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Variational Approach for the Quantum Zakharov System.

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Variational approach for the quantum Zakharov system

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Plan of the talk

- Time-dependent variational solutions for the Gross-Pitaevskii equation
- Quantum Zakharov system
- Associated variational formalism
- Time-dependent variational solution for the quantum Zakharov system
- Conclusions

Time-independent variational method

- Variational solution for energy spectra: extremize

$$< \psi \,|\, H \,|\, \psi >$$

for a given class of wave-functions and Hamiltonian operator

- Then we have some parameters extremizing the expectation value of *H*.
- Some physical intuition about the form of the wave-function must be known in advance

Time-dependent variational solutions

- Widespread application
- Intuitive, approximate solutions
- There is the need for a Lagrangian function
- Very popular for Bose-Einstein condensates (BECs), for instance

Variational solution for Bose-Einstein condensates

- Gross-Pitaevskii equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+g\,|\,\psi\,|^2\,\psi+V(x,t)\psi=i\hbar\frac{\partial\psi}{\partial t}$$

- Usually, a harmonic external potential (laser field)
- Small coupling parameter *g (*dilute systems)

Action functional for BECs

$$S[\psi,\psi^*]=\int dt\, dx\, \psi^*\left(-i\hbar\frac{\partial}{\partial t}-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x,t)+\frac{g}{2}|\psi|^2\right)\psi$$

$$\frac{\delta S}{\delta\psi^*}=0 \;\Rightarrow\; equation \quad for \;\; \psi$$

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Time-dependent Gaussian *ansatz*

$$\psi \equiv \left(\frac{N}{\pi^{1/2}\alpha(t)} \right)^{1/2} \exp\left(-\frac{x^2}{2\alpha(t)^2} \right) \exp\left(i\beta(t) x^2 \right)$$

$$prob.\ density: |\psi|^2 = Gaussian$$

$$velocity\ \ field: \frac{i\hbar}{2m|\psi|^2} \left(\psi \frac{\partial \psi^*}{\partial x} - \psi^* \frac{\partial \psi}{\partial x} \right) = \frac{2\hbar}{m} \beta x$$

- It matches the ground-state of the quantum harmonic oscillator (quadratic external potential)
- Good for small coupling parameter and for the system close to thermodynamic equilibrium
- Non-stationary case: modulations around the ground-state solution

$$V = \frac{m\omega^2 x^2}{2} \Rightarrow S = \int dt\ L\ ,$$

$$L = L(\alpha, \beta, \dot{\beta}) = \frac{N}{4m}\left[\frac{\hbar^2}{\alpha^2} + \frac{\sqrt{2}gmN}{\sqrt{\pi}\alpha} + \alpha^2(4\hbar^2\beta^2 + m[m\omega^2 + 2\hbar\dot{\beta}])\right]$$

$$\delta S = 0 \Rightarrow \beta = \frac{m}{2\hbar}\frac{\dot{\alpha}}{\alpha},$$

$$\ddot{\alpha} + \omega^2 \alpha = \frac{\hbar^2}{m^2 \alpha^3} + \frac{gN}{\sqrt{2\pi}\, m\, \alpha^2}.$$

Quantum Zakharov system

- Zakharov equations: coupling between high frequency (Langmuir) and low frequency (ion-acoustic) modes
- Derivation: two-fluid model, electron-ion plasma, cold electrons, two time-scale method
- Ref: L. G. Garcia *et al.,* PoP **12**, 012302 (2005)

System of equations:

- *E* = envelope electric field
- *n* = density fluctuation
- *H* = dimensionless quantum parameter

$$i\frac{\partial E}{\partial t}+\frac{\partial^2 E}{\partial x^2}-H^2\frac{\partial^4 E}{\partial x^4}=nE$$

$$\frac{\partial^2 n}{\partial t^2}-\frac{\partial^2 n}{\partial x^2}+H^2\frac{\partial^4 n}{\partial x^4}=\frac{\partial^2}{\partial x^2}|E|^2$$

Lagrange formulation:

- Fields: *E*, *E**, *u*
- Ref: F. Haas, PoP **14**, 042309 (2007)

$$L = \frac{i}{2}\left(E^*\frac{\partial E}{\partial t} - E\frac{\partial E^*}{\partial t}\right) - \left|\frac{\partial E}{\partial x}\right|^2 - \frac{\partial u}{\partial x}|E|^2$$

$$+\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 - H^2\left|\frac{\partial^2 E}{\partial x^2}\right|^2 - \frac{H^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)^2,$$

$$n = \frac{\partial u}{\partial x}$$

Conservation laws

- Besides Hamiltonian and momentum functionals,

$$N = \frac{1}{\sqrt{\pi}} \int |E|^2 dx = number\ of\ \ quanta$$

Classical case: Langmuir ``soliton´´ solution

- Variational method in this case: B. Malomed, D. Anderson, M. Lisak, M. L. Quiroga-Teixeiro and L. Stenflo, Phys. Rev. E **55**, 962 (1997)
- Isolated Langmuir solitons do not decay
- Langmuir solitons ~ particle trapping

Quantum case: Gaussian time-dependent trial solution

$$E = A(t)\exp\left(-\frac{x^2}{2a(t)^2} + i\phi(t) + i\kappa(t)x^2\right),$$

$$n = -\frac{M}{s(t)^2}\exp\left(-\frac{x^2}{s(t)^4}\right)$$

- *a(t)* and *s(t)* ~ width of the Gaussians
- A linear stability analysis of the fixed points shows instability for sufficiently large quantum effects

