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Gravitational waves
Lecture 3

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These notes are also more detailed than the material I will cover!

— Scott Hughes

How to measure a gravitational wave

1. Indirectly: Use the fact that GWs carry energy, look for a system in which "backreaction" of waves on system's characteristics can be measured.

Example: Binary star systems. Consider stars sufficiently far apart that orbital dynamics is well-described by Newtonian gravity. For example, if orbits are circular, then

$$E_{orb} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - \frac{G m_1 m_2}{r}$$

$$= -\frac{1}{2} \frac{G \mu M}{r} \qquad M = m_1 + m_2$$

$$\mu = m_1 m_2 / M$$

Ω = orbital angular frequency

$$= \sqrt{\frac{GM}{r^3}} \qquad (r = \text{orbital separation})$$

Use quadrupole formula to compute waves:

$$h_{ij}^{TT} = \frac{2}{R} G \frac{d^2}{dt} I_{kl} \left(P_{ki} P_{lj} - \frac{1}{2} P_{kl} P_{ij} \right)$$

$$(R = \text{distance from binary to measurement})$$

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where $I_{kl} = \int \rho \, x'_k x'_l \, d^3x'$

$$\doteq \begin{bmatrix} \mu r^2 \cos^2 \Omega t & \mu r^2 \cos \Omega t \sin \Omega t & 0 \\ \mu r^2 \cos \Omega t \sin \Omega t & \mu r^2 \sin^2 \Omega t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note: oriented coordinates such that binary is in the x-y plane.

Also, recall $P_{ij} = \delta_{ij} - n_i n_j$
where n_i represents components of a unit vector pointing from the source to an observer. Useful to write this in spherical coordinates:

$$n_i \doteq \left(\sin\theta \cos\phi, \ \sin\theta \sin\phi, \ \cos\theta \right)$$

Now, straightforward to assemble gravitational waveform observed by someone at arbitrary θ, ϕ with respect to the binary.

Next, apply Isaacson's stress energy tensor to compute total energy loss due to GW emission:

$$\frac{dE^{GW}}{dt} = \int R^2 \, d\Omega \left(T^{00} \right)^{GW}$$

$$= \int R^2 \, d\Omega \ \langle \partial_t h_{ij} \, \partial_t h_{ij} \rangle$$

For quadrupole GWs, result is

$$\boxed{\begin{aligned}\frac{dE^{GW}}{dt} &= \frac{G}{5}\left\langle \dddot{I}_{ij}\, \dddot{I}_{ij} \right\rangle \\ &= \frac{32G}{5}\mu^2 r^4 \Omega^6\end{aligned}}$$

← General quadrupole

← circular binary

Now, require conservation of total energy:

$$\frac{dE^{orb}}{dt} + \frac{dE^{GW}}{dt} = 0$$

Allow radius of orbit to slowly shrink:

$$\boxed{\frac{dE^{orb}}{dt} = -\frac{G\mu M}{2r^2}\,\frac{dr}{dt}}$$

Now, combine

- result for dE^{GW}/dt
- result for dE^{orb}/dt
- $\Omega = \sqrt{GM/r^3} = 2\pi/P$

  where P = orbital period

to find $\frac{dP}{dt}$ ≡ rate of change of orbital period due to GW emission

$$= -\frac{96}{5}\, P\, \frac{G^3 \mu M^2}{r^4}$$

(More realistic: binaries with eccentricity. Easy to fix: detailed analysis shows that

$$\frac{dP}{dt} = -\frac{96}{5} P \frac{G^3 \mu M^2}{a^4} f(e)$$

$a \equiv$ semi-major axis of orbit

$$f(e) = \frac{\left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4\right)}{(1 - e^2)^{7/2}}$$

$e \equiv$ orbital eccentricity.)

Now about 5 binary neutron star systems found in which this period decay has been observed! Key is that at least one member of the binary is a <u>pulsar</u>: emits very regular radio pulses, acting as a clock.

