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School on Stochastic Geometry, the Stochastic Lowener Evolution, and Non-Equilibrium Growth Processes

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Domain Walls in Random Potts.

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Domain walls in Random Potts

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Trieste, July 18, 2008

The q-states Potts model (q=3)

$$\mathscr{Z} = \sum_{\{S_i\}} \mathrm{e}^{-\beta \mathscr{H}[\{S\}]}$$

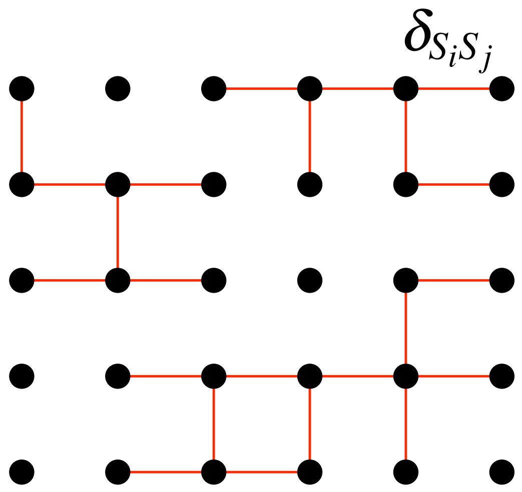
$$\mathscr{H}[\{S\}] = J \sum_{\langle ij \rangle} \delta_{S_i S_j}$$

$$S_i \in \{0, 1, 2\} = \{\text{red}, \text{green}, \text{blue}\}$$

The 3-states Potts model has a critical point on the square lattice at (Kramers Wannier duality)

$$\exp(\beta_c J) = 1 + \sqrt{q}\ , \qquad q = 3$$

Cluster expansion (Fortuin Kasteleyn = FK clusters)



$$\mathscr{Z} = \sum_{\{S_i\}} \mathrm{e}^{\beta J \sum_{\langle ij \rangle} \delta_{S_i S_j}}$$

$$= \sum_{\{S_i\}} \prod_{\langle ij \rangle} \left[1 + \left(\mathrm{e}^{\beta J} - 1 \right) \delta_{S_i S_j} \right]$$

$$\propto \sum_{\{S_i\}} \prod_{\langle ij \rangle} \left[1 - p + p \delta_{S_i S_j} \right] \;, \quad 1 - p = \mathrm{e}^{-\beta J}$$

each cluster has a unique color

$$\mathscr{Z} = \sum_{\mathscr{G}} p^{|\mathscr{G}|} (1-p)^{|\overline{\mathscr{G}}|} q^{||\mathscr{G}||}$$

graphs

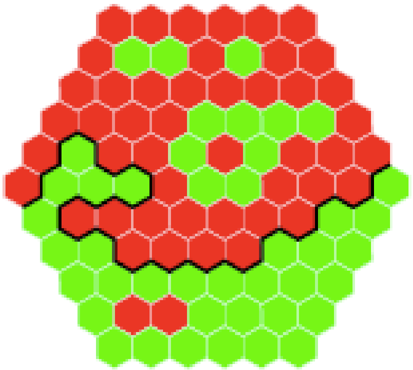
number of edges

number of edges in complement

number of connected components

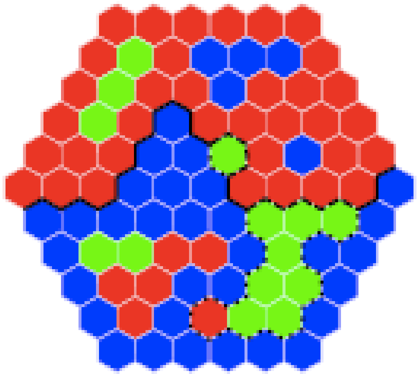
Domain walls and boundary conditions (bc)