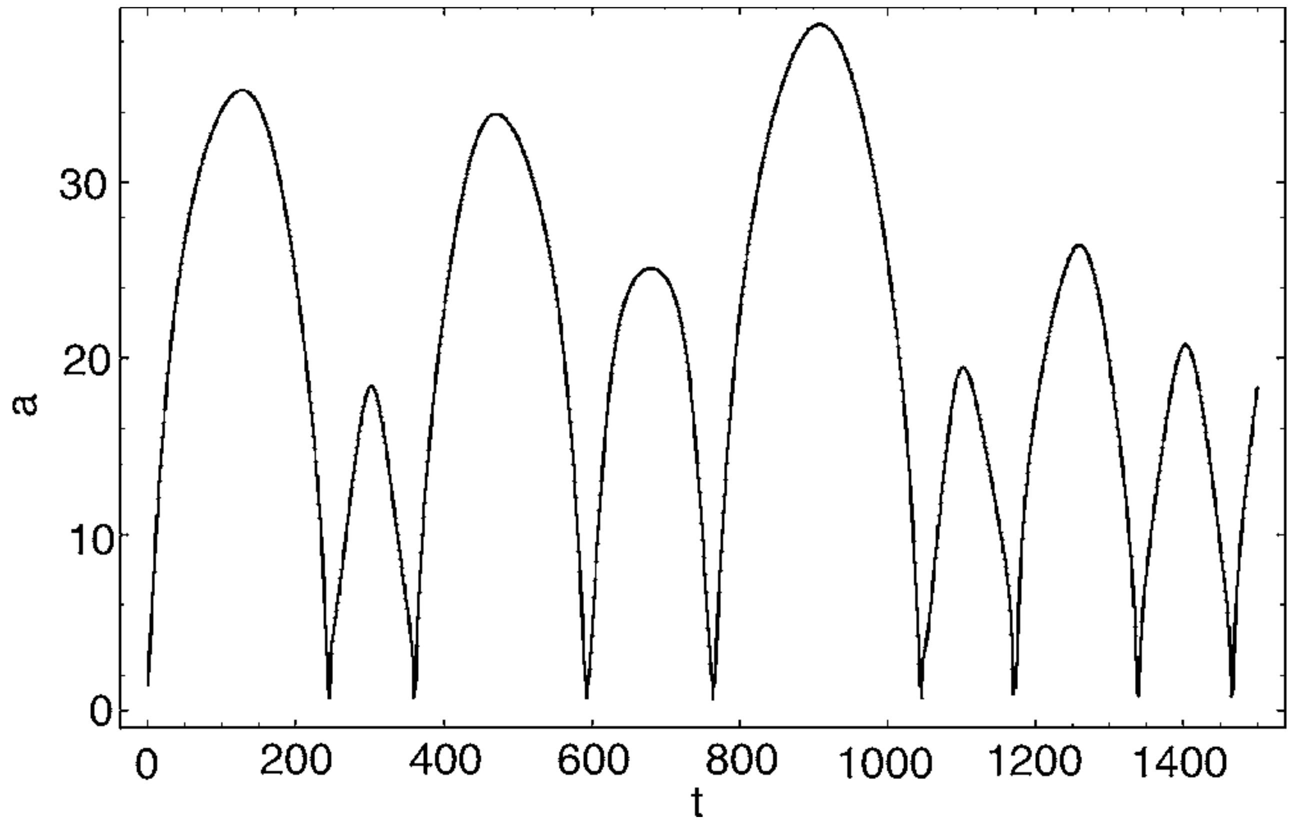

FIG. 2. Simulation for the semiclassical system (20)–(22) showing $a(t)$. Parameters, $M=N=3$, $H=0.3$. Initial condition, $(a_0, s_0, \dot{a}_0, \dot{s}_0) = (1.52, 1.36, 0, 0.62)$.

