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Conformally invariant random simple curves on Riemann surfaces (Conformally invariant restriction measures on Riemann Surfaces III: Theta functions and their application)

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CONFORMALLY INVARIANT RESTRICTION MEASURES ON RIEMANN SURFACES III: THETA FUNCTIONS AND THEIR APPLICATION

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Notes for lectures delivered at the ICTP School on Stochastic Geometry, the Stochastic Loewner Evolution and Non-Equilibrium Growth Processes, 7–18 July 2008.

1. Riemann's theta function

Our notation follows J. Fay, "Theta functions on Riemann surfaces."

Let $z \in \mathbb{C}^g$, τ a symmetric $g \times g$-matrix with complex entries, whose real part is negative definite, $\Re\tau < 0$. Denote Γ the lattice generated by the columns of the $g \times 2g$-matrix $(2\pi i I, \tau)$. Here I is the $g \times g$ identity matrix. Then, if $e \in \mathbb{C}^g$, there exists a unique pair of vectors $\delta, \epsilon \in \mathbb{R}^g$ so that

$$e = (\epsilon, \delta) \begin{pmatrix} 2\pi I \\ \tau \end{pmatrix}.$$

$\begin{bmatrix} \delta \\ \epsilon \end{bmatrix}$ are called the *characteristics* of e and we write

$$e = \{{}^{\delta}_{\epsilon}\}_\tau, \quad [e] = \begin{bmatrix} \delta \\ \epsilon \end{bmatrix}.$$

Riemann's theta function is

$$\theta(z) = \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z) = \sum_{m \in \mathbb{Z}^g} \exp\left(\frac{1}{2} m\tau m^t + mz^t\right), \quad z \in \mathbb{C}^g.$$

Since $\Re\tau < 0$, this sum converges absolutely and uniformly on compacts, defining an entire function of τ and z. The theta function with characteristics $\begin{bmatrix} \delta \\ \epsilon \end{bmatrix}$ is defined by

$$\begin{aligned} \theta_\tau \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} (z) &= \exp\left(\frac{1}{2}\delta\tau\delta^t + (z + 2\pi i\epsilon)\delta^t\right) \theta(z + e) \\ &= \sum_{m \in \mathbb{Z}^g} \exp\left(\frac{1}{2}(m+\delta)\tau(m+\delta)^t + (z + 2\pi i\epsilon)(m+\delta)^t\right). \end{aligned}$$

It is quasi-periodic for the lattice Γ:

$$\theta \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} (z_1, \ldots, z_j + 2\pi i, \ldots, z_g) = \exp(2\pi i \delta_j)\ \theta \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} (z),$$

and

$$\theta \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} (z_1 + \tau_{j1}, \ldots, z_g + \tau_{jg}) = \exp(-\frac{1}{2}\tau_{jj} - z_j - 2\pi i \epsilon_j)\ \theta \begin{bmatrix} \delta \\ \epsilon \end{bmatrix} (z).$$

If $2\alpha, 2\beta \in \mathbb{Z}^g$, then $\{{}^{\alpha}_{\beta}\} \in \mathbb{C}^g$ is called a half-period. Half-periods are even or odd if $4\ \alpha \cdot \beta \equiv 0$ or $1 \pmod 2$. For our purposes it is enough to consider reduced half-periods, where $\alpha_i, \beta_i \in \{0, 1, 2\}$.

Example 1. If $g = 1$, then there are three even (reduced) half-periods

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$

and one odd half-period

$$\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}.$$

It is straightforward to check that $\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is an even/odd function of z if its characteristics are even/odd.

