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cNLS Equation versus cKdV Equation in the Madelung's Fluid representation

R. Fedele

Universita' Federico II di Napoli
Italy

cNLS Equation *versus* cKdV Equation in the Madelung's Fluid representation

Renato Fedele

Dipartimento di Scienze Fisiche, Università Federico II
and Istituto Nazionale di Fisica Nucleare, Sezione di Napoli,
Complesso Universitario di M.S. Angelo, Napoli, Italy

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In collaboration with:

- **S. De Nicola, C.N.R. Pozzuoli, Napoli**
- **D. Grecu, NIPNE, Buchurest, Romania**
- **P.K. Shukla, Ruhr-Universität Bochum, Germany**
- **A. Visinescu, NIPNE, Buchurest, Romania**

THE MADELUNG'S FLUID PICTURE

- **Proposed in 1926 by Madelung (first proposal of hydrodynamical model of quantum mechanics)**
- **A second proposal came in 1927 by Korn**

$$\frac{\partial \rho}{\partial s} + \frac{\partial}{\partial x}(\rho V) = 0 \,,$$

$$\left(\frac{\partial}{\partial s} + V\frac{\partial}{\partial x}\right) V = -\frac{\partial U}{\partial x} + \frac{\alpha^2}{2}\frac{\partial}{\partial x}\left[\frac{1}{\rho^{1/2}}\frac{\partial^2 \rho^{1/2}}{\partial x^2}\right]$$

"DENSITY":

$$\rho = |\Psi|^2$$

"CURRENT VELOCITY":

$$V(x,s)$$

$$i\alpha \frac{\partial \Psi}{\partial s} + \frac{\alpha^2}{2} \frac{\partial^2 \Psi}{\partial x^2} - \mathbf{U}[|\Psi|^2]\Psi = 0$$

$$\Psi = \sqrt{\rho(x,s)} \exp\left[\frac{i}{\alpha}\Theta(x,s)\right]$$

$$\boxed{\begin{aligned} &\frac{\partial \rho}{\partial s} + \frac{\partial}{\partial x}(\rho V) = 0\,, \\ &\left(\frac{\partial}{\partial s} + V\frac{\partial}{\partial x}\right) V = -\frac{\partial U}{\partial x} + \frac{\alpha^2}{2}\frac{\partial}{\partial x}\left[\frac{1}{\rho^{1/2}}\frac{\partial^2 \rho^{1/2}}{\partial x^2}\right], \end{aligned}}$$

$$V(x,s) = \frac{\partial \Theta(x,s)}{\partial x}\,.$$

BASIC EQUATIONS

$$\frac{\partial \rho}{\partial s} + \frac{\partial}{\partial x}(\rho V) = 0 \quad (1)$$

$$\left(\frac{\partial}{\partial s} + V\frac{\partial}{\partial x}\right)V = -\frac{\partial U}{\partial x} + \frac{\alpha^2}{2}\frac{\partial}{\partial x}\left[\frac{1}{\rho^{1/2}}\frac{\partial^2 \rho^{1/2}}{\partial x^2}\right] \quad (2)$$

- BY MULTIPLYING (1) BY V:

$$\rho\left(\frac{\partial}{\partial s} + V\frac{\partial}{\partial x}\right)V \ = \ -V\frac{\partial \rho}{\partial s} - V^2\frac{\partial \rho}{\partial x} + \rho\frac{\partial V}{\partial s} \quad (3)$$

- MULTIPLYING (3) BY ρ, COMBINING THE RESULT WITH (2) AND OBSERVING THAT:

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho^{1/2}}\frac{\partial^2 \rho^{1/2}}{\partial x^2}\right)=\frac{1}{\rho}\left(\frac{1}{2}\ \frac{\partial^3 \rho}{\partial x^3}-4\ \frac{\partial \rho^{1/2}}{\partial x}\ \frac{\partial^2 \rho^{1/2}}{\partial x^2}\right)$$

