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Gravitational waves
Lecture 4

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Astrophysical sources of gravitational waves

Key criterion to be an interesting source:

Need high mass/energy density moving at very high speeds

Most sources thus involve compact objects, esp. black holes and neutron stars.

One that does not: "primordial GWs". Consider standard FRW spacetime of cosmology, but add a perturbation:

$$ds^2 = -d\tau^2 + a^2(\tau)\left[dx^2 + dy^2 + dz^2\right] + h_{\mu\nu}\,dx^\mu dx^\nu$$

$$= a^2(\eta)\left[-d\eta^2 + dx^2 + dy^2 + dz^2\right] + h_{\mu\nu}\,dx^\mu dx^\nu$$

Note: for simplicity, writing the metric as spatially flat; not necessary.

Note 2: Very convenient to change to "conformal time", defined by $d\tau = a\,d\eta$ where a is the scale factor.

Run this through Einstein field equations ... $a(\eta)$ governed by usual Friedmann equations. To find equations governing $h_{\mu\nu}$, 1st change notation a bit:

$$h_{\mu\nu} = e_{\mu\nu} \frac{\mu(\eta)}{a(\eta)} \exp(i \vec{n} \cdot \vec{x})$$

↳ spatial vectors

basis tensor: Describes polarization states of $h_{\mu\nu}$

↳ Amplitude of waves is governed by function $\mu(\eta)$.

Einstein tells us

$$\left[\frac{d^2}{d\eta^2} + \left(n^2 - \frac{a''}{a} \right) \mu \right] = 0$$

where $a'' = d^2 a / d\eta^2$.

Note nature of solutions:

- If $n^2 > a''/a$, μ is roughly oscillatory: μ remains of roughly constant magnitude, $h \sim \mu/a$ falls off as universe expands
- If $n^2 < a''/a$, μ will tend to grow exponentially: $h \sim \mu/a$ grows (α remains constant if a grows exponentially)

Translating these conditions into physical space, we find

$$\left(\frac{n^2}{a''/a} \right) \sim \left(\frac{r_H}{\lambda_{GW}} \right)^2$$

where $r_H = a/(da/d\tau) \equiv$ "Horizon scale"

Putting this together, we find

- If $\lambda_{GW} < r_H$, the GW mode dies as the universe expands: "Modes <u>inside the horizon</u> die"

- If $\lambda_{GW} > r_H$, the mode can <u>grow</u> as the universe expands: "Modes outside the horizon grow"

This becomes particularly interesting during inflation, since a mode can then move from inside the horizon to outside:

$$\frac{a(t_2)}{a(t_1)} = e^{H_I (t_2 - t_1)}$$

H_I → Expansion rate during inflation

$$\frac{\lambda(t_2)}{\lambda(t_1)} = \frac{a(t_2)}{a(t_1)}$$ → wavelength grows exponentially

$$r_H = \frac{a}{\dot{a}} = H_I^{-1}$$ → Horizon scale is fixed.

Suppose the initial state is random and low amplitude - e.g., corresponding to some initial random vacuum ground state. DURING INFLATION, THIS STATE CAN BE SIGNIFICANTLY AMPLIFIED!

→ Inflationary cosmology can provide a background of GWs much stronger than the initial amplitude: modes "move outside" the horizon, are amplified, then move back in as inflation ends and the universe evolves in a "normal" way.

Astrophysics & cosmology of binary systems:

A pair of compact stars or black holes in orbit about one another: system with rapidly varying mass quadrupole, very strong GW radiator.

Order-of-magnitude estimates: Use Newtonian gravity to model binary orbital dynamics, use quadrupole formula to estimate rate at which binary's characteristics change:

Key observable is frequency of gravitational waves generated by system:

$$\Omega_{orbit} = \sqrt{\frac{G M_{TOT}}{r^3}} \qquad r = \text{orbital separation}$$

$\frac{d\Omega_{orbit}}{dt}$ = rate of change of Ω_{orbit} due to GW emission

$$= \frac{96}{5}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3} \Omega_{orbit}^{11/3}$$

where

$$\mathcal{M} = \mu^{3/5} M^{2/5} = \frac{m_1^{3/5} m_2^{3/5}}{(m_1 + m_2)^{1/5}}$$

(μ: reduced mass; M: total mass)

Fold in the fact that

$$\Omega_{gw} = 2\Omega_{orbit}$$

(factor of 2 for quadrupole radiation)

$$= 2\pi f_{gw}$$

Using that set of definitions and formulae, we can plug in some fiducial parameters and see how rapidly binary evolves:

Stellar mass binaries: $m_1 \simeq m_2 \simeq 1\,M_\odot = m$

$$\rightarrow \boxed{\frac{df_{GW}}{dt} = \frac{10\text{ Hz}}{\text{sec}} \left(\frac{m}{1\,M_\odot}\right)^{5/3} \left(\frac{f}{100\text{ Hz}}\right)^{11/3}}$$

In band of ground-based detectors ($10\text{ Hz} \lesssim f \lesssim 1000\text{ Hz}$) binary rapidly evolves.