Denote C a compact Riemann surface of genus $g > 0$ (number of handles). Fix a canonical homology basis $A_1, \ldots, A_g$, $B_1, \ldots, B_g$, i.e. a basis such that the intersection form is given by

$$\#(A_i, A_j) = \#(B_i, B_j) = 0, \quad \#(A_i, B_j) = \delta_{ij}.$$

Associated to this basis is a basis $v_1, \ldots, v_g$ for the space of holomorphic differentials, normalized by

$$\frac{1}{2\pi i} \int_{A_j} v_k = \delta_{jk}.$$

If we set

$$\tau_{jk} = \int_{B_j} v_k,$$

then $\tau_{jk} = \tau_{kj}$ and $\Re\tau < 0$, by the Riemann bilinear relations. Define

$$J(C) = \mathbb{C}^g/(2\pi i I, \tau) \qquad \text{Jacobian of } C.$$

For $x, y \in C$, set

$$\int_x^y v = \left(\int_x^y v_1, \ldots, \int_x^y v_g \right) \in \mathbb{C}^g.$$

The value of this integral depends on the path of integration. If the path is altered, then the integral is changed by a period. Thus

$$\int_x^y v/(2\pi i I, \tau) \in J(C)$$

is independent of the path. For fixed x we get a holomorphic map from C into $J(C)$ (Abel map). Consider

$$y \mapsto \theta[\alpha] \left(\int_x^y v \right). \tag{1}$$

This map is locally single-valued, globally multi-valued, but its zeros are well defined. Suppose that $[\alpha] = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix}$ is an odd half-period which is non-singular, i.e $\theta[\alpha]$ is not identically zero. Non-singular half-periods exist for every τ and it can be shown that then the map given in (1) has zeros for $y = x$ and $y = p_1, \dots, p_{g-1}$ and that the holomorphic differential

$$h_\alpha^2 \equiv \sum_{j=1}^{g} \frac{\partial \theta[\alpha]}{\partial z_j}(0) v_j(x)$$

has $g-1$ double zeros at $p_1, \dots, p_{g-1}$ and no others. Define

$$E(x,y) = \frac{\theta[\alpha]\left(\int_x^y v\right)}{h_\alpha(x) h_\alpha(y)} \quad \text{Prime Form.}$$

The prime form has its only zero at $x = y$ and it is of order 1. It plays on C the role $x - y$ plays on $\mathbb{C}$.

Example 2. (i) If f is a meromorphic function on C with zeros at $a_1, \dots, a_n$ and poles at $b_1, \dots, b_n$ (counted with multiplicities), then

$$f(x) = const. \prod_{j=1}^{n} \frac{E(x, a_j)}{E(x, b_j)}.$$

(ii) If ω_{a-b} is the differential of the third kind with zero A-periods, a single pole at a with residue $= 1$ and a single pole at b with residue -1, and regular everywhere else, then

$$\omega_{a-b}(x) = d_x \log \frac{E(x,a)}{E(x,b)}.$$

(iii) Construction of differentials of the second kind. Set

$$\omega(x,y) = d_x d_y \log E(x,y)$$

$\omega(x,y)$ is a bilinear differential. If we fix a local uniformizer ζ at $y = a$, then ω restricts to a linear differential

$$\omega_{a,\zeta}(x)$$

which has zero A-periods, a double pole at a, and is regular everywhere else.

2. An application to SLE$_{8/3}$

Consider the Schottky-double of a multiply connected domain D, with analytic boundary components $\Gamma_0, \dots, \Gamma_n$. Choose a canonical homology basis $A_1, \dots, A_n, B_1, \dots, B_n$, with $A_k = \Gamma_k, k = 1, \dots, n$. If $v_1, \dots, v_n$ are the normalized differentials, then the period matrix

$$\tau = \left(\int_{B_j} v_k \right)$$

is a *real* $n \times n$-matrix, and the normalized differential of the second kind, $\omega(x, y)$ is real on Γ_k. Choose a uniformizer at $q \in \Gamma_0$. Then $\int \omega_q(x)$ maps D to $\mathbb{C} \cup \{\infty\}$, so that

$$q \mapsto \infty, \quad \Gamma_0 \backslash \{q\} \to \mathbb{R}, \quad \Im(\text{image of } \Gamma_k) = const.$$

That is the image of D is the upper half-plane with n horizontal slits. The following example is a very rough sketch, some of the constants will need to be adjusted, but the main ideas hopefully become clear.