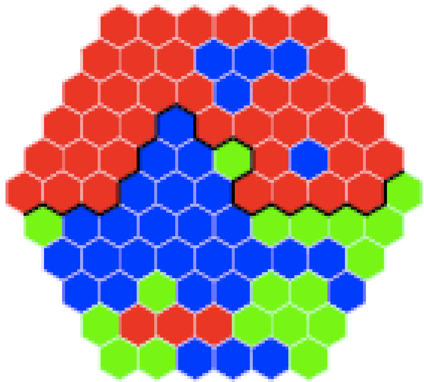
Ising



fixed bc

Potts



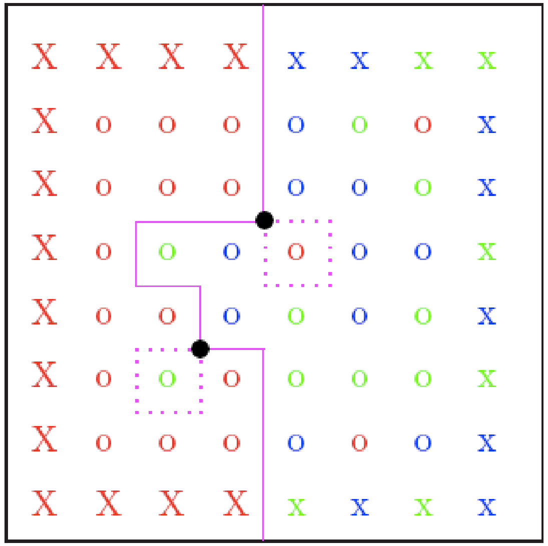


Gamsa Cardy

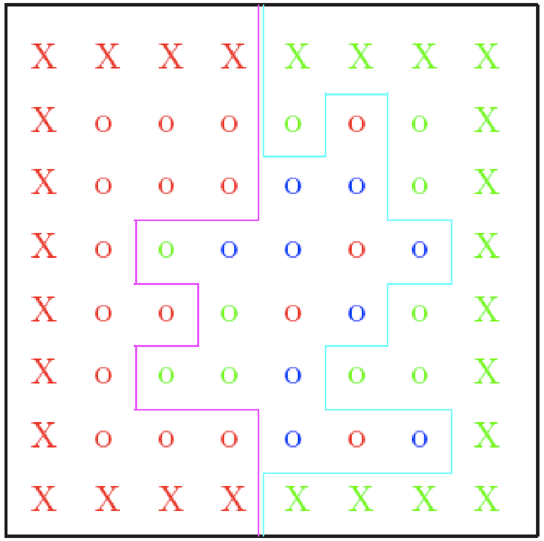
fixed bc

fluctuating bc

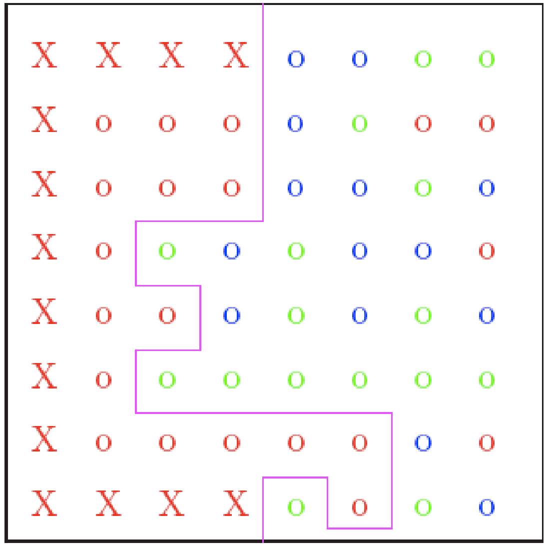
More boundary conditions



fluctuating



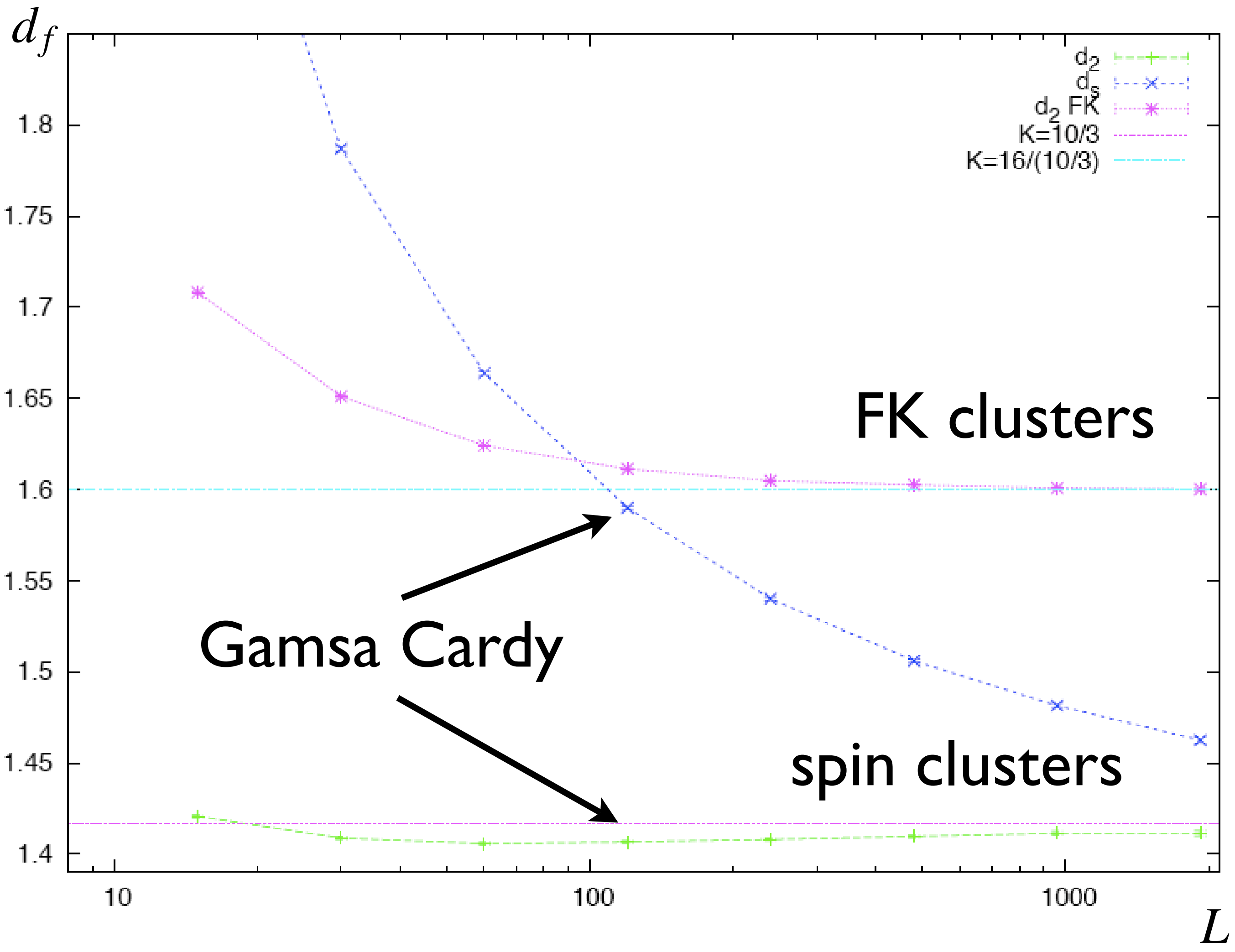
fixed



fixed - free

Potts without disorder

simulations by Marco Picco



$$\kappa = \frac{24}{5}$$

$$d_f = \frac{8}{5} = 1.6$$

CFT/SLE

$$\kappa = \frac{10}{3}$$

$$d_f = \frac{17}{12} = 1.417$$

3-states Potts and CFT/SLE

$$c = \frac{4}{5}$$

$$\kappa = \frac{10}{3}$$

$$\kappa' = \frac{24}{5}$$

2/3 ψ	1/15 σ		
1/8 R_1	1/40 R_2	21/40 R'_2	13/8 R'_1
0 Id	2/5 ε	7/5 ε'	3 W

- Ψ: Parafermionic currents (Z_3 symmetry)
- W: W symmetry
- R : non-abelian elements of D_3 dihedral group