WE OBTAIN:

$$\rho\left(\frac{\partial}{\partial s}+V\frac{\partial}{\partial x}\right)V=$$

$$=-\frac{\partial U}{\partial x}\rho+\frac{\alpha^2}{4}\ \frac{\partial^3 \rho}{\partial x^3}-2\alpha^2\ \frac{\partial \rho^{1/2}}{\partial x}\ \frac{\partial^2 \rho^{1/2}}{\partial x^2}\ (4)$$

- COMBINING (4) WITH (3):

$$-V\frac{\partial \rho}{\partial s} \; - \; V^2\frac{\partial \rho}{\partial x} \; + \; \rho\frac{\partial V}{\partial s} =$$

$$= -\frac{\partial U}{\partial x}\rho + \frac{\alpha^2}{4}\,\frac{\partial^3 \rho}{\partial x^3} - 2\alpha^2\,\frac{\partial \rho^{1/2}}{\partial x}\,\frac{\partial^2 \rho^{1/2}}{\partial x^2} \qquad (5)$$

- BY INTEGRATING (2) WITH RESPECT TO x AND MULTIPLYING THE RESULTING EQUATION BY

$$\rho^{1/2}\;\left(\partial \rho^{1/2}/\partial x\right)$$

WE HAVE:

$$-2\alpha^2 \, \frac{\partial \rho^{1/2}}{\partial x} \, \frac{\partial^2 \rho^{1/2}}{\partial x^2} = -2 \frac{\partial \rho}{\partial x} \int \left(\frac{\partial V}{\partial s} \right) \, dx -$$

$$-V^2 \frac{\partial \rho}{\partial x} \, -2 \, U \frac{\partial \rho}{\partial x} \, + \, 2 \, c_0(s) \, \frac{\partial \rho}{\partial x} \qquad (6)$$

WHERE $c_0(s)$ IS AN ARBITRARY FUNCTION OF s.

- FINALLY, BY COMBINING (5) AND (6) WE OBTAIN THE FOLLOWING EQUATION:

$$-\left(\frac{\partial V}{\partial s}\right)\rho + V\frac{\partial \rho}{\partial s} + 2\left[c_0(s) - \int\left(\frac{\partial V}{\partial s}\right)\ dx\right]\frac{\partial \rho}{\partial x} -$$

$$-\left(\frac{\partial U}{\partial x}\rho + 2\ U\frac{\partial \rho}{\partial x}\right) + \frac{\alpha^2}{4}\frac{\partial^3 \rho}{\partial x^3} = 0 \qquad (7)\ .$$

$$\frac{\partial \rho}{\partial s} + \frac{\partial}{\partial x}(\rho V) = 0$$

IN PARTICULAR FOR:

$$U[\rho] = q_0 \rho^\beta$$

$$-\left(\frac{\partial V}{\partial s}\right)\rho + V\frac{\partial \rho}{\partial s} + 2\left[c_0(s) - \int \left(\frac{\partial V}{\partial s}\right)\, dx\right]\frac{\partial \rho}{\partial x} - (\beta + 2)\, q_0\, \rho^\beta \frac{\partial \rho}{\partial x} + \frac{\alpha^2}{4}\frac{\partial^3 \rho}{\partial x^3} = 0$$

$$\frac{\partial \rho}{\partial s} + \frac{\partial}{\partial x}(\rho V) = 0$$

FOR DETAILS, SEE:

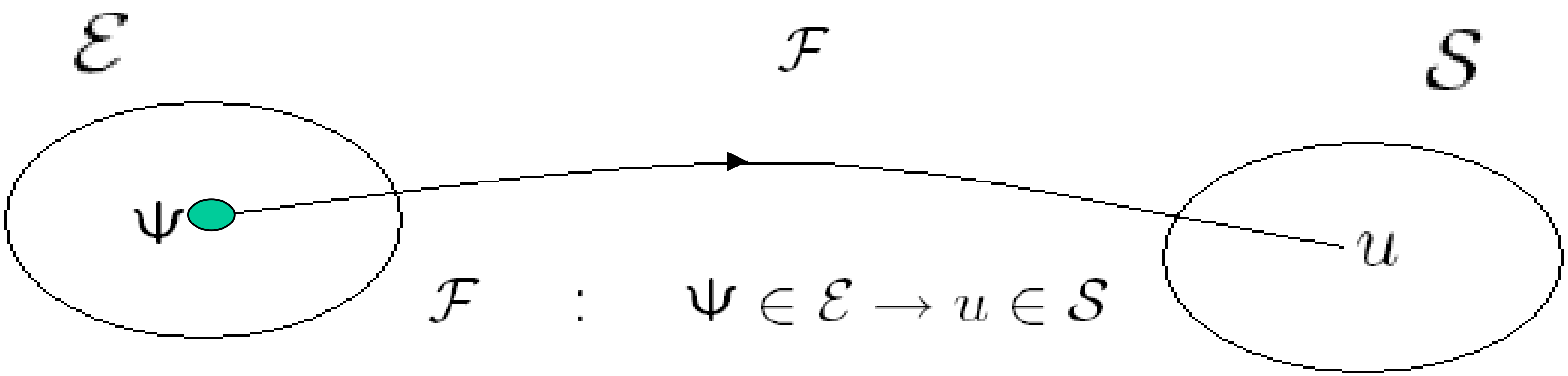
R. Fedele and H. Schamel, Eur. Phys. J. B **27**, 313 (2002);

R. Fedele, Physica Scripta **65**, 502 (2002)

R. Fedele, H. Schamel and P.K. Shukla,Physica Scripta T98, 18 (2002)

R. Fedele, H. Schamel, V. I. Karpman and P K Shukla J. Phys. A: Math. Gen. **36** 1169 (2003)

A correspondence between soliton-like and envelope soliton-like solutions, in the form of travelling waves, of wide families of generalized Korteweg-de Vries equation (gKdVE) and generalized nonlinear Schrödinger equation (gNLSE), respectively, has been constructed within the framework of the Madelung's fluid and used to find solitonlike solutions of NLSE with nonlinearities more complicated than the cubic one.



$$\mathcal{E} = \{\Psi\}$$

the set of all the stationary-profile envelope solutions of the gNLSE:

$$i\alpha \frac{\partial \Psi}{\partial s} + \frac{\alpha^2}{2}\frac{\partial^2 \Psi}{\partial x^2} - \mathbf{U}[\,|\Psi|^2]\Psi = 0$$

$$\Psi(x,s) = \sqrt{\rho(x - u_0 s)}\, \exp\left[i\Theta(x,s)/\alpha\right]$$

$$\mathcal{S} = \{u(\xi) \geq 0\}$$

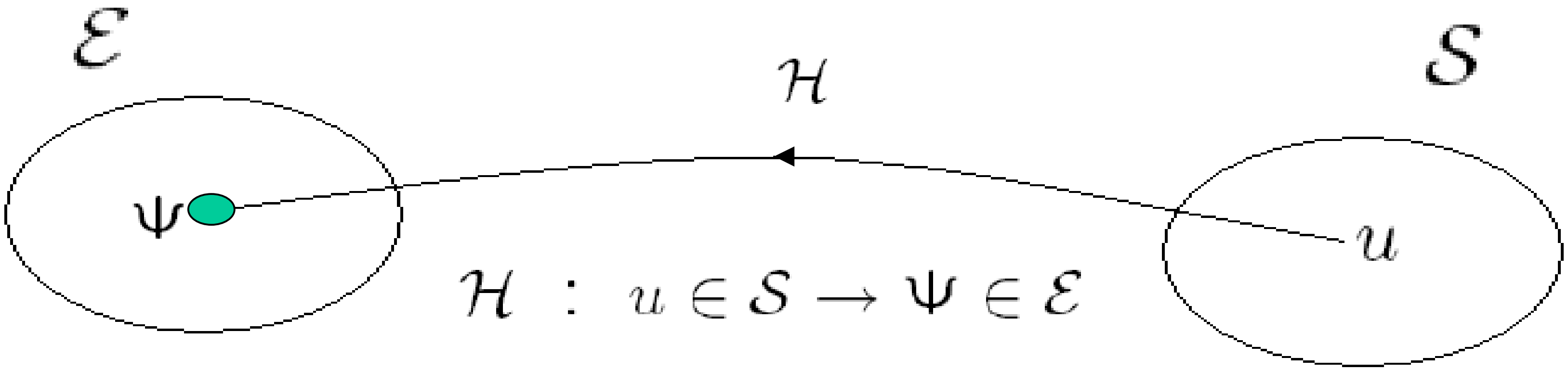
the set of all non-negative stationary-profile solutions of the gKdVE:

$$a\,\frac{\partial u}{\partial s} - G[u]\,\frac{\partial u}{\partial x} + \frac{\nu^2}{4}\frac{\partial^3 u}{\partial x^3} = 0$$

$$u = u(x - u_0 s)$$

$$G[\rho] \equiv \rho\frac{dU[\rho]}{d\rho} + 2U[\rho]$$

$$u = \mathcal{F}[\Psi] = |\Psi|^2 = \rho(\xi)$$



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$$u = u(x - u_0 s)$$

$$G[\rho] \;\equiv\; \rho \frac{dU[\rho]}{d\rho} + 2U[\rho]$$

$$\Psi = \mathcal{H}[u] = \sqrt{u(\xi)}\; \exp\left\{\frac{i}{\nu}\left[\phi_0 - \left(c_0 + u_0^2\right) s + u_0 x + A_0 \int \frac{d\xi}{u(\xi)}\right]\right\}$$

gNLSE

$$i\alpha \frac{\partial \Psi}{\partial s} + \frac{\alpha^2}{2} \frac{\partial^2 \Psi}{\partial x^2} - \mathbf{U}[\,|\Psi|^2]\Psi = 0$$

$\longrightarrow$

gKdVE

$$a \frac{\partial u}{\partial s} - G[u] \frac{\partial u}{\partial x} + \frac{\nu^2}{4} \frac{\partial^3 u}{\partial x^3} = 0$$

$$G[\rho] \equiv \rho \frac{dU[\rho]}{d\rho} + 2U[\rho]$$

$$\nu = \alpha$$

gNLSE

$$i\alpha \frac{\partial \Psi}{\partial s} + \frac{\alpha^2}{2}\frac{\partial^2 \Psi}{\partial x^2} - \mathbf{U}[\,|\Psi|^2]\Psi = 0$$

gKdVE

$$a \frac{\partial u}{\partial s} - G[u]\frac{\partial u}{\partial x} + \frac{\nu^2}{4}\frac{\partial^3 u}{\partial x^3} = 0$$

$$U[u] = \frac{1}{u^2}\left[K_0 + \int G[u]\; u\; du\right]$$

$$\alpha = \nu$$

APPLICATIONS

1. $$U[\rho] = q_0|\Psi|^{2\beta} = q_0\rho^\beta$$

$$2E\,\frac{d\rho}{d\xi} - (\beta+2)\; q_0\;\rho^\beta\frac{d\rho}{d\xi} + \frac{\alpha^2}{4}\frac{d^3\rho}{d\xi^3} = 0$$

WHERE $E = c_0 - \frac{V_0^2}{2}$.

- **Bright solitons (β >0)**
- **Dark solitons (β =1)**

2. $U = a_1|\Psi|^2 + a_2|\Psi|^4 = a_1\rho + a_2\rho^2$

BY USING THE EQUATION:

$$-\left(\frac{\partial V}{\partial s}\right)\rho + V\frac{\partial \rho}{\partial s} + 2\left[c_0(s) - \int\left(\frac{\partial V}{\partial s}\right)\, dx\right]\frac{\partial \rho}{\partial x} -$$

$$-\left(\frac{\partial U}{\partial x}\rho + 2\,U\frac{\partial \rho}{\partial x}\right) + \frac{\alpha^2}{4}\frac{\partial^3 \rho}{\partial x^3} = 0 \qquad (7)$$