In all cases, general relativistic formula for dP/dt is confirmed, typically with errors $< 1\%$.

2. Directly: Look for the influence of GWs on matter.
Two interesting cases:
   1. Free "fluid" (actually, plasma) of early universe
   2. Freely falling masses.

Consider fluid/plasma first: To analyze, we consider the equation of motion for matter,

$$\nabla_\alpha T^{\alpha\beta} = 0$$

↳ Covariant derivative: accounts for how geometry varies as we move in our curved manifold.

$$\nabla_\alpha T^{\alpha\beta} = \partial_\alpha T^{\alpha\beta} + T^{\mu\beta}\Gamma^\alpha_{\alpha\mu} + T^{\alpha\mu}\Gamma^\beta_{\alpha\mu}$$

"connection" from Lecture 2.

Zeroth order: "Perfect" fluid: a fluid with no viscosity, heat flow, or anisotropy:

$$T^{\alpha\beta} = (\rho + P)\,u^\alpha u^\beta + P g^{\alpha\beta}$$

ρ = density of fluid
P = Pressure of fluid
u^α = velocity of fluid element
$g^{\alpha\beta}$ = spacetime metric.

In particular, compare spacetime with GW to spacetime without:

$$\nabla_\alpha T^{\alpha\beta} = 0 \quad \rightarrow \quad g_{\alpha\beta} \text{ includes GWs}$$

$$\hat{\nabla}_\alpha \hat{T}^{\alpha\beta} = 0 \quad \rightarrow \quad g_{\alpha\beta} \text{ does not include GWs.}$$

Compare dynamics of fluid with & without GWs

Result: NO DIFFERENCE!

$\longrightarrow$ Gravitational waves do not couple to a homogeneous perfect fluid: Need some kind of inhomogeneity for effects to appear.

Better example: Viscous fluid:

$$T^{\alpha\beta} = (\rho + P)u^\alpha u^\beta + P g^{\alpha\beta} - 2\eta\sigma^{\alpha\beta}$$

$\eta =$ shear viscosity

$\sigma^{\alpha\beta} \equiv$ rate of shear of fluid.

Detailed analysis shows $\sigma_{ij} = \partial_t h_{ij}^{GW}$: Leads to energy transfer between GW and fluid.

Analysis of this kind important in setting foundations for interaction of GWs with primordial plasma in the early universe; observable imprint by impact of primordial GWs on polarization of cosmic microwave background.

Next, examine free-floating masses:

How would a gravitational wave impact these masses?

Good spacetime for this:

$$ds^2 = -dt^2 + (1+h)\,dx^2 + (1-h)\,dy^2 + dz^2$$

$h = h(t-z)$ = GW propagating down z-axis.

Naive calculation: calculate the motion of masses using geodesic equation in this spacetime:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$$

Before GW: $\frac{dx^\mu}{d\tau} = (1,0,0,0)$ for both masses: initially at rest.

After GW: $\frac{dx^\mu}{d\tau} = (1,0,0,0) + \mathcal{O}(h)$

shifted by an amount of order the GW.

Focus on spatial motions:

$$\frac{d^2x^j}{d\tau^2} + \Gamma^j{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

$$\rightarrow \quad \frac{d^2x^j}{d\tau^2} + \Gamma^j{}_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} + \mathcal{O}(h^2) = 0$$

In principle, neglected terms of form here:

$$\underbrace{\Gamma}_{\mathcal{O}(h)} \times \underbrace{\frac{dx^0}{d\tau}}_{\mathcal{O}(1)} \times \underbrace{\frac{dx^i}{d\tau}}_{\mathcal{O}(h)}$$

For this metric, $\Gamma^j{}_{00} = 0 \rightarrow$

$$\boxed{\frac{d^2x^j}{d\tau^2} = 0 + \mathcal{O}(h^2)}$$

→ The masses <u>apparently</u> don't move to leading order in GW amplitude h!