Binary neutron stars sweep through band in roughly 5-15 minutes ($m \simeq 1.35\,M_\odot$, $f_{low} \sim 10$ Hz, $f_{high} \sim 10^3$ Hz); Binary black holes sweep through band in 1-5 minutes ($m \simeq 3-10\,M_\odot$, same frequencies).

Massive black hole binaries:

$$\boxed{\frac{df_{GW}}{dt} \simeq \frac{4\times10^{-3}\text{ Hz}}{\text{day}} \left(\frac{m}{10^6\,M_\odot}\right)^{5/3} \left(\frac{f}{10^{-3}\text{ Hz}}\right)^{11/3}}$$

Binaries consisting of black holes in the $\sim 10^6\,M_\odot$ range sweep through band of space-based detectors ($\sim 10^{-4}\text{ Hz} \lesssim f \lesssim 0.1\text{ Hz}$) over a timescale of months.

Relativity & Astrophysics view of binaries:

Independent of mass, fairly universal characterisation of binary evolution:

INSPIRAL: Binary consists of two widely separated, distinct bodies, slowly evolving due to backreaction of GW emission.

"Newtonian gravity + quadrupole" approximation described earlier gives essence of how binary evolves through this regime. More accurate description provided by <u>post-Newtonian</u> expansion of general relativity.

Roughly speaking, inspiral waves are only sensitive to masses and spins of members of binary - higher order structure contributes negligibly to orbital evolution and hence GW emission.

MERGER: Transition from two bodies into one. <u>Cannot</u> approximate: full machinery of general relativity needed to model dynamics. Structure of bodies clearly important! If bodies are black holes, things are simple: Black holes fully characterized by mass and spin. If neutron stars, gets complicated: must model fluid dynamics, nuclear physics of matter, magnetic fields, etc ... Not easy!

RINGDOWN: Last gasp of merger, if the merged remnant is a black hole. Once the system settles down, spacetime must be described by Kerr black hole solution of general relativity.

Ringdown waves carry away the last deviations from Kerr solution - enforce general relativity's rule that black holes are described only by mass and spin.

Waves completely described by expanding spacetime around black hole metric:

$$g_{\mu\nu} = g_{\mu\nu}^{BH} + h_{\mu\nu}$$

Use Einstein to build eqvation governing $h_{\mu\nu}$.

Result is that waves take form of damped sinusoid:

$$h \sim e^{-t/\tau_R} \cos\left[2\pi f_R t\right]$$

$$f_R \simeq \frac{c^3}{2\pi G M_f}\left[1 - 0.63(1 - a_f)^{3/10}\right]$$

$$Q_R \equiv \pi f_R \tau_R = 2(1 - a_f)^{-9/20}$$

M_f = final remnant mass

$$a_f \equiv \frac{c}{G}\frac{|\vec{S}_f|}{M_f^2}$$

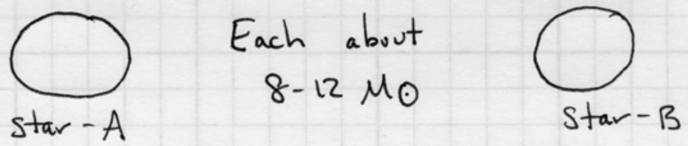
$\vec{S}_f$ = final remnant spin

Relativity description of waves more-or-less the same for all masses ... Astrophysical description quite different! Diffeent masses correspond to very different formation scenarios — and hence very different information should we measure these sources.

Stellar mass range: Binaries form with members that are remnants from stellar collapse. Example: Binary pulsars appear to be left from a pair of normal stars in a binary; each member explodes, leaves a neutron star behind.

Very complicated process! Typical evolutionary sequence:

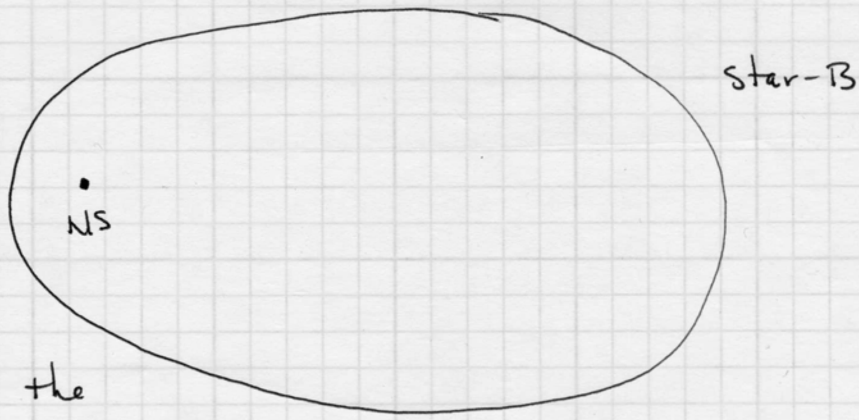
1:



2:

star-A undergoes supernova! Binary disrupted about 50% of the time.

