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Gravitational Waves - Lecture 2

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Note: These lecture notes include much more detail than I intend to cover during my lecture!

I have included the extra detail to fill in the gaps of material that I cannot cover in detail during the lecture hour.

Material from page 14 onward is higher than the scope of my intended lecture, but is included for interested students.

Scott Hughes

Review / summary Key notions of general relativity

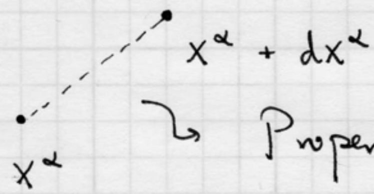
"Gravitational field" $\rightarrow$ "Geometry of spacetime"

Spacetime : a 3 + 1 dimensional manifold of events.

Label events in the spacetime with coordinates x^α.

If we've chosen time coordinates, then we'll label the remaining spatial degrees of freedom x^i :

Greek indices for spacetime, Latin for space.



↳ Proper spacetime distance between events:

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$$

"metric" of spacetime.

Examples:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

"Flat" spacetime of special relativity

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \frac{dr^2}{1 - 2GM/rc^2} + r^2 d\Omega^2$$

Spacetime of a non-rotating black hole

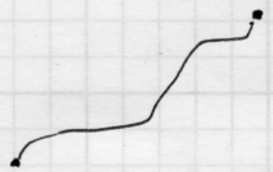
$$ds^2 = -dt^2 + (1 + h(t-z))\, dx^2 + (1 - h(t-z))\, dy^2 + dz^2$$

Flat spacetime + a gravitational wave.

Suppose a body moves through spacetime, tracing out the <u>worldline</u> $z^\alpha(\tau)$

↳ "proper time": Time as measured by that body

Very special worldline: <u>Geodesic</u>. Represents an <u>extremum</u> of spacetime distance between two events:



B

A

$$S = \int_A^B \sqrt{g_{\alpha\beta}\, dx^\alpha\, dx^\beta}$$

Extremize: $\delta S = 0$; result is

$$\boxed{\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = 0}$$

where

$$\boxed{\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu} \left(\partial_\gamma g_{\mu\beta} + \partial_\beta g_{\gamma\mu} - \partial_\mu g_{\beta\gamma} \right)}$$

$\Gamma^\alpha{}_{\beta\gamma}$ is the spacetime's "connection".

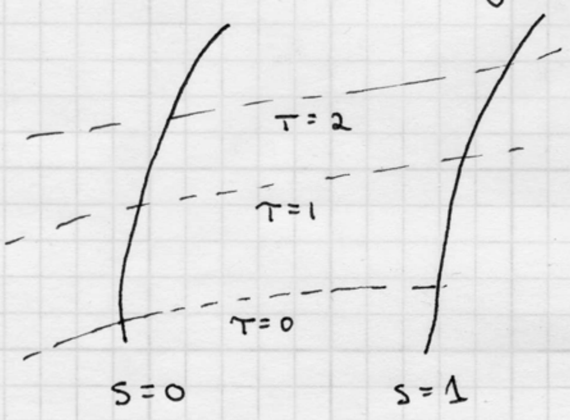
Importance of this result:

$$\boxed{\text{FREELY FALLING BODIES FOLLOW GEODESICS IN GENERAL RELATIVITY}}$$

This is how motion due to gravity is determined in general relativity: Sources of gravity determine spacetime metric, geodesic eqration tells how bodies move in that spacetime.

Note: trajectory independent of mass of body! Notion of principle of equivalence: free-fall equivalent, locally, to uniform acceleration in flat spacetime.

Consider two nearby geodesics:



τ : proper time along geodesics

s : parameter that smoothly varies from 1 curve to other.

$u^\alpha = \dfrac{dx^\alpha}{d\tau}$ $\equiv$ tangent along geodesics.

$\xi^\alpha = \dfrac{dx^\alpha}{ds}$ $\equiv$ tangent along curves of constant τ that connect geodesics.