Representation of a conformal field theory with $c < 1$
... away from free field theory: Coulomb gas I

stress energy tensor $$\mathscr{T}(z) = -\frac{1}{4} :\partial_z\phi(z)\partial_z\phi(z): + i\alpha_0\partial_z^2\phi(z)$$

short-distance expansion

$$\langle \mathscr{T}(z)\mathscr{T}(z')\rangle = \frac{c}{2(z-z')^4}$$ with central charge $$c = 1 - 24\alpha_0^2 = 1 - 3\frac{(\kappa-4)^2}{2\kappa}$$

dimension of vertex operators

$$\mathscr{T}(z) :\mathrm{e}^{i\alpha\phi(z')}: = \left[\frac{h_\alpha}{(z-z')^2} + \frac{1}{z-z'}\partial_{z'}\right] :\mathrm{e}^{i\alpha\phi(z')}:$$

$V_\alpha(z)$

$$h_\alpha = \alpha^2 - 2\alpha_0\alpha \equiv h_{2\alpha_0-\alpha} = h_{\overline{\alpha}}$$

2-point function of a vertex operator with himself

$$\langle V_\alpha(z)V_{2\alpha_0-\alpha}(z')\rangle = \frac{1}{(z-z')^{2h_\alpha}}\frac{1}{L^{4\alpha_0^2}}$$ is zero for large L

add additional "charge" at ∞

$$\langle V_\alpha(z)V_{2\alpha_0-\alpha}(z')\rangle_{-2\alpha_0} = \lim_{R\to\infty} R^{8\alpha_0^2}\langle V_\alpha(z)V_{2\alpha_0-\alpha}(z')V_{-2\alpha}(R)\rangle = (z-z')^{-2h_\alpha}$$

Coulomb Gas 2

Hamiltonian $$\mathscr{H}[\phi] = \mathscr{H}_0 - \int \mathrm{d}^2 z\, \mu_- V_-(z) + \mu_+ V_+(z)$$

marginal with $$V_\pm(z) =: \mathrm{e}^{i\alpha_\pm \phi(z)}:$$

$$\alpha_+ \alpha_- = -1$$
$$\alpha_+ + \alpha_- = 2\alpha_0$$

primary operators are represented as vertex operators

$$\Phi_{nm}(z) \longrightarrow \begin{cases} V_{nm}(z) \\ \overline{V}_{nm}(z) \end{cases}$$

$$V_{nm}(z) =: \mathrm{e}^{i\alpha_{nm}\phi(z)}:$$
$$\overline{V}_{nm}(z) =: \mathrm{e}^{i\alpha_{\overline{nm}}\phi(z)}:$$

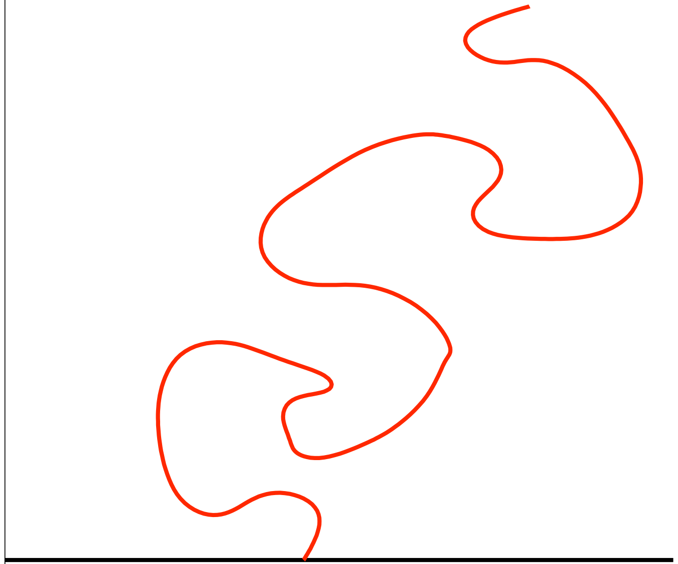
in order to close algebra, charges must be quantized

$$\alpha_{n,m} = \frac{1-n}{2}\alpha_- + \frac{1-m}{2}\alpha_+$$

charges $V_\pm$ allow for more possibilities for neutral objects

Curve-Creating Operators

curve-creating operator
on the boundary



$\Phi_{12}(z)$

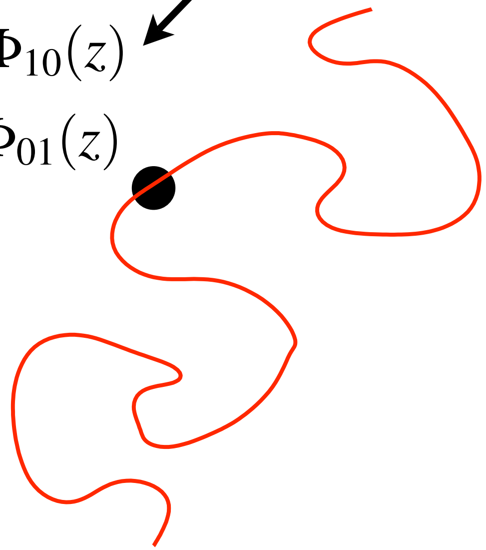
in the bulk

FK clusters

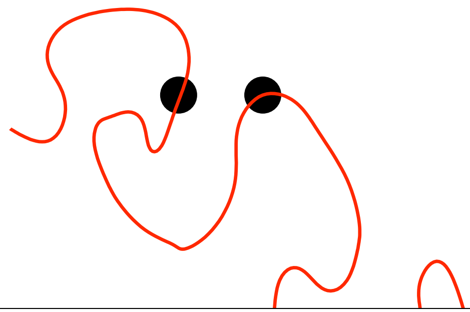
spin
clusters

$\Phi_{10}(z)$

$\Phi_{01}(z)$



fractal
dimension
of curve



$$\langle \Phi_{10}(z)\Phi_{10}(z')\rangle^c \sim \frac{1}{|z-z'|^{4h_\varepsilon}}$$

