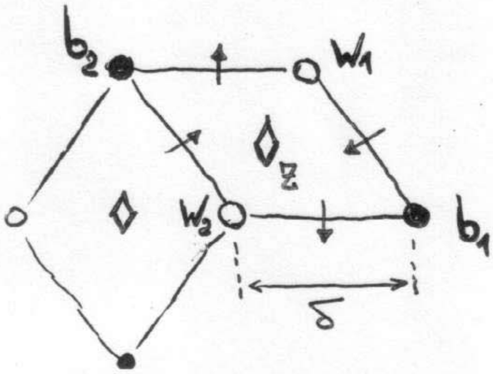


EXERCISES:

① F - δ-HOL $\Rightarrow$ F - HOL

② F - δ-HOL $\Rightarrow$ $H := \mathrm{Im} \int^{\delta} (F(z))^2 \, d^{\delta} z$ is well-defined

separately on •'s and o's:



$$H(b_2) - H(b_1) := \mathrm{Im}\left[(F(z))^2 \cdot (b_2 - b_1)\right]$$

$$H(w_2) - H(w_1) := \mathrm{Im}\left[(F(z))^2 \cdot (w_2 - w_1)\right]$$

③ H can be defined simultaneously on •'s and o's:

for each pair $w \sim b$

$$H(b) - H(w) := 2\delta \left| \mathrm{Pr}\left[F(z); \frac{1}{\sqrt{i(w-b)}}\right]\right|^2$$

ⓐ well-defined:

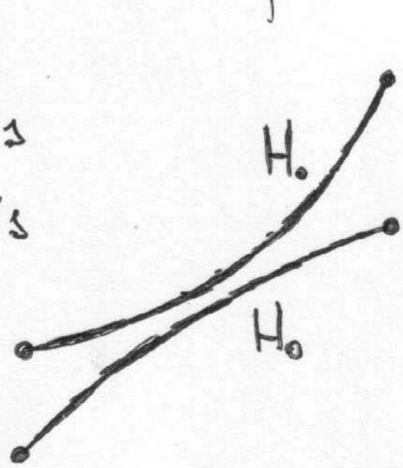
$$(H(b_2) - H(w_2)) + (H(b_1) - H(w_1)) = (H(b_2) - H(w_1)) + (H(b_1) - H(w_2))$$

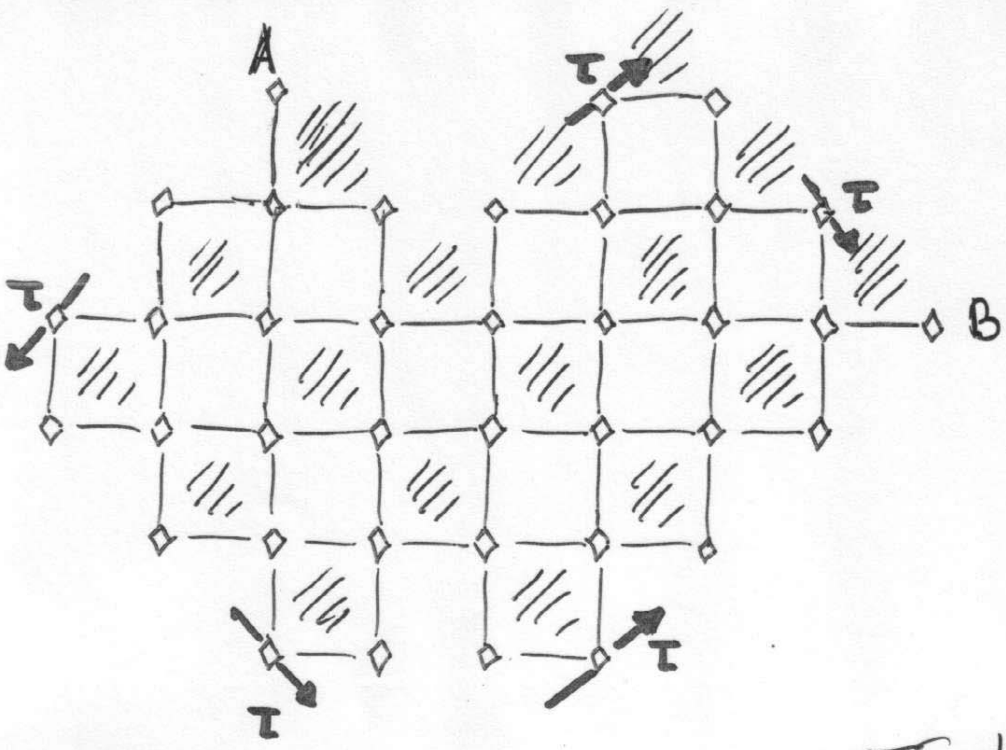
Hint: " $2\delta |F(z)|^2$ "