WE GET THE FOLLOWING STATIONARY MKdVE:

$$2E'\,\frac{d\rho}{d\xi} - 4a_2\,(\rho - \rho_0)^2\,\frac{d\rho}{d\xi} + \frac{\alpha^2}{4}\frac{d^3\rho}{d\xi^3} = 0$$

WHERE:

$$E' = E + \frac{9}{32}\frac{a_1^2}{a_2}\,,\ \ \rho_0 = -\frac{3}{8}\frac{a_1}{a_2}$$

- **Bright solitons**
- **Dark solitons**
- **Grey solitons**

ENVELOPE SOLITONS OF NLSE WITH AN "ANTI-CUBIC" NON-LINEARITY

The method presented above can be also used to find envelope soliton-like solutions of the following modified NLSE containing, besides the cubic and quintic nonlinearities, an **anti-cubic** nonlinearity (i.e. $|\Psi|^{-4}\Psi$):

$$i\alpha\frac{\partial\Psi}{\partial s} + \frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - Q_0\,|\Psi|^{-4}\Psi + \left[q_1\,|\Psi|^2 + q_2\,|\Psi|^4\right]\Psi = 0$$

where Q_0 is a real constant.

Recently we used MA to discuss generalized derivative NLS eqs. (D. Grecu, A.T. Grecu, A. Visinescu, R. Fedele, S. DeNicola - 2008)

$$i\alpha \frac{\partial \Psi}{\partial t} + \frac{\alpha^2}{2} \frac{\partial^2 \Psi}{\partial x^2} + i\gamma \frac{\partial}{\partial x} \left(U(|\Psi|^2)\Psi \right) = 0 \quad gdNLS - 1$$

$$i\alpha \frac{\partial \Psi}{\partial t} + \frac{\alpha^2}{2} \frac{\partial^2 \Psi}{\partial x^2} + i\gamma U(|\Psi|^2) \frac{\partial \Psi}{\partial x} = 0 \qquad gdNLS - 2$$

For $U(|\Psi|^2) = |\Psi|^2$ they transform into completely integrable equations.

- Generalized NLS Equation

$$i\frac{\partial\Psi}{\partial t} + \frac{1}{2}\frac{\partial^2\Psi}{\partial x^2} + 2i\Psi^2\frac{\partial\Psi^*}{\partial x} + \alpha U(|\Psi|^2)\Psi = 0$$

- Bright Soliton Solution
- Periodic Solutions
- Solution Stability

D. Grecu, A.T. Grecu, A. Visinescu, R. Fedele, S.DeNicola, *J. Nonlinear Math. Phys* - to be published (Proc. NEEDS-2007)

A. Visinescu, S. DeNicola, R. Fedele, D. Grecu, *Proc.Int. Conf. FARPhys 2007, Iassy* - to be published

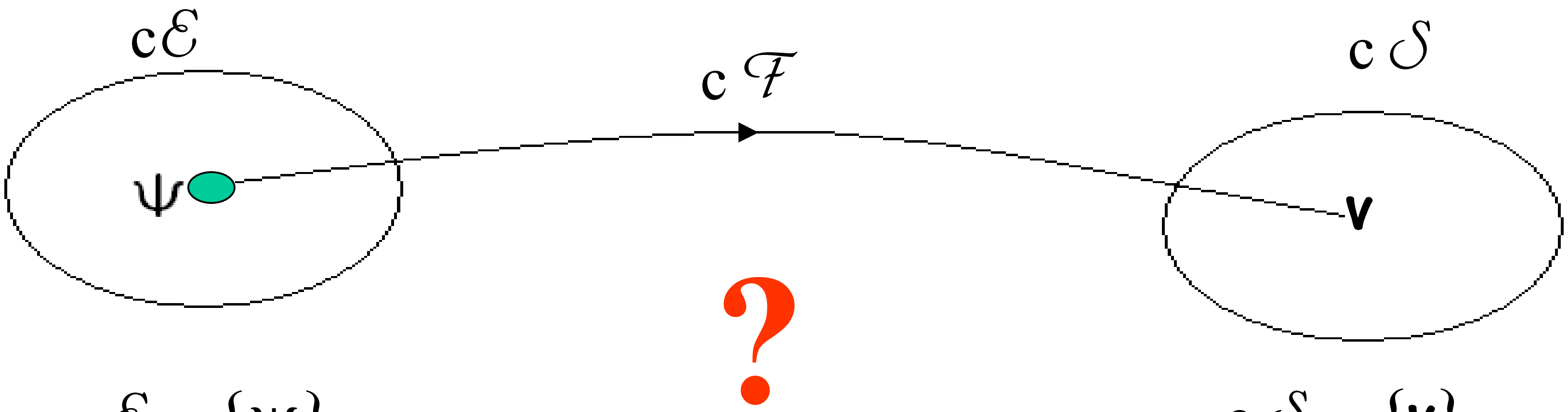
cNLSE Equation *versus* cKdVE in the Madelung's Fluid Approach