What's going on??? Geodesic equation describes motion IN A GIVEN COORDINATE SYSTEM ... in this case, the coordinates are essentially comoving with the masses! (Note that masses are at $x=0$ & $x=L$ at all times.)

Better analysis: Compute *proper* separation of masses.

$$D = \text{Separation of masses}$$

$$= \int_0^L ds = \int_0^L g_{xx}\, dx$$

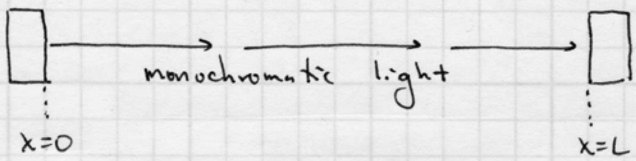
$$= \int_0^L \sqrt{1 + h_+}\, dx$$

$$\simeq \int_0^L \left(1 + \frac{h_+}{2}\right) dx$$

$$\rightarrow \quad D = L + \frac{h_+ L}{2}$$

Proper separation varies with *strain* $h_+/2$.

Even better analysis, though slightly beyond scope of these lectures: Compute phase shift imparted to light that propagates down the "arm" formed by the two masses:



$$\Delta\Phi = \int \omega \, dt$$

ω ↳ frequency of laser

$$= \int k_0 \, dt$$

$k_0 \equiv$ time component of wave vector for light

$$= g_{tt} k^0 = -k^0.$$

$$k^0 = \hat{k}^0 + \delta k^0$$

$\hat{k}^0$ ↳ wave vector component in absence of gravitational wave.

Shift in wave vector can be computed by integrating the geodesic equation:

$$\frac{d\delta k^0}{d\chi} + \delta\Gamma^0{}_{\alpha\beta} \hat{k}^\alpha \hat{k}^\beta = 0$$

$\delta\Gamma^0{}_{\alpha\beta} \equiv$ shift in connection due to GW

$\chi \equiv$ "affine parameter" along light ray

$$\rightarrow \quad \frac{d\,\delta k^0}{dx} = \frac{1}{2}\,\dot{h}_+\,\hat{k}^x\,\hat{k}^x$$

For radiation, $\hat{k}^x = \hat{k}^0$ in units with $c=1$.

Also, $\vec{\hat{k}} = \dfrac{d\vec{x}}{dx}$, so $\hat{k}^0 = \dfrac{dt}{dx}$;

thus,

$$\frac{d\,\delta k^0}{dt} = \frac{1}{2}\,\dot{h}_+\,\hat{k}^0$$

$$\rightarrow \quad \delta k^0 = \frac{1}{2}\,h_+\,\hat{k}^0$$

since $\hat{k}^0 =$ constant.

$$\rightarrow \quad k^0 = \hat{k}^0 + \frac{1}{2}\,h_+\,\hat{k}^0$$

$$\boxed{\rightarrow \quad \Delta\Phi = -\int \hat{k}^0\left(1+\tfrac{1}{2}h_+\right)dt}$$

Phase shift imposed on laser picks up a term scaling linealy in GW amplitude!

Direct measureable of GW!

Can we actually measure this effect? Yes... provided we keep noise low enough.

2 flavors of noise:

1. Laser "phase noise": No laser is perfectly monochromatic! Indeed, phase is subject to uncertainty associated with number of photons we measure. Issue: to measure GW, need

$$\Delta\Phi_{GW} > \Delta\Phi_{Laser}$$

2. "Acceleration noise": Masses aren't really free-falling ... a more accurate description of them is