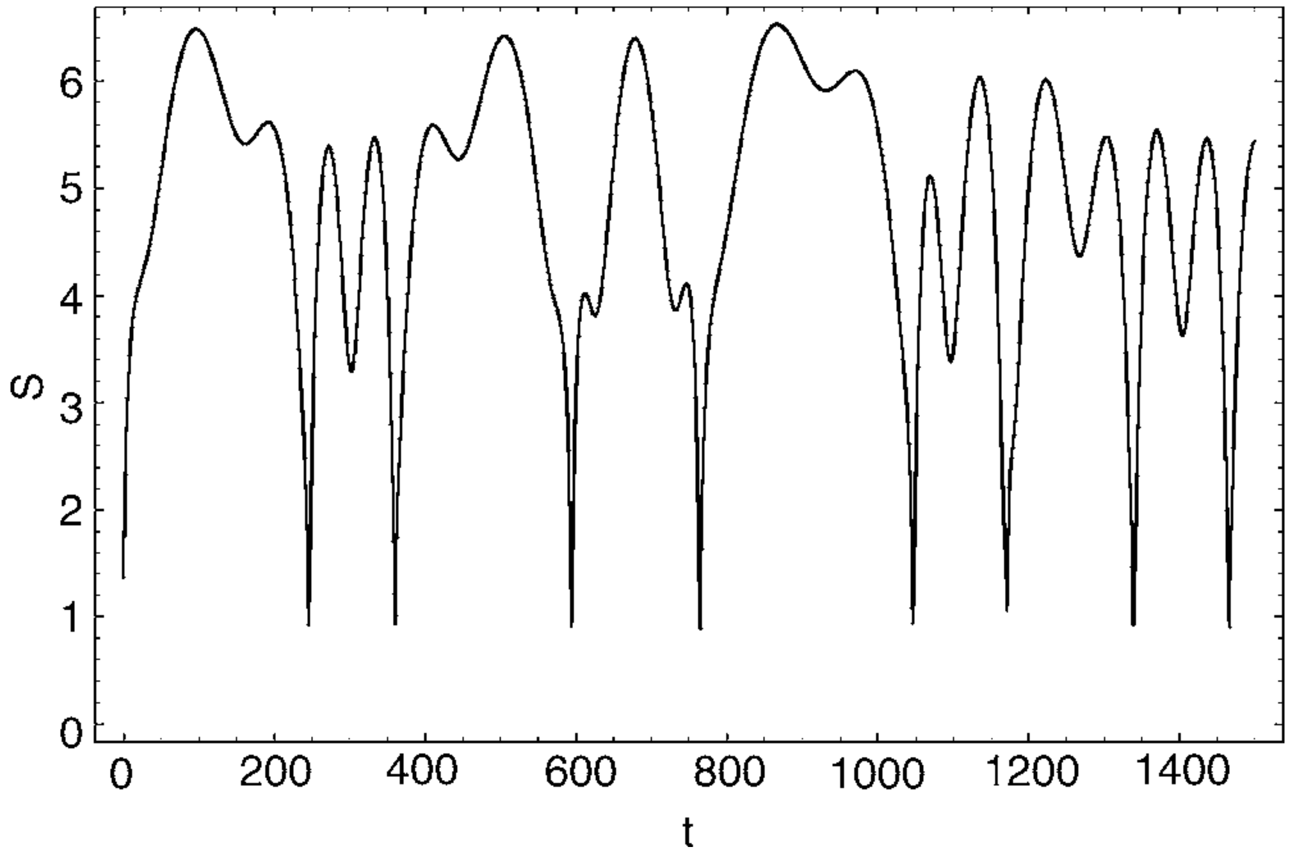

FIG. 3. Simulation for the semiclassical system (20)–(22) showing $s(t)$. Parameters, $M=N=3$, $H=0.3$. Initial condition, $(a_0, s_0, \dot{a}_0, \dot{s}_0) = (1.52, 1.36, 0, 0.62)$.

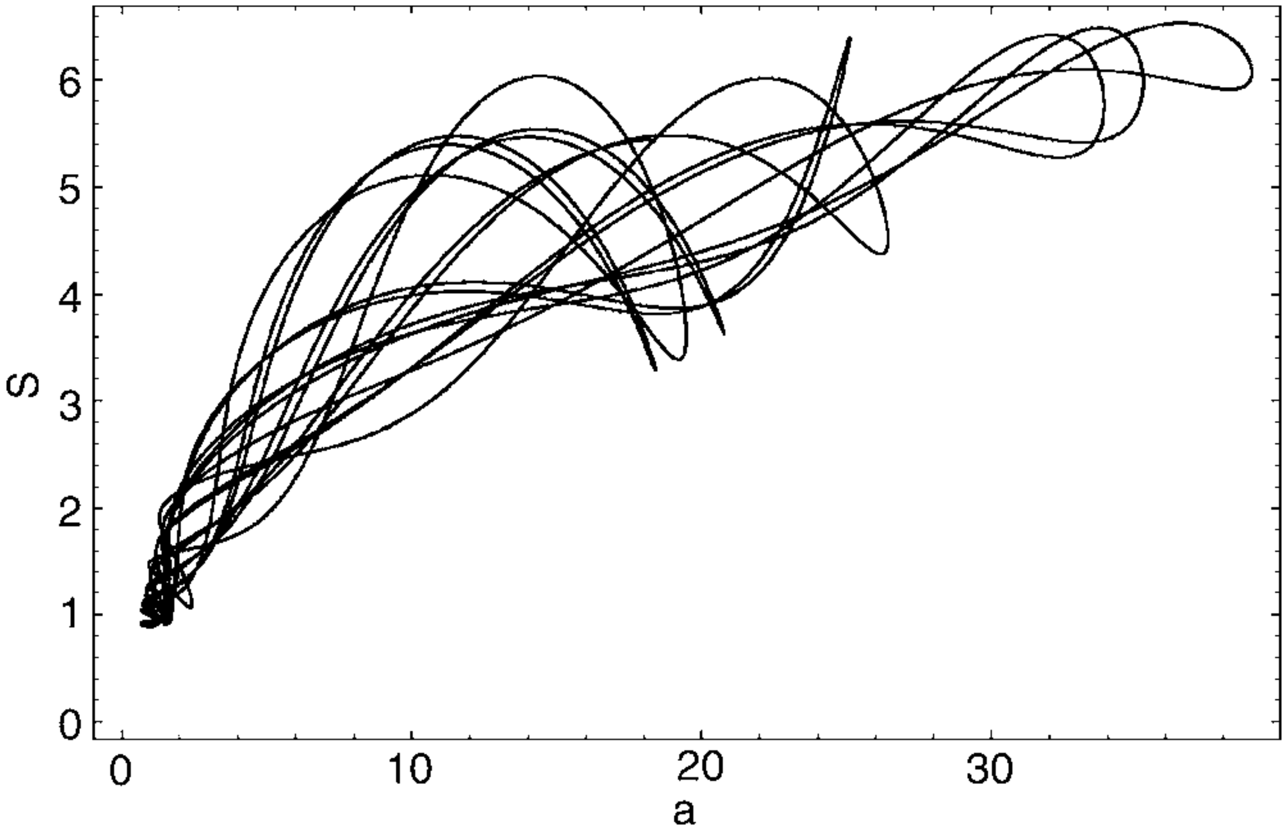

FIG. 4. Trajectory for (20)–(22) in configuration space. Parameters, $M=N=3$, $H=0.3$. Initial condition, $(a_0, s_0, \dot{a}_0, \dot{s}_0)=(1.52, 1.36, 0, 0.62)$.