$$\Longrightarrow \quad d_f = 2 - 4h_\varepsilon$$

Potts + random temperature disorder

$$\mathscr{H} = \mathscr{H}_{\text{pure}} + \int \mathrm{d}^2 z\, \varepsilon(z)\, \delta t(z)$$

$\delta t(z)$ is a quenched Gaussian random temperature

$$\overline{\delta t(z)} = 0\,, \qquad \overline{\delta t(z)\delta t(z')} = g_0 \delta^2(z - z')$$

coupling to the energy density $\varepsilon(z) = \Phi_{12}(z)$

Replicated Hamiltonian

take n copies in the limit of n to 0, and average (replica trick)

$$\overline{\exp\left(-\sum_{a=1}^{n} \mathscr{H}^a\right)} = \exp\left(-\sum_{a=1}^{n} \mathscr{H}^a_{\text{pure}} + g_0 \sum_{a,b=1}^{n} \varepsilon^a(z)\varepsilon^b(z)\right)$$

Conformal perturbation theory

Ludwig 1987

Using Coulomb gas, we can construct a continuous family of models parameterized by $p = \dfrac{\pi}{2\arccos(\frac{\sqrt{q}}{2})}$ q-states Potts

Ising $p = q = 2$ 3-states Potts $p = q = 3$

Operators defined for all p. Disorder g_0 is marginal for p=2. Do an expansion in p-2.

$$\delta\mathscr{H} = -g_0 \sum_a \varepsilon^a(z)^2 - g_0 \sum_{a\neq b} \varepsilon^a(z)\varepsilon^b(z)$$

irrelevant

dimension is $O(p\text{-}2)$

Renormalization of the coupling

OPE

$$\sum_{a\neq b}\epsilon^a(Z)\epsilon^b(Z)\sum_{c\neq d}\epsilon^c(Z')\epsilon^d(Z') \to 4\sum_{d\neq a\neq b}\epsilon^b(Z)\epsilon^d(Z')\left\langle\epsilon(Z)\epsilon(Z')\right\rangle_0+\ldots$$

$$= 4(n-2)\sum_{b\neq d}\epsilon^b(Z)\epsilon^d(Z)\left\langle\epsilon(Z)\epsilon(Z')\right\rangle_0+\ldots$$

with

$$\left\langle\epsilon(Z)\epsilon(Z')\right\rangle_0=\frac{1}{|Z-Z'|^{4\Delta_\epsilon}}$$

$$\Delta_\epsilon=\Delta_{12}=\frac{p+1}{2(2p-1)}$$
$$=\frac{1}{2}-\frac{p-2}{6}+\frac{1}{9}(p-2)^2+O\left((p-2)^3\right)$$

Integrate over space

$$gL^{4\Delta_\epsilon-2}=g_0+\frac{g_0^2}{2}\times 4(n-2)\times\int_{|Z|<L}\frac{1}{|Z-Z'|^{4\Delta_\epsilon}}$$
$$=g_0+4\pi(n-2)g_0^2\int_0^L \mathrm{d}r\, r^{1-4\Delta_\epsilon}$$
$$=g_0+4\pi(n-2)g_0^2\frac{L^{2-4\Delta_\epsilon}}{2-4\Delta_\epsilon}$$

gives β function

$$L\partial_L g=(2-4\Delta_\epsilon)g+4\pi(n-2)g^2+\ldots$$

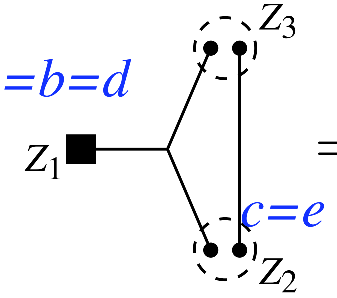
fixed point

$$g^*=\frac{1-2\Delta_\epsilon}{4\pi}$$

Renormalization of the operator $\Phi_{10}(z)$

$$\Phi_{10}^a(z_1)\frac{1}{2!}\left[g_0\sum_{b\neq c}\int_{z_2}\varepsilon^b(z_2)\varepsilon^c(z_2)\right]\left[g_0\sum_{d\neq e}\int_{z_3}\varepsilon^d(z_3)\varepsilon^e(z_3)\right]\longrightarrow\Phi_{10}^a(z_1)$$

need 2 groups of replicas, e.g: $a{=}b{=}d$ and $c{=}e$



$$=\frac{g_0^2}{2!}4(n-1)\int_{Z_2,Z_3}\langle\varepsilon(Z_2)\varepsilon(Z_3)\rangle_0\left(\Phi_{10}(Z_1)\varepsilon(Z_2)\varepsilon(Z_3)\middle|\Phi_{10}(Z_1)\right)$$