$\equiv$ notion of displacement.

How does the displacement evolve? Governed by equation of geodesic deviation:

$$\boxed{\frac{D^2 \xi^\alpha}{d\tau^2} = R^\alpha{}_{\beta\gamma\delta}\, u^\beta u^\gamma \xi^\delta}$$

where
$$\boxed{\begin{aligned} R^\alpha{}_{\beta\gamma\delta} &= \partial_\gamma \Gamma^\alpha{}_{\beta\delta} - \partial_\delta \Gamma^\alpha{}_{\beta\gamma} \\ &\quad + \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta} - \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma} \end{aligned}}$$

is the Riemann curvature tensor: Encodes the deviation of spacetime from a flat geometry.

Also encodes tidal gravitational effects: Rate at which neighboring geodesics diverge tells us how free-fall varies over regions of spacetime.

Variants of curvature tensor:

$$R_{\alpha\beta} \equiv R^{\mu}{}_{\alpha\mu\beta} \qquad \text{"Ricci curvature"}$$

symmetric on indices α & β.

$$R \equiv R^{\mu}{}_{\mu} = g^{\mu\beta} R_{\beta\mu}$$

"Ricci scalar"

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \qquad \text{"Einstein curvature"}$$

So far, everything just concerns geometry ... also need to describe matter, fields, etc ... tool for this is the <u>stress-energy</u> tensor:

$$T^{\mu\nu} \equiv \text{Stress-energy}$$
$$= \text{flux of momentum density } p^{\mu} \text{ in } x^{\nu} \text{ direction}$$

Physical meaning of components:

T^{00} = energy density

T^{0i} = energy flux $\rightarrow$ $T^{0i} = T^{i0}$ in units with $c = 1$!

T^{i0} = momentum density

T^{ij} = momentum flux (T^{ii} = Pressure)

Stress energy tensor acts as our <u>source</u> for spacetime via <u>Einstein field equation</u>:

$$\boxed{G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

Note structure: $G_{\mu\nu} \sim \partial^2 g$: Two derivatives of metric = source.

Reminiscent of Newtonian field equation: $\nabla^2 \phi = 4\pi G \rho$

→ Note association of "potential" with metric.

Weak gravity: Consider limit in which spacetime is nearly flat:

$$\boxed{g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}}$$

Components $\| h_{\alpha\beta} \| \ll 1$.

Now, linearize in h: For example,

$$h^{\alpha\beta} \equiv g^{\alpha\mu} g^{\beta\nu} h_{\mu\nu} = \eta^{\alpha\mu} \eta^{\beta\nu} h_{\mu\nu} + \mathcal{O}(h^2)$$

Note that combining this with definition

$$g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}{}_{\gamma}$$

implies $\boxed{g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta} + \mathcal{O}(h^2)}$

Important detail: Suppose we change coordinates. How does our representation of the metric change?

Simple: If we change from x^α to y^α, then

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\nu} \, g_{\mu\nu}$$

representation in new coordinates

representation in old coordinates

Suppose coordinates are just slightly shifted:

$$y^\alpha = x^\alpha + \xi^\alpha$$

where $\partial \xi^\alpha / \partial x^\mu \ll 1$. Then,

$$\frac{\partial y^\alpha}{\partial x^\mu} = \delta^\alpha{}_\mu + \partial_\mu \xi^\alpha$$

$$\longrightarrow \quad \frac{\partial x^\alpha}{\partial y^\mu} = \delta^\alpha{}_\mu - \partial_\mu \xi^\alpha + \mathcal{O}(\partial \xi^2)$$

Now, examine how representation of weak gravity metric changes:

$$g'_{\mu\nu} = \left(\frac{\partial x^\alpha}{\partial y^\mu}\right)\left(\frac{\partial x^\beta}{\partial y^\nu}\right) g_{\alpha\beta}$$

$$\rightarrow \quad g'_{\mu\nu} = \left(\delta^\alpha{}_\mu - \partial_\mu \xi^\alpha\right)\left(\delta^\beta{}_\nu - \partial_\nu \xi^\beta\right)\left(\eta_{\alpha\beta} + h_{\alpha\beta}\right)$$

$$= \eta_{\mu\nu} + h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

We can write this as

$$\boxed{h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu}$$

Gauge transformation! Just like

$$A'_\mu = A_\mu - \partial_\mu \Lambda$$

in Maxwell's theory.