ⓑ compatible with standard definition

④ H is subharmonic on •'s and superharmonic on o's

[i.e., $\Delta^{\delta} H_{\bullet} \geq 0$
$\Delta^{\delta} H_{\circ} \leq 0$]





BOUNDARY CONDITIONS

$F(z) \uparrow\uparrow 1/\sqrt{\tau}$

⇓

$H_\circ \equiv \text{const}$ on AB

$H_\bullet \equiv \text{const}$ on BA

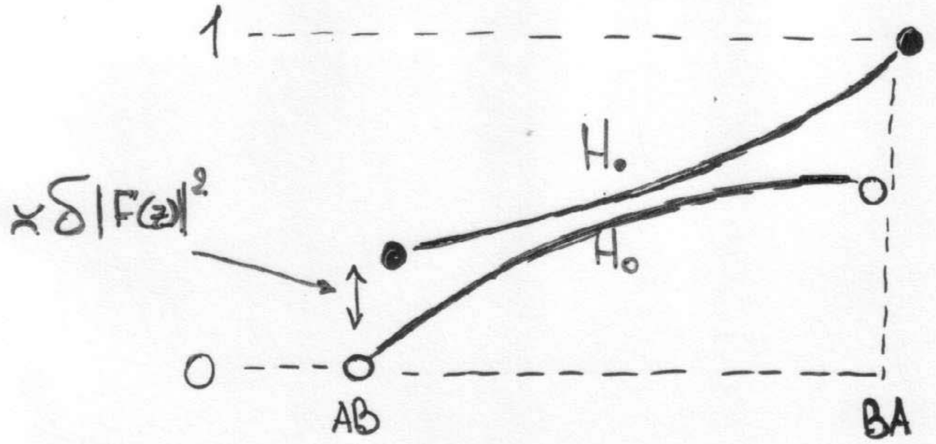
NORMALIZATION

$|F(A)| = F(B) = 1/\sqrt{2\delta}$

⇒

$H_\circ \equiv 0$ on AB

$H_\bullet \equiv 1$ on BA



REMARK:

ON THE SQUARE LATTICE magnetisation estimates are available

⇐

estimate of $|F(z)|$ on the boundary $\Longrightarrow$ $H \to \omega(\cdot, BA)$

PROPOSITION:

$H = \operatorname{Im} \int F^2 dz$

$$\left.\begin{array}{l} F \text{ - } \delta\text{-HOL IN } \Omega \\ |H_\bullet|, |H_\circ| \leq M \quad \text{IN } \Omega \end{array}\right\} \Rightarrow$$

$$\Rightarrow \boxed{|F(z)|^2 \leq \frac{\text{const}}{\text{dist}(z, \partial\Omega)} \cdot M}$$

Remark: The same as if F^2-hol.

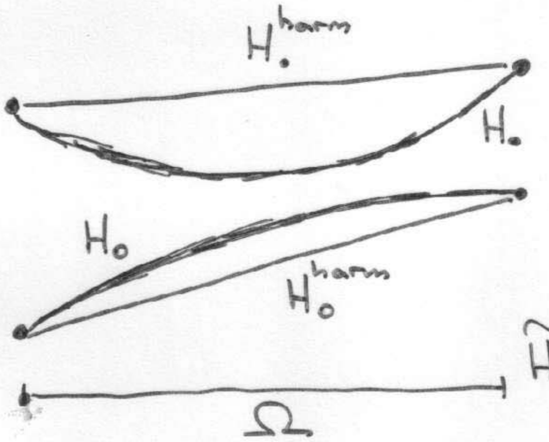
[In particular,

$$|H_\bullet - H_\circ| \leq \text{const} \cdot \frac{\delta}{d} M$$

d ← dist to $\partial\Omega$]

SKETCH OF PROOF:

Let $\Omega = B(z, d)$ [some nice approximation]



$$H_\bullet = H_\bullet^{harm} - \widetilde{H}_\bullet$$

$$H_\circ = H_\circ^{harm} + \widetilde{H}_\circ$$

$\widetilde{H}_\circ, \widetilde{H}_\bullet$: superharmonic

$= 0$ on $\partial\Omega$

$\leq 2M$ inside

(1) $$\widetilde{H}_\bullet = \iint_\Omega^\delta G_\Omega(\cdot, u)(\Delta^\delta \widetilde{H}_\bullet)(u)\, dm^\delta(u)$$

$$\widetilde{H}_\bullet \leq 2M \Rightarrow \boxed{\|\Delta^\delta \widetilde{H}_\bullet\|_{L^1(\Omega^r)} \leq \text{const} \cdot M}$$