Coulomb gas: need additional vertex-operator $V_+(y)$

$$\left(\Phi_{10}(z_1)\varepsilon(z_2)\varepsilon(z_3)\middle|\Phi_{10}(z_1)\right)=|z_1-z_2|^{-4\Delta_{12}}G(U)\ ,\qquad U=\frac{z_1-z_3}{z_1-z_2}$$

$$G(U)=\mu_+|U|^{-\frac{2p}{2p-1}}|U-1|^{\frac{2p}{2p-1}}F(U)$$

$$F(U)=\int_Y|Y|^{\frac{4p}{2p-1}}|Y-1|^{-\frac{4p}{2p-1}}|U-Y|^{-\frac{4p}{2p-1}+2\eta}$$ ← regulator

Dotsenko magic formula (DMF)

we want:

$$I = \int |x|^{2a}|x-1|^{2b}|y|^{2a'}|y-1|^{2b'}|x-y|^{4g}d^2x d^2y$$

DMF:

$$I = s(b)s(b')\left[J_1^+J_1^- + J_2^+J_2^-\right] + s(b)s(b'+2g)J_1^+J_2^- + s(b+2g)s(b')J_2^+J_1^-$$

$$J_1^+ = J(a,b,a',b',g)\ ;\ J_2^+ = J(b,a,b',a',g)$$
$$J_1^- = J(b,-2-a-b-2g,b',-2-a'-b'-2g,g)$$
$$J_2^- = J(-2-a-b-2g,b,-2-a'-b'-2g,b',g)$$

$$J(a,b,a',b',g) = \int_0^1 du \int_0^1 dv\, u^{a+a'+2g+1}(1-u)^b v^{a'}(1-v)^{2g}(1-uv)^{b'}$$

$$= \frac{\Gamma(2+a+a'+2g)\Gamma(1+b)\Gamma(1+a')\Gamma(1+2g)}{\Gamma(3+a+a'+b+2g)\Gamma(2+a'+2g)} \sum_{k=0}^{\infty} \frac{(-b')_k(2+a+a'+2g)_k(1+a')_k}{k!(3+a+a'+b+2g)_k(2+a'+2g)_k}$$

$$(a)_k = a(a+1)...(a+k-1)$$

relations:

$$s(2g+a+b)J_1^- + s(a+b)J_2^- = \frac{s(a)}{s(2g+a'+b')}\left(s(a')J_1^+ + s(2g+a')J_2^+\right)$$

$$s(2g+a'+b')J_2^- + s(a'+b')J_1^- = \frac{s(a')}{s(2g+a+b)}\left(s(a)J_2^+ + s(2g+a)J_1^+\right)$$

Dimension of FK clusters in Random Potts

$$\int_{Z_2,Z_3} \left(\Phi_{10}(Z_1)\,\epsilon(Z_2)\,\epsilon(Z_3)\Big|\Phi_{10}(Z_1)\right)\left(\epsilon(Z_2)\epsilon(Z_3)\big|1\right) = \mu^+ \frac{\pi L^{4-8\Delta_{12}}}{2(1-2\Delta_{12})} K_1 = -7.07101 \frac{L^{4-8\Delta_{12}}}{(1-2\Delta_{12})}$$

gives finally the dimension of Φ_{10} :

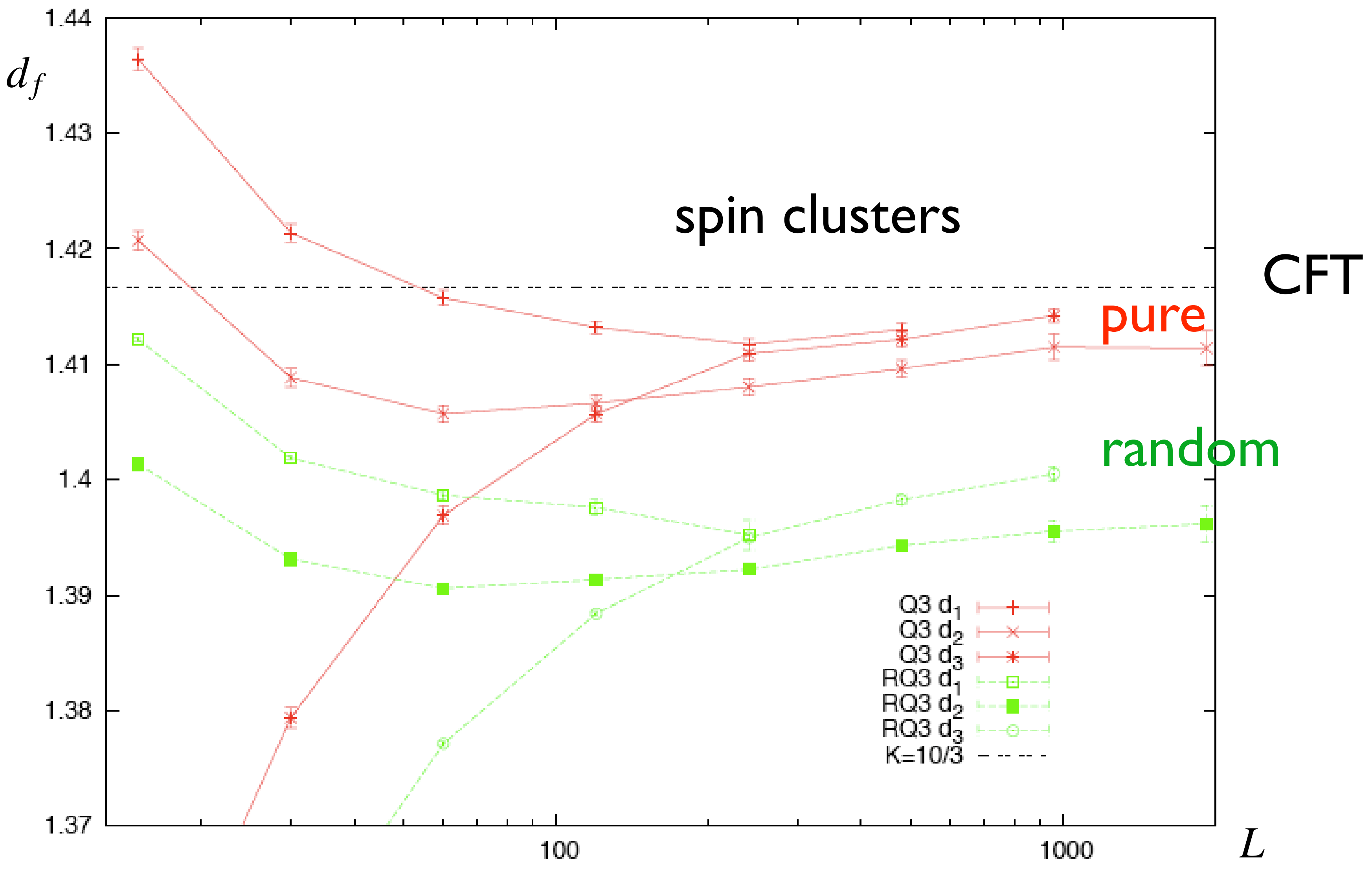
$$\begin{aligned}
\dim_L(\Phi_{10}) &= -2\Delta_{10} + \frac{(1-2\Delta_\epsilon)^2}{2\pi^2} \times 7.07101 \\
&= \frac{1-p}{2p-1} + \frac{(p-2)^2}{2(2p-1)^2\pi^2} \times 7.07101 \\
&\overset{p=3}{=} -\frac{2}{5} + 0.0143289
\end{aligned}$$