Denote γ a chordal $\text{SLE}_{8/3}$ in the upper half-plane from x to ∞. Let

$$D = \mathbb{H} - \text{ slits.}$$

we would like to calculate $P^{\mathbb{H}}_{x \to \infty}(\gamma \subset D)$. This probability depends on the configuration of slits described by $3n$ parameters (e.g. the left-end-point of each slit plus its length), and also on x. The configuration of slits in turn is a function of the period matrix τ, although this dependence is hard to make explicit. In any event, the desired probability can be written as a function

$$F(x, \tau).$$

Idea: find a martingale, use it to derive a PDE for F. As usual, the martingale is obtained by conditioning,

$$\begin{aligned}
&P^{\mathbb{H}}_{x \to \infty}(\gamma \subset D | \gamma[0, t]) \\
&= 1\{\gamma[0, t] \subset D\} P^{\mathbb{H} \backslash \gamma[0,t]}_{\gamma_t \to \infty}(\gamma \subset D) \quad \text{(domain Markov)} \\
&= 1\{\gamma[0, t] \subset D\} P^{\mathbb{H}}_{x_t \to \infty}(\gamma \subset g_t(D)). \quad \text{(conformal invariance)}
\end{aligned}$$

So $P^{\mathbb{H}}_{x_t \to \infty}(\gamma \subset g_t(D))$ is a martingale on $\gamma[0, t] \subset D$. Map $g_t(D)$ to a standard domain D_t, i.e. the upper half-plane minus n horizontal slits (use map from above). The location and lengths of the slits will have changed, because the moduli of D and $g_t(D)$ are not the same. But because the map Φ_t, which maps $g_t(D)$ to D_t, cannot be extended to $\mathbb{H}$ we have

$$P^{\mathbb{H}}_{x_t \to \infty} \neq P^{\mathbb{H}}_{\Phi_t(x_t) \to \infty}(\gamma \subset D_t).$$

To proceed we need an additional property of the measure, namely restriction (what we did so far applies to all SLEs). For $\text{SLE}_{8/3}$ it follows from an argument of Beffara that

$$P^{\mathbb{H}}_{x_t \to \infty}(\gamma \subset g_t(D)) = \left(\Phi_t'(x_t)\Phi_t'(\infty)\right)^{5/8} P^{\mathbb{H}}_{\Phi_t(x_t) \to \infty}(\gamma \subset D_t),$$

where $\Phi_t(\infty) = \infty$. In particular,

$$\left(\Phi_t'(x_t)\Phi_t'(\infty)\right)^{5/8} F(\tau_t, \Phi_t(x_t))$$

is a martingale. To extract a PDE for F, we need a Markov process inside it. Unfortunately, $\Phi_t(x_t)$ is non-Markov, because Φ_t depends on the entire history of the path x_t. We will try to turn $\Phi_t(x_t)$ into a Markov process by an appropriate change of measure using Girsanov's theorem. The stochastic differential equation for $y_t = \Phi_t(x_t)$ is

$$dy_a = \sqrt{\kappa} dB_a + 2\frac{\partial}{\partial y} \log \theta_{\tau_a}[\alpha] \left(\int_\infty^{y_a} v \right) da,$$

where $da = \Phi_t'(x_t)^2 dt$. It is the drift term's dependence on τ_a which renders the process y_a non-Markov. Now a miracle happens. The Girsanov transform with the martingale (approximate expression)

$$M_a = \theta_{\tau_a}[\alpha] \left(\int_\infty^{y_a} v \right)^\beta \exp \left(2 \int_.^a \left(\frac{\partial^2}{\partial y^2} \log \theta \right) (\int^{y_b} v, \tau_b) \, db \right)^\delta$$

eliminates the drift. This uses in a crucial way the heat-equation for θ functions, and then leads to a PDE for F.

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