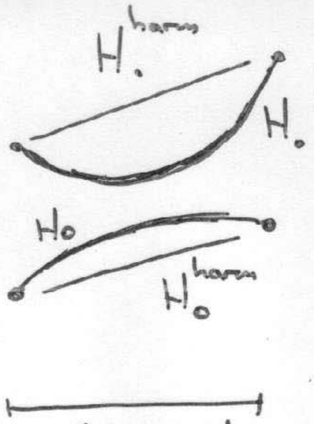
$\Omega^r = B(z, d/2)$

"Proof":

$u \in B(z, d/2) \Rightarrow$

$\Rightarrow \|G_\Omega(\cdot, u)\|_{L^1(\Omega)} \geq \text{const} \cdot d^2$

② Consider Ω' :



$H_\circ = H_\circ^{harm} - \widetilde{H}_\circ$

$H_o = H_o^{harm} + \widetilde{H}_o$

$\Omega' = B(z, d/2)$

$$\|\Delta^\delta \widetilde{H}\|_{L^1(\Omega')} \leq \text{const}$$

[Note: $\Delta^\delta \widetilde{H} = \Delta^\delta H$ doesn't depend on Ω]

$$\nabla \widetilde{H} = \iint_{\Omega'}^{\delta} \nabla G_{\Omega'}(\cdot, u)(\Delta^\delta \widetilde{H})(u)\, dm^\delta(u)$$

$$\Rightarrow \|\nabla \widetilde{H}\|_{L^1(\Omega'')} \leq \iint_{\Omega'}^{\delta} \underbrace{\|\nabla G_{\Omega'}(\cdot, u)\|_{L^1(\Omega'')}}_{\leq \text{const}\cdot d} |(\Delta^\delta \widetilde{H})(u)|\, dm^\delta(u)$$

So,

$$\boxed{\|\nabla \widetilde{H}\|_{L^1(\Omega'')} \leq \text{const}\cdot d \cdot M}$$

[USE ASYMPTOTICS OF GREEN'S FUNCTION [KENYON]]

③ BY HARNACK'S ESTIMATE THE SAME HOLDS FOR H^{harm} :

$$\|\nabla H^{harm}\|_{L^1(\Omega'')} \leq \text{const}\cdot d\, \|\nabla H^{harm}\|_{L^2(\Omega'')}$$

$$\leq \text{const}\, \|H^{harm}\|_{L^2(\Omega)} \leq \text{const}\cdot d \cdot M$$

$$\|\nabla H\|_{L^1(\Omega'')} = \|F^2\|_{L^1(\Omega'')}$$

$$= \left(\|F\|_{L^2(\Omega'')}\right)^2 \leq \text{const} \cdot d \cdot M$$

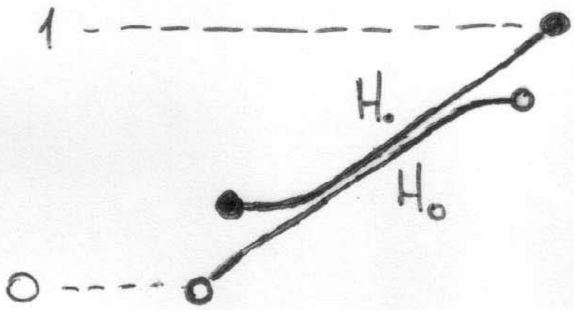
MEAN VALUE PROPERTY
FOR HOLOMORPHIC FUNCTIONS:

$$|F(z)|^2 \leq \frac{\text{const}}{d^2} \iint^{\delta} |F(z)|^2 \leq \frac{\text{const}}{d} \cdot M$$

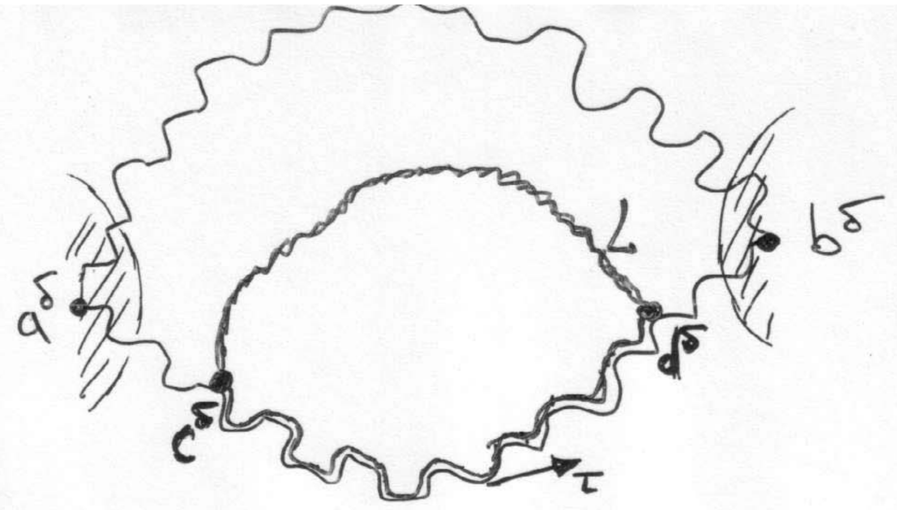
☒

— " ——— " ——— " ——— " ——— " ——— " ———

In particular, $H_\cdot$ and H_0 are uniformly close to each other inside Ω:



We still need to control $|F(z)|$ on the boundary.