Cylindrical nonlinear Schrödinger equation (cNLSE)

$$i\frac{\partial\psi}{\partial t} + a\frac{\partial^2\psi}{\partial x^2} + b|\psi|^2\psi + \frac{i}{2t}\psi = 0$$

Cylindrical Korteweg-de Vries equation (cKdVE)

$$\frac{\partial v}{\partial s} + a'\, v\frac{\partial v}{\partial y} + b'\frac{\partial^3 v}{\partial y^3} + \frac{1}{2s}v = 0$$



$c\,\mathcal{E} = \{\psi\}$

the set of solutions of the cNLSE:

$$i\frac{\partial\psi}{\partial t} + a\frac{\partial^2\psi}{\partial x^2} + b|\psi|^2\psi + \frac{i}{2t}\psi = 0$$

$c\,\mathcal{S} = \{\mathbf{v}\}$

the set of solutions of the cKdVE:

$$\frac{\partial v}{\partial s} + a'\,v\frac{\partial v}{\partial y} + b'\frac{\partial^3 v}{\partial y^3} + \frac{1}{2s}v = 0$$

Let us observe that the cNLSE has a non-Hermitian potential term. To apply the Madelung's fluid approach starting from the cNLS equation, one has first transform this equation into another one in which the potential is Hermitian.

A simple transformation leads to a NLSE with a time-dependent coefficient of the potential term.
This time-dependence introduces some difficulties to perform a direct transformation to obtain the cKdVE. These difficulties are related to the fact that the desired transformation should control also the phase of ψ (namely V) and simultaniously has to provide a solution which satisfies both continuity and motion equation (3rd-order differential equation for ρ).

It turns out that, at least for the time being, the transformation from cNLSE to cKdVE should be restricted to a sub-class of solutions that involves a step in which a cNLSE is transformed into a standard NLSE. This step is, of course, possible and well known. Another important step of the transformation is the one that transforms a standard KdVE into a cKdVE.

OBSERVATION 1: link cNLSE - NLSE

Under the transformation

$$\psi(x,t) = -\frac{1}{t}\exp\left[i\frac{x^2}{4at}\right]\ \Psi\left(\xi = -\frac{x}{t}, \tau = -\frac{1}{t}\right)$$

the cNLSE

$$i\frac{\partial \psi}{\partial t} + a\frac{\partial^2 \psi}{\partial x^2} + b|\psi|^2\psi + \frac{i}{2t}\psi = 0$$

becomes the standard NLSE

$$i\frac{\partial \Psi}{\partial \tau} + a\frac{\partial^2 \Psi}{\partial \xi^2} + b|\Psi|^2\Psi = 0$$