I will show data by Picco, who has

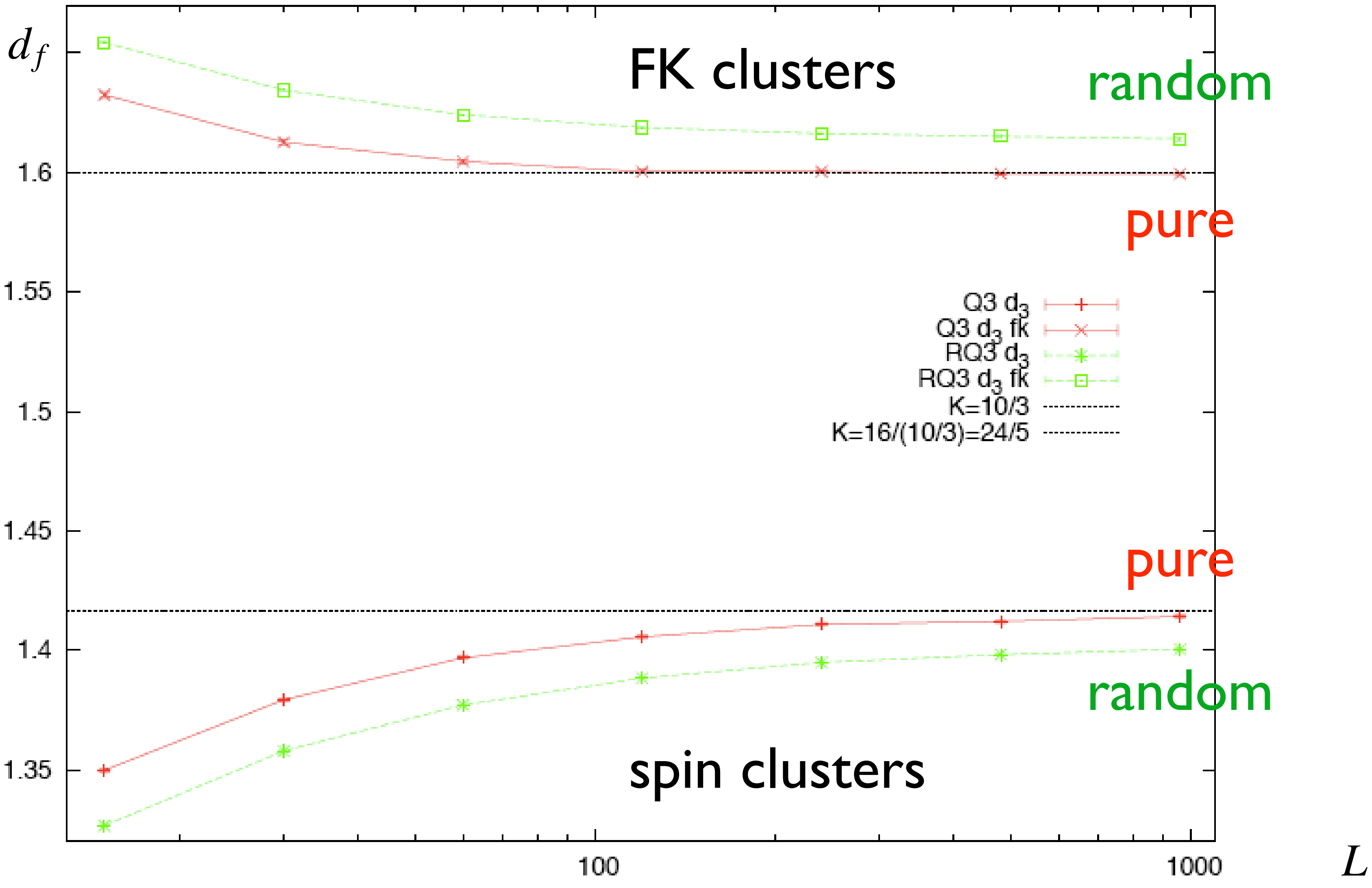
$$-\frac{2}{5} + 0.015$$

in nice agreement.