CONSIDER $\displaystyle\oint_L^\delta (F^\delta(z))^2\, d^\delta z$

① Boundary conditions $F(z) \uparrow\uparrow 1/\sqrt{\tau}$

$$\Rightarrow \int_{c^\delta d^\delta}^\delta (F^\delta(z))^2\, d^\delta z = \int_{c^\delta d^\delta}^\delta |F^\delta(z)|^2\, |d^\delta z|$$

② $|F^\delta(z)|^2 \leq \text{const} \cdot \frac{1}{d}$, $d = \text{dist}(z, \partial\Omega^\delta)$

$$\Rightarrow \left| \int^\delta_{\substack{\text{inner}\\ \text{part of } L}} (F^\delta(z))^2\, d^\delta z \right| \leq \text{const} \cdot |\log\delta|$$

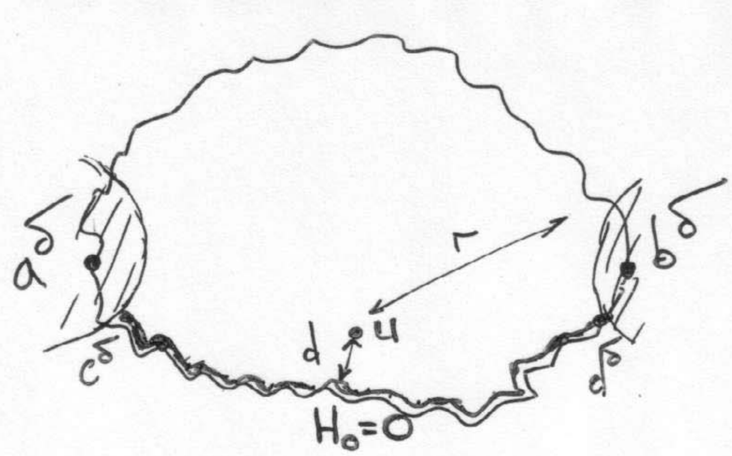
③ FORTUNATELY,

$$\int_L^\delta (F^\delta(z))^2\, d^\delta z = -\sum (\Delta^\delta H_\bullet)(b)\, m^\delta(b) \leq 0$$

$\sum$ over black vertices inside L

THUS:

$$\boxed{\int_{c^\delta d^\delta}^\delta |F^\delta(z)|^2\, |d^\delta z| \leq \text{const} \cdot |\log\delta|}$$



very rough bound, but WE DIDN'T USE "EXTERNAL" INFORMATION

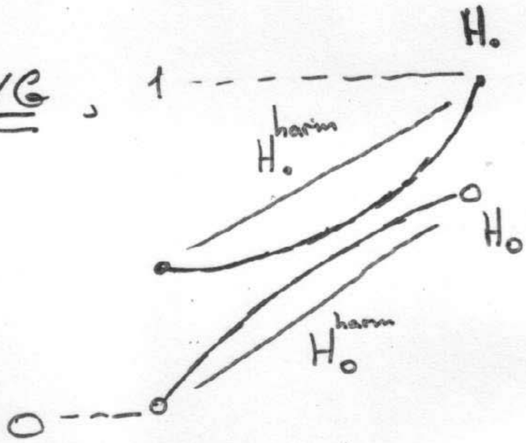
$$\underbrace{\int_{c^\delta d^\delta} |F^\delta(z)|^2 \, |d^\delta z|}_{\asymp} \le \text{const}\cdot|\log\delta|$$

(✓) ← ?

$$\frac{1}{2\delta}\int_{c^\delta d^\delta} H_\bullet(z)\,|d^\delta z| \asymp \sum_{c^\delta d^\delta} H_\bullet .$$

↑ closest to the boundary

APPLYING WEAK - BEUERLING ,



$$H_\bullet^{harm}(u) \le \text{const}\cdot \underbrace{\frac{\delta^\beta |\log\delta|}{d^\beta}}_{\downarrow\, \delta\downarrow 0 \;\; 0}$$

influence of $a^\delta b^\delta$

$$+ \; \text{const}\cdot \underbrace{\left(\frac{d}{r}\right)^\beta}_{\downarrow\, d\to 0 \;\; 0}$$

influence of $b^\delta a^\delta$ and neighborhoods of a^δ, b^δ

COMPACTNESS ARGUEMENTS

GIVE THE CONVERGENCE

OF $F^\delta(z)$ TO $\sqrt{\varphi'(z)}$