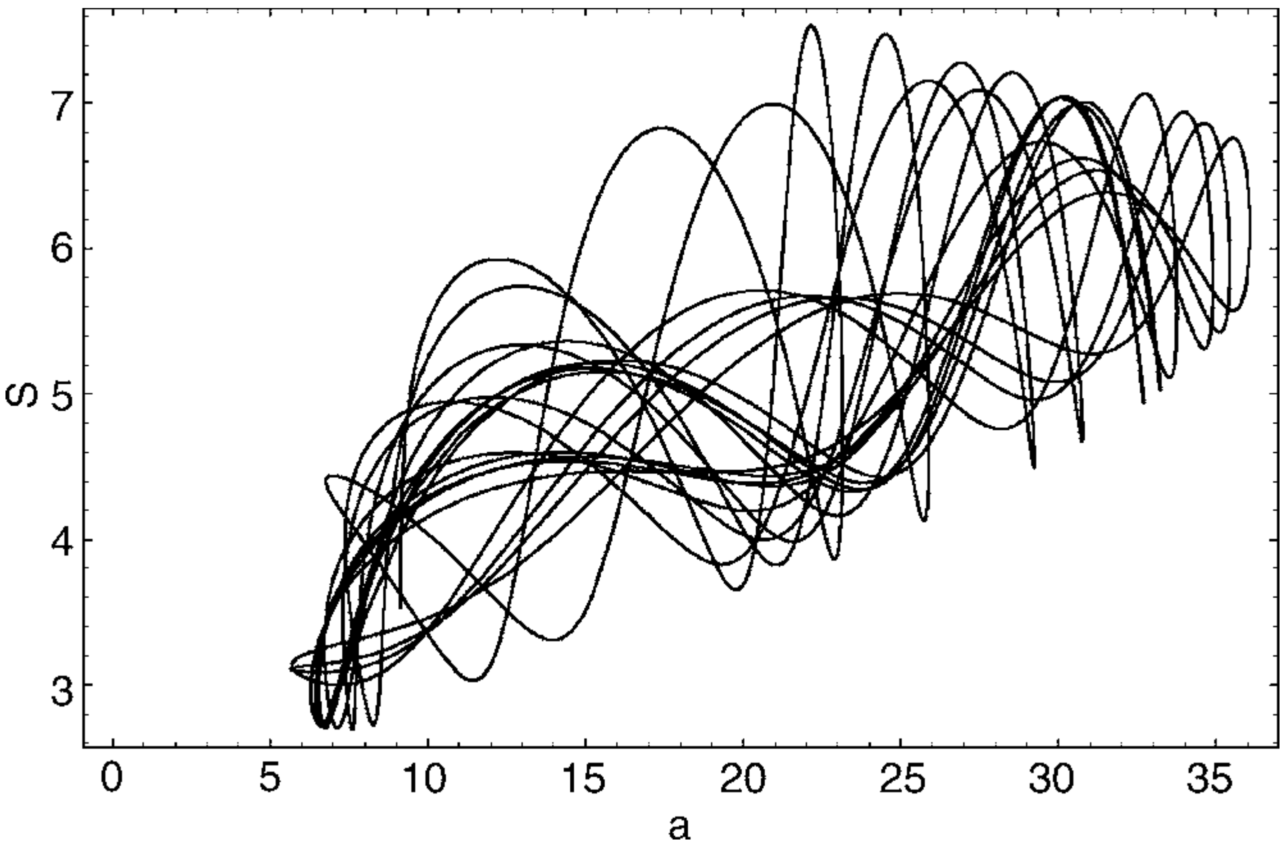

FIG. 5. Simulation for the full dynamical system (18)–(20) showing a and s. Parameters, $M=N=1$, $H=5$. Initial condition at $(\kappa,a,s,\dot{s})=(0,9.15,3.53,0.20)$.