Potts with disorder



Spin and SK clusters



A correlation function for afficionados

The 0-vector condition (martingale!) for Φ_{12} gives the PDE

$$\Big[-\frac{3}{2(2\Delta_{12}+1)}\frac{\partial^2}{\partial z^2}+\frac{\Delta_{10}}{(z-z_1)^2}+\frac{1}{z-z_1}\frac{\partial}{\partial z_1}+\frac{\Delta_{12}}{(z-z_2)^2}+\frac{1}{z-z_2}\frac{\partial}{\partial z_2}$$
$$+\frac{\Delta_{10}}{(z-z_4)^2}+\frac{1}{z-z_4}\frac{\partial}{\partial z_4}\Big]\langle\Phi_{10}(Z_1)\epsilon(Z_2)\epsilon(Z)\Phi_{10}(Z_4)\rangle=0$$

With the ansatz

$$\langle\Phi_{10}(Z_1)\epsilon(Z_2)\epsilon(Z)\Phi_{10}(Z_4)\rangle\ =|Z_4|^{-4\Delta_{10}}\frac{1}{|Z_1-Z_2|^{4\Delta_{12}}}G\left(U=(u=\frac{z-z_1}{z_2-z_1},\bar u=\frac{\bar z-\bar z_1}{\bar z_2-\bar z_1})\right)$$

this becomes

$$G(u)=(1-u)^{\frac{p+1}{1-2p}}u^{\frac{p-1}{2p-1}}F(u)$$

$$\frac{2(p-1)F(u)}{(1-2p)^2}+\left(2-\frac{4(p-1)u}{2p-1}\right)F'(u)+(1-u)uF''(u)=0$$

with solutions

$$F(u)=c_1\,{}_2F_1\left(-\frac{1}{2p-1},\frac{2p}{2p-1}-\frac{2}{2p-1};2;u\right)+c_2G^{2,0}_{2,2}\left(u\,\middle|\,\begin{matrix}\frac{1}{2p-1},1+\frac{1}{2p-1}\\-1,0\end{matrix}\right)$$

A correlation function for afficionados (2)

solution was from last slide:

$$F(u) = c_1\, {}_2F_1\left(-\frac{1}{2p-1}, \frac{2p}{2p-1} - \frac{2}{2p-1}; 2; u\right) + c_2 G^{2,0}_{2,2}\left(u \,\middle|\, \begin{matrix} \frac{1}{2p-1}, 1+\frac{1}{2p-1} \\ -1, 0 \end{matrix}\right)$$

$$F(u) =: {}_2F_1\left(-\frac{1}{2p-1}, \frac{2p}{2p-1} - \frac{2}{2p-1}; 2; u\right)\left[c_1 + c_2 \ln(u)\Gamma(\tfrac{1}{2p-1})\Gamma(1+\tfrac{1}{2p-1})\right] + c_2 R(u)$$

regular

$$G(U)\Big|_{p=2} = \frac{\Gamma(\frac{1}{3})^6}{27\pi^2}\frac{|U|^{\frac{2}{3}}}{|1-U|^2}\left|{}_2F_1\left(-\frac{1}{3}, \frac{2}{3}; 2; u\right)\right|^2$$

$$+\frac{\Gamma(\frac{1}{3})^8}{54\sqrt{3}\pi^3}\frac{|U|^{\frac{2}{3}}}{|1-U|^2}\left[{}_2F_1\left(-\frac{1}{3}, \frac{2}{3}; 2; u\right) G^{2,0}_{2,2}\left(\overline{u} \,\middle|\, \begin{matrix} \frac{1}{3}, \frac{4}{3} \\ -1, 0 \end{matrix}\right) + c.c.\right]$$

The missing correlation function

Coulomb gas for $\langle \Phi_{01}(Z_1)\epsilon(Z)\epsilon(Z_2)\Phi_{01}(Z_3)\rangle$ fails miserably!

Try PDE due to 0-vector condition for Φ_{12}

$$\left[-\frac{3}{2(2h_{12}+1)}\frac{\partial^2}{\partial z^2}+\sum_{i=1}^{3}\frac{h_i}{(z-z_i)^2}+\frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right]\langle \Phi_{01}(Z_1)\epsilon(Z)\epsilon(Z_2)\Phi_{01}(Z_3)\rangle = 0$$

With the same parameterization as above

$$G(u,\bar{u}) = a(\bar{u})\frac{(1-u)^{2/3}}{u^{5/6}} + b(\bar{u})\frac{\sqrt{u}\,{}_2F_1\left(-\frac{1}{3},1;\frac{7}{3};u\right)}{1-u}$$

This can be combined to (and only to) for general p

$$\begin{aligned} G(U) &= A|1-U|^{1+\frac{1}{2p-1}}\,|U|^{\frac{2p+1}{1-2p}} + \frac{(2p+1)^2}{4p^2}|1-U|^{\frac{2(p+1)}{1-2p}}\,|U|\left|{}_2F_1\left(1,\tfrac{1}{1-2p};2+\tfrac{1}{2p-1};U\right)\right|^2 \\ &= A|1-U|^{1+\frac{1}{2p-1}}\,|U|^{\frac{2p+1}{1-2p}} \\ &\quad + \frac{(2p+1)^2}{4p^2}|1-u|^{\frac{2p+2}{1-2p}}\,|u|\left|\frac{(1-u)^{\frac{2p+1}{2p-1}}\Gamma\left(\frac{2p+1}{1-2p}\right)\Gamma\left(2+\frac{1}{2p-1}\right)u^{\frac{2p}{1-2p}}}{\Gamma\left(\frac{1}{1-2p}\right)} + \frac{2p\,{}_2F_1\left(1,\frac{1}{1-2p},\frac{2}{1-2p},1-u\right)}{2p+1}\right|^2 \end{aligned}$$

The second term has a non-trivial monodromy around 1, which is not canceled by first term!