OBSERVATION 2: link KdVE - cKdVE

Under the transformation

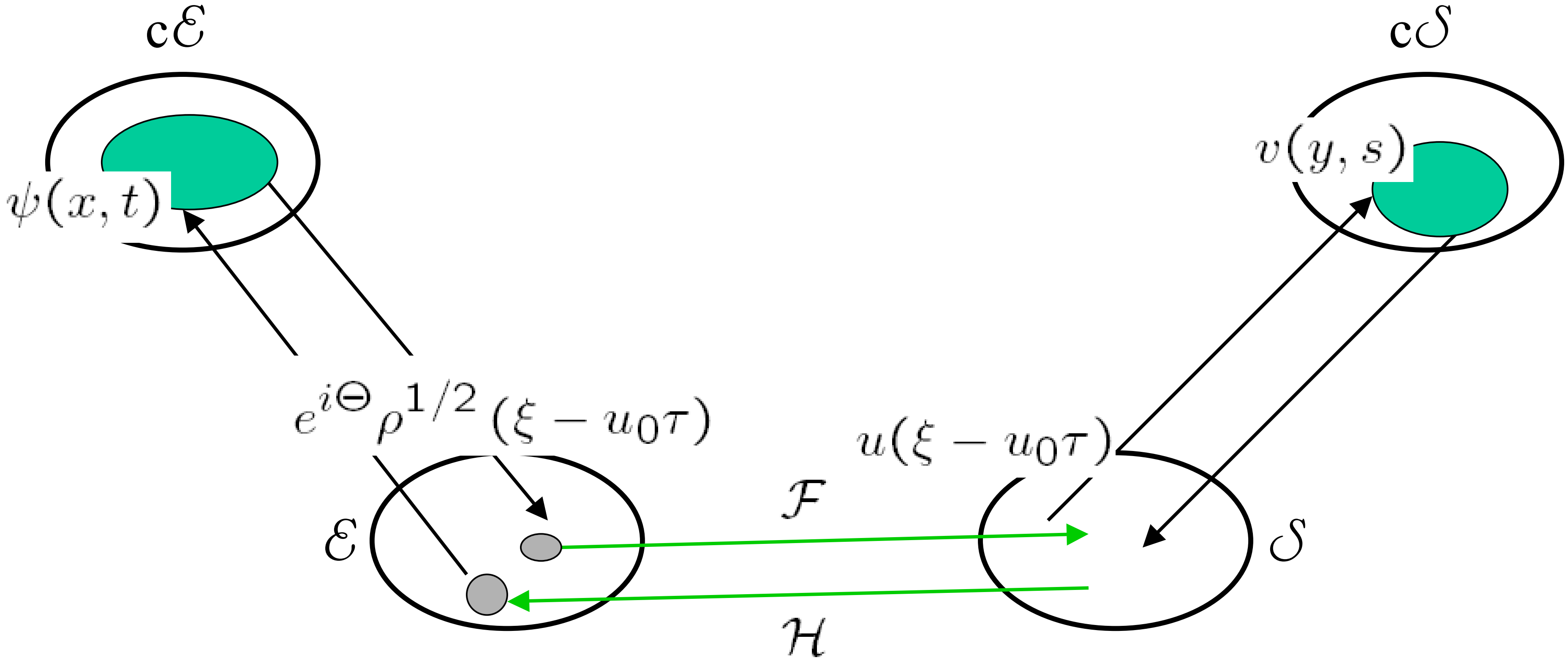
$$v(y,s) = \frac{1}{s}\,u\left(\xi = s^{-1/2}y, \tau = -2s^{-1/2}\right) + \frac{y}{2a's}$$

the standard KdVE

$$\frac{\partial u}{\partial \tau} + a'\,u\frac{\partial u}{\partial \xi} + b'\frac{\partial^3 u}{\partial \xi^3} = 0$$