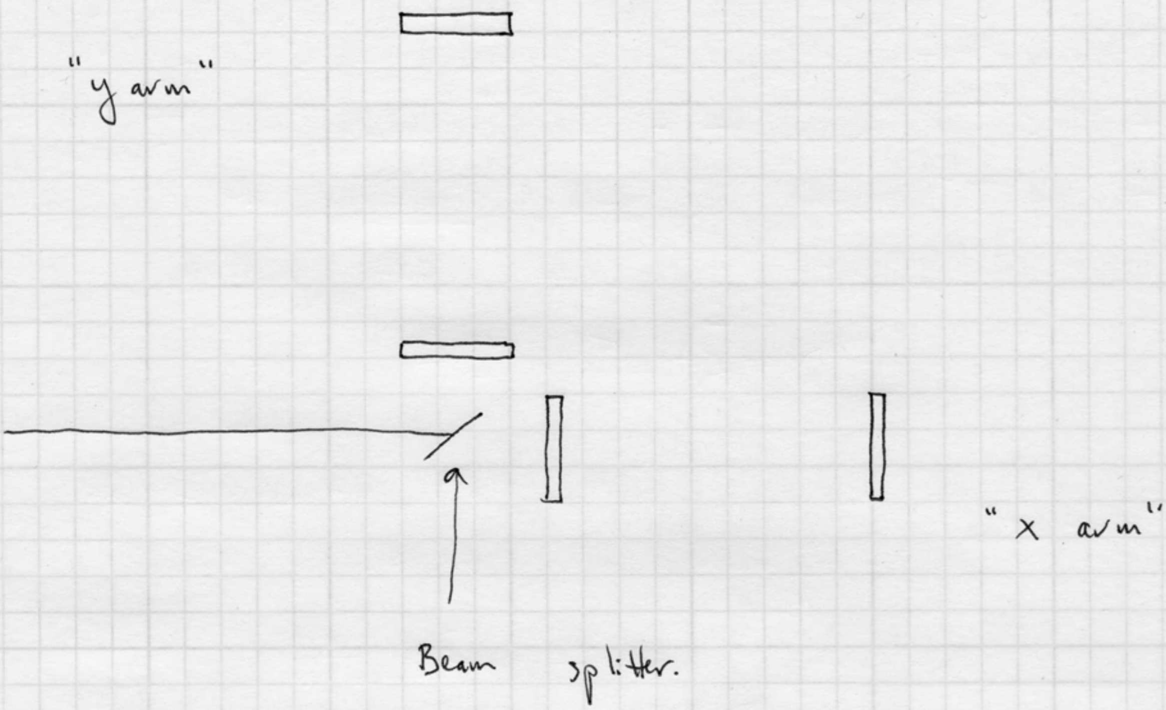
$$\frac{d^2x^j}{d\tau^2} + \Gamma^j{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = f^j$$

spurious forces arising from noisy elements of the experiment.

Need to control these forces to keep their influence smaller than the impact of a GW you hope to measure.

Phase error: 1st, need to have a better picture of interferometry:

Key element: Use two arms, orthogonal to one another.



Utility of this: GWs are tidal. The phase shift imposed on light in the x-arm due to the gravitational wave will have opposite sign of phase shift imposed on light in the y-arm:

$$\Delta\Phi_x^{GW} = k L h \times B \quad \hookrightarrow \text{number of bounces}$$

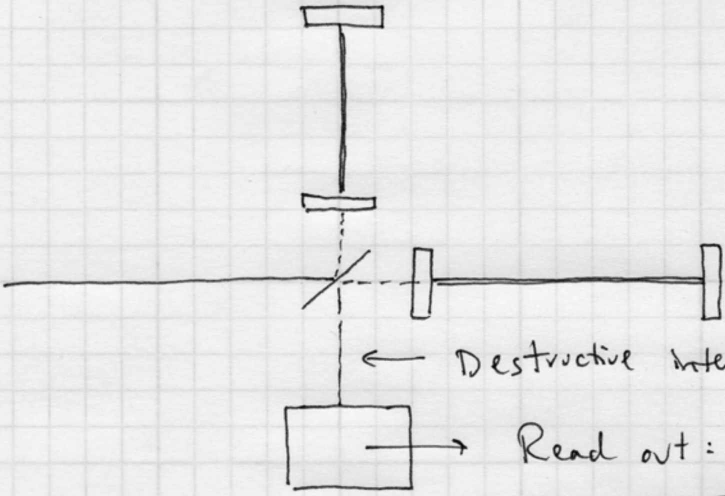
$$= \frac{2\pi}{\lambda_L} L h B$$

$$\Delta\Phi_y^{GW} = -\frac{2\pi L h B}{\lambda_L}$$

Suppose there is some phase noise due to the laser. It will be the same in each arm, since the same laser state is sent into the arms by the beam splitter:

$$\Delta\Phi_x^{\text{Noise}} = \Delta\Phi_y^{\text{Noise}}$$

Now, arrange the interferometer so that one reads out the difference in phase between the two arms:



← Destructive interference of light from arms

→ Read out:

$$\Delta\Phi^{\text{READ-OUT}} = \Delta\Phi_x - \Delta\Phi_y$$

$$= \left(\Delta\Phi_x^{\text{GW}} + \Delta\Phi_x^{\text{Noise}}\right) - \left(\Delta\Phi_y^{\text{GW}} + \Delta\Phi_y^{\text{Noise}}\right)$$

$$= \Delta\Phi_x^{\text{GW}} + \Delta\Phi_x^{\text{Noise}} + \Delta\Phi_x^{\text{GW}} - \Delta\Phi_x^{\text{Noise}}$$

→ $\boxed{\Delta\Phi^{\text{READ-OUT}} = 2\Delta\Phi_x^{\text{GW}}}$: L-shaped topology sensitive to GW, not to laser noise.

More fundamental "laser noise": shot noise.

This is the finite resolution with which phase can be determined due to measuring a finite number of photons:

$$\Delta\Phi_{shot} \simeq \frac{1}{\sqrt{N_{phot}}}$$

where N_{phot} = number of photons gathered during one's measurement.

$$N_{phot} = \underbrace{\frac{I}{\underbrace{hc/\lambda_L}_{\text{energy of a single photon}}}}_{\text{laser power}} \cdot \underbrace{\frac{1}{2f}}_{\text{timescale on power is gathered}} \cdot \eta \rightarrow \text{quantum efficiency of photodiode.}$$

$$\rightarrow \quad \delta\Phi = \left[\frac{2hcf}{I\eta\lambda_L}\right]^{1/2}$$

→ CAUTION: Planck, not GW amplitude...

This will be "confused" for a GW with

$$h_{noise} = \frac{1}{4\pi BL}\left[\frac{2hcf\lambda_L}{\eta I}\right]^{1/2}$$

Goal: Need $h_{noise} < h_{astrophysics}$ to have a detector which can measure a gravitational wave!

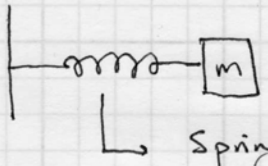
Suppose: 1. Want $h_{noise} \simeq 10^{-22}$

2. $B \simeq 100$, $L \simeq 4$ km, $\lambda \simeq 1\,\mu m$ (parameters for ground-based detectors)

$$\rightarrow \boxed{\text{Need } \eta I \simeq 100 \text{ Watts}}$$

Acceleration noise: Key result here is the "Fluctuation-dissipation theorem":

Suppose you have a damped oscillator in a thermal bath:



Bath of temperature T

Spring constant k

System has damping constant γ

Motion of oscillator governed by equation

$$m\ddot{q} + \gamma\dot{q} + kq = \delta F$$

Where δF is the random force that buffets the oscillator due to the system's interaction with the bath.

Fluctuation dissipation theorem tells us that the <u>spectral density</u> of the random force δF is set by the damping constant γ and the temperature:

$$S_{\delta F} = 4\gamma k_B T$$

(Spectral density gives a precise description of random processes. Key point for our discussion: the rms amplitude of the random force δF is given by the integral of the spectral density over the frequency band of the measurement:

$$\sigma_{\delta F}^2 = \int_{f_{LO}}^{f_{HI}} S_{\delta F} \, df$$

Hence, fluctuation - dissipation gives us a complete description of the thermal contribution to acceleration noises in our detector.)