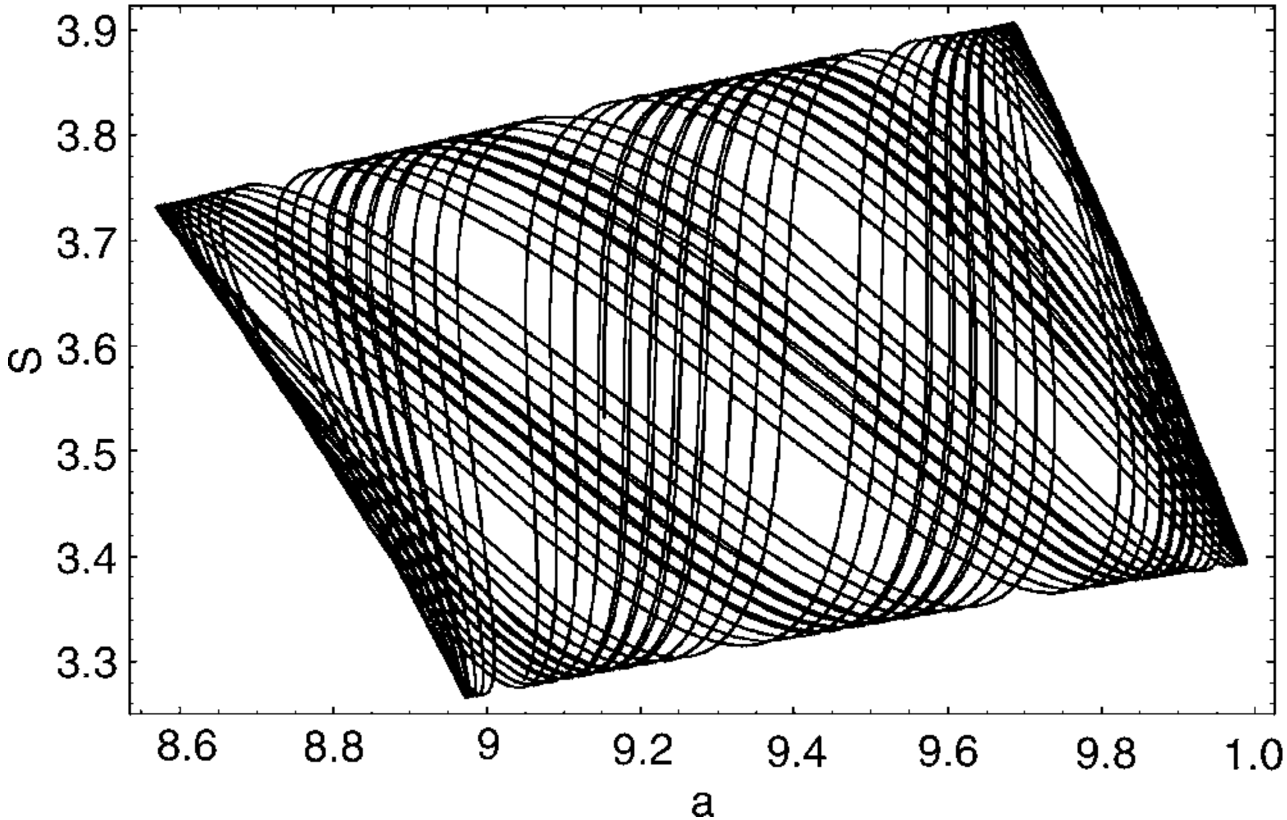

FIG. 6. Simulation for the full dynamical system (18)–(20) showing a and s. Parameters, $M = N = 1$, $H = 5$. Initial condition at $(\kappa, a, s, \dot{s}) = (0, 9.15, 3.53, 0.05)$.

Hyperchaos in the quantum Zakharov system

- Ref: A. P. Misra, D. Ghosh and A. R. Chowdhury, Phys. Lett. A **372**, 1469 (2008)
- Hyperchaos: at least two positive Lyapounov exponents

Conclusions

- Time-dependent variational approach: a powerful general, qualitative method
- Quantum effects destroy localizability in the case of the quantum Zakharov system
- Destabilizing influence
- Tunneling effect