With a little effort, not too difficult to show that under gauge transformation

$$R_{\alpha\mu\beta\nu} \longrightarrow R_{\alpha\mu\beta\nu}$$

In words,

$$\boxed{\text{The curvature tensor is invariant to adjusting the gauge of our spacetime}}$$

Go through definitions of tensors, assemble Einstein tensor:

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$$

$$= \frac{1}{2}\Big\{ \partial_\alpha \partial^\mu h_{\mu\beta} + \partial_\beta \partial^\mu h_{\mu\alpha} - \partial_\alpha \partial_\beta h$$

$$- \Box h_{\alpha\beta} + \eta_{\alpha\beta} \Box h - \eta_{\alpha\beta} \partial^\mu \partial^\nu h_{\mu\nu} \Big\}$$

where $h = \eta^{\mu\nu} h_{\mu\nu} \equiv$ The "trace" of $h_{\mu\nu}$.

Simplification 1: Introduce

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} h$$

Notice: $\eta^{\alpha\beta} \bar{h}_{\alpha\beta} = h - \frac{1}{2} \eta^{\alpha\beta} \eta_{\alpha\beta} h = -h$

Insert $h_{\alpha\beta} = \bar{h}_{\alpha\beta} + \frac{1}{2} \eta_{\alpha\beta} h \longrightarrow$

$$G_{\alpha\beta} = \frac{1}{2}\Big\{ \partial_\alpha \partial^\mu \bar{h}_{\mu\beta} + \partial_\beta \partial^\mu \bar{h}_{\mu\alpha} - \eta_{\alpha\beta} \partial^\mu \partial^\nu \bar{h}_{\mu\nu}$$

$$- \Box \bar{h}_{\alpha\beta} \Big\}$$

Simplification 2: Adjust gauge so that

$$\partial^\mu \bar{h}_{\mu\nu} = 0 \qquad \text{"Lorenz gauge"}$$

Then,

$$G_{\alpha\beta} = -\frac{1}{2} \Box \bar{h}_{\alpha\beta}$$

Einstein's field equation becomes

$$\boxed{\Box \bar{h}_{\alpha\beta} = -\frac{16\pi G}{c^4} T_{\alpha\beta}}$$

Identical structure to Maxwell wave equation for potential A_μ! Solution "obvious":

$$\boxed{\bar{h}_{\alpha\beta} = 4G \int \frac{T_{\alpha\beta}(\vec{x}', t - |\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|} d^3x'}$$

Putting $c = 1$ for convenience.

Note: This solution makes <u>all</u> components of $h_{\alpha\beta}$ look radiative. Consequence of the gauge choice!

Theorem: Given a solution $h_{\alpha\beta}$ to the linearized field equations only the <u>spatial</u>, <u>transverse</u>, and <u>traceless</u> components h^{TT}_{ij} encode the radiation content in a gauge invariant manner.

(Proof: Sec 2.2 of "The basics of gravitational wave theory", New Journal of Physics, vol 7, p 204, 2005; gr-qc/0501041)

Transverse means $\partial_i h^{TT}_{ij} = 0$

(Note: $\partial^i = \partial_i$ since spatial metric is Euclidean.)

Traceless means $\delta_{ij} h^{TT}_{ij} = 0$.

Solution for spatial components:

$$\bar{h}_{ij} = \frac{4G}{r} \int T_{ij}(t-r, \vec{x}') \, d^3x'$$

(Working in distant limit.)