becomes the cKdVE

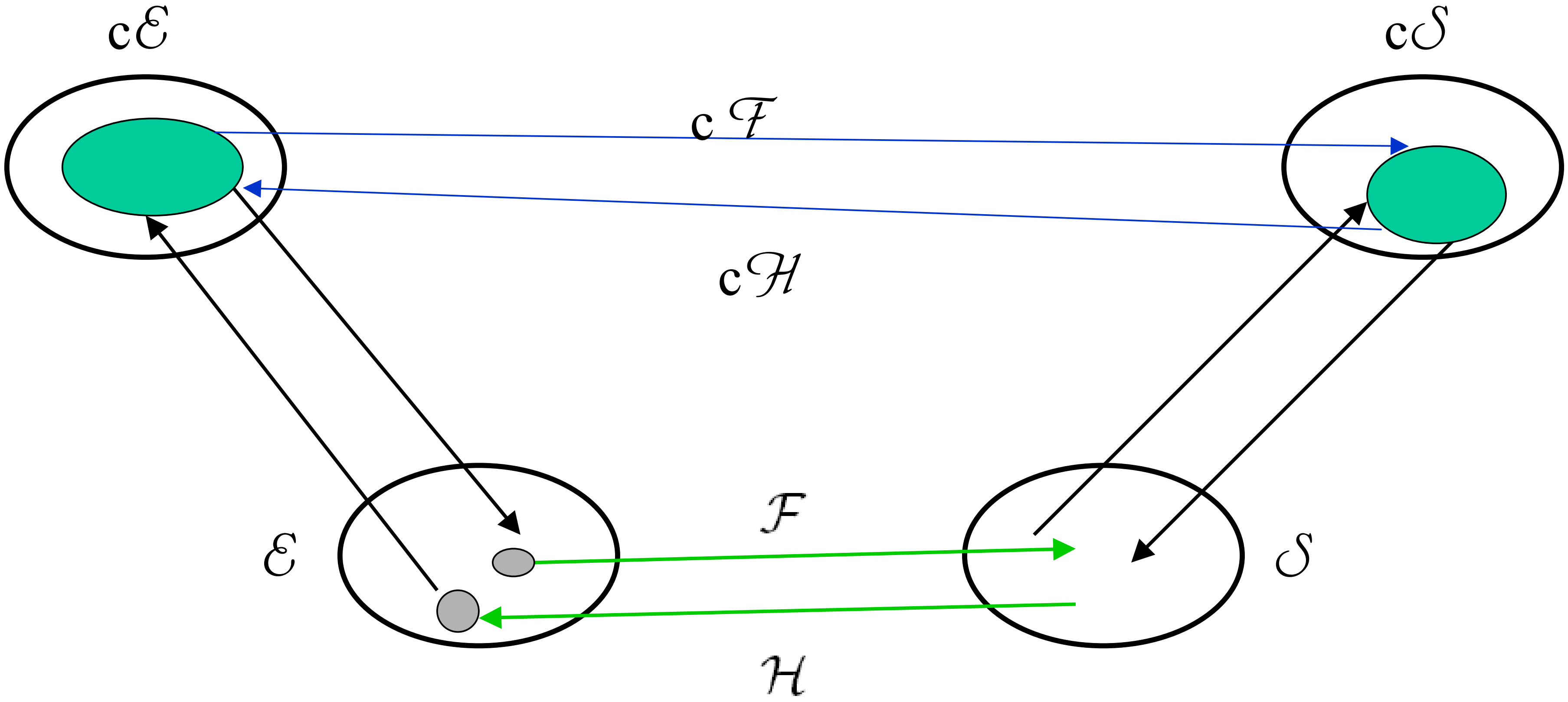
$$\frac{\partial v}{\partial s} + a'\,v\frac{\partial v}{\partial y} + b'\frac{\partial^3 v}{\partial y^3} + \frac{1}{2s}v = 0$$



$$\psi(x,t) = -\frac{1}{t}\exp\left[i\frac{x^2}{4at}\right] e^{i\Theta}\rho^{1/2}(\xi - u_0\tau)_{\xi=-\frac{x}{t},\,\tau=-\frac{1}{t}}$$

$$u(\xi - u_o\tau) = \rho(\xi - u_0\tau) = |\Psi(\xi - u_0\tau)|^2$$

$$v(y,s) = \frac{y}{2a's} + \frac{1}{s}u(\xi - u_0\tau)_{\xi=s^{-1/2}y,\,\tau=-2s^{-1/2}}$$



$c\mathcal{E}$
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