3:



Star-B evolves into a red giant. NS orbits inside the "envelope".

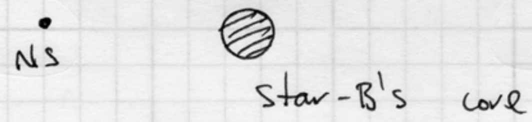
Two possible outcomes at this stage:

A: As the neutron star orbits, hydrodynamic drag decays the orbit so much that the NS ends up at the core of the giant star.

→ No more binary: have a "Thorne-Żytkow" object, or TZO.

B: Orbital energy is sufficient that hydrodynamic drag does not completely decay the orbit. Instead, stellar envelope is unbound; end up with NS in orbit around a dense stellar core:

4:



5: Core explodes in supernova. Probably disrupts binary about 50% of the time; otherwise, get

If initial stars have masses > 12 $M_\odot$, may get black holes in the end state - either NS-BH or BH-BH.

Stellar mass compact binaries, especially ones with black holes, also likely to form in cores of dense stellar clusters through multi-body interactions.

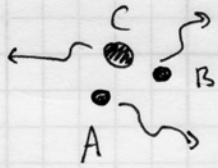
2 body orbit is stable ... 3 body is not!

Sketch of what can be found using computer simulations:

(I)

A + B are bound in a binary; C comes along and interacts. Let $m_A < m_B < m_C$. Then,

(II)



Have an interval of very complicated "trinary" interaction...

(III)

... end up ejecting binary, leaving more massive members in a tight binary.

Particularly exciting source: binary in which each member is a black hole with mass $> 10^5 M_\odot$: Binaries formed in the merger of galaxies and "protogalaxies"

Theoretical logic

1. Galaxies merge! Observed to happen quite often, especially when the universe was younger. Now understood to play a major role in building large galaxies from smaller galaxies and structures: hierarchical assembly from small to large.

2. Galaxies host black holes: Very solid measurements in nearby galaxies; more circumstantial cases made for more distant objects.

1 + 2 → galaxy growth by mergers likely to produce binary black holes.

Expectation is that rate of binary formation should closely track rate of galaxy mergers. Should get high rate from early epochs of structure growth, when mergers were particularly common.

Modeling binaries: Two important techniques, post-Newtonian (pN) expansion and numerical relativity.

pN: Recall how we did linearized theory:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \; ; \quad \partial^\mu \bar{h}_{\mu\nu} = 0$$

$$\longrightarrow \quad \Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$$

Suppose we define our "perturbation" as

$$h^{\mu\nu} \equiv \sqrt{-g}\, g^{\mu\nu} - \eta^{\mu\nu}$$

$\hookrightarrow$ determinant of $g_{\mu\nu}$.

Impose the condition $\partial_\alpha h^{\alpha\beta} = 0$

It can then be shown that the <u>EXACT</u> Einstein field equations can be written

$$\Box h^{\alpha\beta} = 16\pi G \tau^{\alpha\beta}$$

where $\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$ is the <u>flat</u> spacetime wave operator.

Solution is just $h^{\alpha\beta} = -4G \int \frac{\tau^{\alpha\beta}\, d^3x'}{|\vec{x} - \vec{x}'|}$

Exact solution of general relativity ??

Problem: Source is $\tau^{\alpha\beta}$, not $T^{\alpha\beta}$:

$$\tau^{\alpha\beta} = (-g)T^{\alpha\beta} + \frac{\Lambda^{\alpha\beta}}{16\pi G}$$

$\Lambda^{\alpha\beta}$ is a nonlinear collection of terms in $h^{\alpha\beta}$:

$$\Lambda^{\alpha\beta} = N^{\alpha\beta}[h,h] + M^{\alpha\beta}[h,h,h] + L^{\alpha\beta}[h,h,h,h] + \cdots$$

Horrible ... but crying out for an iterative solution:

$$h = h^0 + h^1 + h^2 + \cdots$$

↙ Linearized theory result: acts as source for h^1; h^0 & h^1 act as sources for h^2; etc.

Numerical relativity: Recasting the (geometric) equations of general relativity into a form amenable to numerical computation.

Major difference: rather than treating binary as two bodies, regard it as <u>one</u> <u>spacetime</u>. Begin with initial state containing two bodies - evolve forward to let those bodies do whatever gravity wants.

Recast metric slightly:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)$$

α = "lapse" - tells how quickly clocks tick as a function of position in spacetime.

β^i = "shift" - tells how rulers slide around as the spacetime evolves.

γ_{ij} describes the geometry of a spatial slice of the spacetime.

α, β^i give coordinate or gauge degree of freedom: we can adjust them as needed to make calculation work.

Effort is then in how to evolve to make γ_{ij} over the interesting evolution of the system.