Recall that in electrodynamics, the continuity equation allowed us to write

$$\int \vec{A} \; d^3x' = -\frac{\partial}{\partial t} \int \vec{x}' \rho_Q \, d^3x'$$

Similar equation in general relativity:

$$\partial^\mu T_{\mu\nu} = 0 \leftarrow \text{Conservation* of stress-energy}$$

From this, we can derive

$$\int T_{ij} \, d^3x' = \frac{1}{2} \frac{\partial^2}{\partial t^2} \int T_{00} \, x_i' x_j' \, d^3x'$$

Recall T_{00} = mass / energy density $\equiv \rho$.

Define: $I_{ij} = \int \rho \, x_i' x_j' \, d^3x' \rightarrow$ "Quadrupole moment" of source.

Then, $$\bar{h}_{ij} = \frac{2G}{r} \frac{d^2 I_{ij}}{dt^2}$$

* Strictly true only in linearized limit!!

Finally, need to project out the transverse & traceless components of this. Easily done: If radiation is propagating along vector $\vec{n}$, define

$$P_{ij} = \delta_{ij} - n_i n_j$$

tensor that projects out components orthogonal to $\vec{n}$: Any component parallel to $\vec{n}$ is multiplied by zero.

Then,

$$\cancel{h^{TT}_{ij} = \bar{h}_{ij}(}$$

$$\boxed{\begin{aligned} h^{TT}_{ij} &= \bar{h}_{kl}\left(P_{ki}P_{lj} - \frac{1}{2}P_{kl}P_{ij}\right) \\ &= \frac{2G}{r}\frac{d^2 I_{kl}}{dt^2}\left(P_{ki}P_{lj} - \frac{1}{2}P_{kl}P_{ij}\right) \end{aligned}}$$

The "quadrupole formula" for GW emission.

More advanced topics

1. Higher order multipoles: As in electrodynamics, radiation beyond leading multipoles exists. Each moment tends to be suppressed by a factor of v/c from the preceding moment.

Examples:

$$I_{ijk} = \int \rho \, x'_i x'_j x'_k \, d^3x' \qquad \text{"Octupole"}$$

$$\rightarrow \quad h \sim \frac{G}{r} \frac{d^3}{dt^3} I_{ijk} n_k$$

Extra derivative suppresses radiation by $\mathcal{O}(v/c)$ relative to quadrupole.

$$J_{ij} = \int \rho \, x'_i x'_k v'_l \, \epsilon_{jkl} \, d^3x' \qquad \text{"Current quadrupole"}$$

Note explicit factor of v.

$$\rightarrow \quad h \sim \epsilon^{kli} \frac{d^2}{dt^2} J^{jk} n^l$$

2. Curved background: Putting

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

is quite restrictive. More general case:

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} + h_{\alpha\beta}$$

Some slowly varying background spacetime: Metric of our expanding universe, or a black hole, our solar system ...

Same basic idea holds - linearize Einstein about amplitude $h_{\alpha\beta}$, but add one new concept:

$h_{\alpha\beta}$ varies on shorter lengthscales and timescales than $\hat{g}_{\alpha\beta}$:

$$\partial_t \hat{g}_{\alpha\beta} \sim \frac{\hat{g}_{\alpha\beta}}{T}, \quad \partial_r \hat{g}_{\alpha\beta} \sim \frac{\hat{g}_{\alpha\beta}}{\mathcal{R}}$$

$$\partial_t h_{\alpha\beta} \sim \frac{h_{\alpha\beta}}{\tau}, \quad \partial_r h_{\alpha\beta} \sim \frac{h_{\alpha\beta}}{\lambda}$$

$$\tau \ll T, \quad \lambda \ll \mathcal{R}.$$

Allows us to organize problem on multiple scales.