The missing correlation function

Coulomb gas for $\langle \Phi_{01}(Z_1)\epsilon(Z)\epsilon(Z_2)\Phi_{01}(Z_3)\rangle$ fails miserably!

Try PDE due to 0-vector condition for Φ_{12}

$$\left[-\frac{3}{2(2h_{12}+1)}\frac{\partial^2}{\partial z^2}+\sum_{i=1}^{3}\frac{h_i}{(z-z_i)^2}+\frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right]\langle \Phi_{01}(Z_1)\epsilon(Z)\epsilon(Z_2)\Phi_{01}(Z_3)\rangle = 0$$

With the same parameterization as above

$$G(u,\bar{u}) = a(\bar{u})\frac{(1-u)^{2/3}}{u^{5/6}} + b(\bar{u})\frac{\sqrt{u}\,{}_2F_1\left(-\frac{1}{3},1;\frac{7}{3};u\right)}{1-u}$$

This can be combined to (and only to) for general p

$$G(U) = A|1-U|^{1+\frac{1}{2p-1}}\,|U|^{\frac{2p+1}{1-2p}} + \frac{(2p+1)^2}{4p^2}|1-U|^{\frac{2(p+1)}{1-2p}}\,|U|\left|{}_2F_1\left(1,\frac{1}{1-2p};2+\frac{1}{2p-1};U\right)\right|^2$$

IMPOSSIBLE!

The second term has a non-trivial monodromy around 1, which is not canceled by first term!

Kac table for Ising

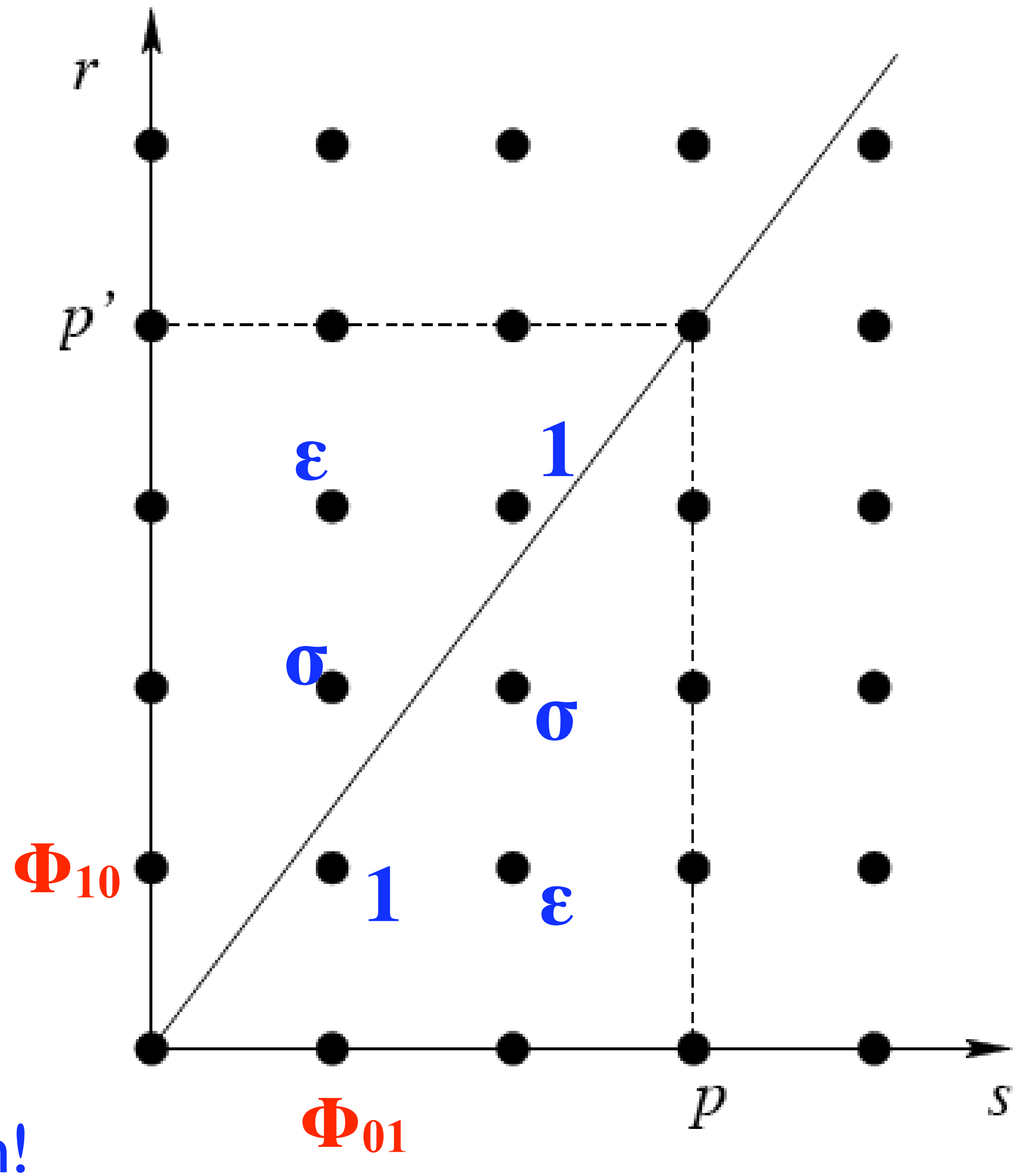
notation: Φ_{rs}

Φ_{11} = **1**

$\Phi_{12} = \varepsilon$

$\Phi_{22} = \sigma$

operators on the boundary of the Kac table are uncharted territory, and worth exploring on their own!



(the bad guy)

Conclusions

- fractal dimension of Random Potts clusters
- SK cluster in agreement of numerical simulations
- spin clusters not yet accessible due to problems in CFT for opeartors outside the Kac table
- multifractal exponents: Is this SLE ?