Find results largely the same, but curved background modifies details: example,

$$G_{\alpha\beta} = -\frac{1}{2}\hat{\Box}\bar{h}_{\alpha\beta} - \hat{R}_{\alpha\mu\beta\nu}\bar{h}^{\mu\nu}$$

Riemann of background

$\hat{\Box}$ ≡ wave operator on curved background

≡ $\hat{g}^{\mu\nu}\hat{\nabla}_\mu\hat{\nabla}_\nu$, where

$\hat{\nabla}_\mu$ ≡ Covariant derivative on background: a derivative that accounts for how basis vectors vary as we move in our manifold.

= ∂_μ ± Terms involving connection coefficients.

3. Non-linearity: One might worry that our linearization procedure throws away nonlinearity: one of the defining characteristics of general relativity!

Not too difficult to derive a totally gauge-invariant, nonlinear wave equation for <u>curvature</u>. (Much harder for metric due to gauge freedom.)

Ingredients:

1. Bianchi identity:

$$\nabla_\alpha R_{\beta\gamma\mu\nu} + \nabla_\beta R_{\gamma\alpha\mu\nu} + \nabla_\gamma R_{\alpha\beta\mu\nu} = 0$$

2. Commutator rule for covariant derivatives:

$$[\nabla_\mu, \nabla_\nu] p_\alpha = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) p_\alpha$$
$$= - p_\beta R^\beta{}_{\alpha\mu\nu}$$

$$[\nabla_\mu, \nabla_\nu] p_{\alpha\beta} = - p_{\gamma\beta} R^\gamma{}_{\alpha\mu\nu} - p_{\alpha\gamma} R^\gamma{}_{\beta\mu\nu}$$

Recipe: Take one more derivative of Bianchi:

$$\nabla^\alpha \nabla_\alpha R_{\beta\gamma\mu\nu} \equiv \Box R_{\beta\gamma\mu\nu} = -\nabla^\alpha \nabla_\beta R_{\gamma\alpha\mu\nu} - \nabla^\alpha \nabla_\gamma R_{\alpha\beta\mu\nu}$$

wave operator for curved spacetime.

Now, apply commutator rule repeatedly. Result becomes

$$\Box R_{\beta\gamma\mu\nu} = \text{(terms that are quadratic in Riemann curvature.)}$$

Note mathematical structure: A wave equation for a field that ~~has~~ has a source that is nonlinear in that field!

If the spacetime is vacuum ($T_{\mu\nu} = 0$), the result is simple:

$$\boxed{\begin{aligned}\Box R_{\alpha\beta\mu\nu} = {} & 2 R_{\mu\gamma\beta\delta} R_{\nu}{}^{\gamma}{}_{\alpha}{}^{\delta} \\ & - 2 R_{\mu\gamma\alpha\delta} R_{\nu}{}^{\gamma}{}_{\beta}{}^{\delta} \\ & + R_{\mu\nu\delta\gamma} R^{\delta\gamma}{}_{\alpha\beta}\end{aligned}}$$

"Penrose wave equation"

4. Energy content of waves: Very subtle point.

Thanks to principle of equivalence, we can always make metric look flat in vicinity of some point:

$$g_{\alpha\beta} \longrightarrow \eta_{\alpha\beta} + \mathcal{O}(\text{Riemann} \times r^2)$$

"Freely falling cordinates"

How can we ascribe energy to wave if we can always set its metric to zero with a clever coordinate choice??

Answer: <u>Non-locality</u>! The wave can only be hidden inside a region whose size is of order λ, the wavelength of the GW.

Isaacson introduced rigorous techniques to define tensors made from quantities averaged over a region of size several $\times\ \lambda$: see

R. Isaacson, Physical Review D, vol 166

p 1263 (1968)

p 1272 (1968)

Key result: A gauge invariant stress-energy tensor describing energy and momentum carried by GWs:

$$\boxed{T^{GW}_{\mu\nu} = \frac{1}{32\pi G} \left\langle \nabla_\mu h_{\alpha\beta} \nabla_\nu h^{\alpha\beta} \right\rangle}$$

where $\langle f \rangle \equiv$ "average f over several wavelengths"

and $h_{\alpha\beta}$ is in a transverse-traceless form.