



*The Abdus Salam  
International Centre for Theoretical Physics*



**1954-15**

**Summer School in Cosmology**

*21 July - 1 August, 2008*

**Gravitational waves  
Lecture 4  
additional material**

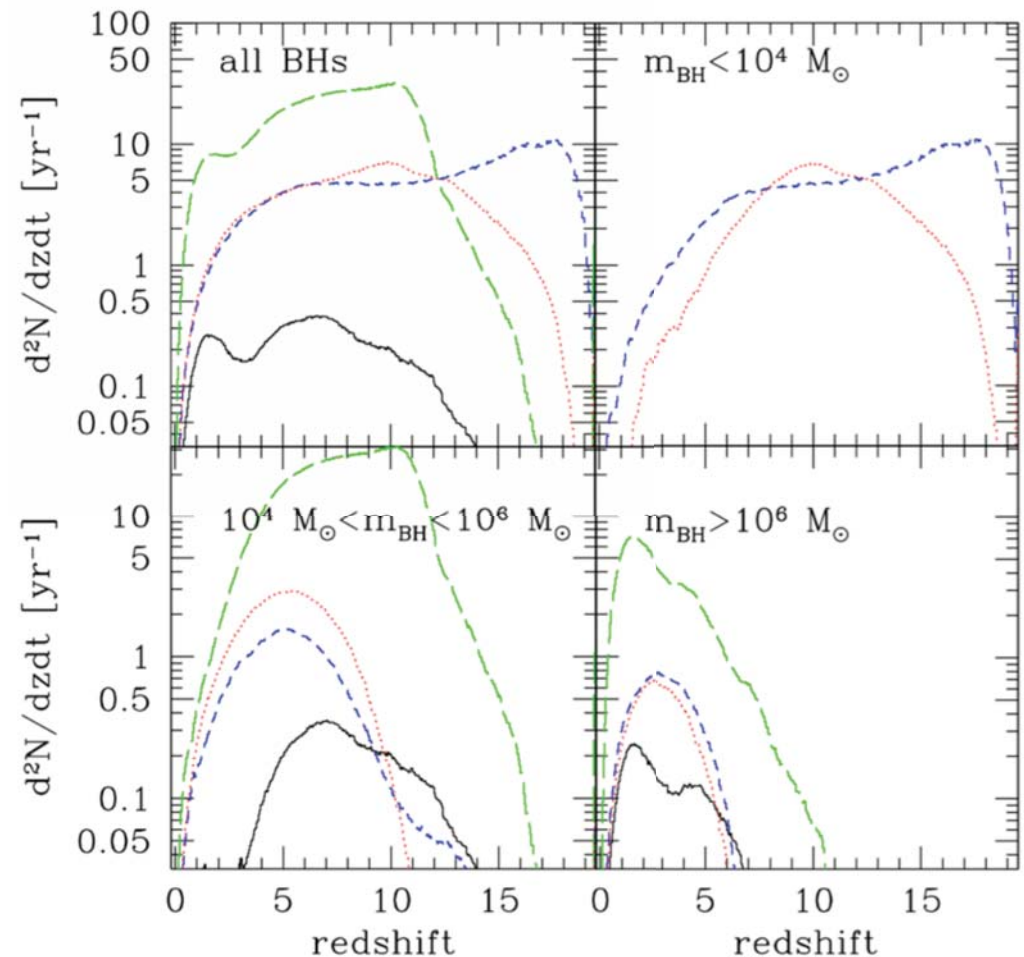
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# Gravitational waves and MBBH

Sources we are interested in: High redshift merger of seeds from hierarchical structure growth.

Model: Black hole growth tracking merger of dark matter halos via merger tree.

Curves: Different assumptions about seed holes and their growth. *All models consistent with AGN optical luminosity for  $1 < z < 6$*



Sesana, Volonteri, & Haardt 2007 MNRAS **377**, 1711

ICTP Summer School, 28 - 31 July 2008

# Equations of motion from post-Newtonian spacetime of a binary

Looking at motion of binary members in post-Newtonian spacetime lets us see how zeroth order Newtonian rule is modified by GR.

$$a_1^i = -\frac{Gm_2 n_{12}^i}{r_{12}^2} + \frac{1}{c^2} \left\{ \left[ \frac{5G^2 m_1 m_2}{r_{12}^3} + \frac{4G^2 m_2^2}{r_{12}^3} + \frac{Gm_2}{r_{12}^2} \left( \frac{3}{2} (n_{12} v_2)^2 - v_1^2 + 4(v_1 v_2) - 2v_2^2 \right) \right] n_{12}^i + \frac{Gm_2}{r_{12}^2} (4(n_{12} v_1) - 3(n_{12} v_2)) v_{12}^i \right\}$$

Lowest order term  
duplicates Newton.

Post-Newton gives corrections in  $v/c$ .

... and more corrections ...

$$\begin{aligned}
& + \frac{1}{c^4} \left\{ \left[ -\frac{57G^3m_1^2m_2}{4r_{12}^4} - \frac{69G^3m_1m_2^2}{2r_{12}^4} - \frac{9G^3m_2^3}{r_{12}^4} \right. \right. \\
& \quad + \frac{Gm_2}{r_{12}^2} \left( -\frac{15}{8}(n_{12}v_2)^4 + \frac{3}{2}(n_{12}v_2)^2v_1^2 - 6(n_{12}v_2)^2(v_1v_2) - 2(v_1v_2)^2 + \frac{9}{2}(n_{12}v_2)^2v_2^2 \right. \\
& \quad \quad \left. \left. + 4(v_1v_2)v_2^2 - 2v_2^4 \right) \right. \\
& \quad + \frac{G^2m_1m_2}{r_{12}^3} \left( \frac{39}{2}(n_{12}v_1)^2 - 39(n_{12}v_1)(n_{12}v_2) + \frac{17}{2}(n_{12}v_2)^2 - \frac{15}{4}v_1^2 - \frac{5}{2}(v_1v_2) + \frac{5}{4}v_2^2 \right) \\
& \quad + \frac{G^2m_2^2}{r_{12}^3} (2(n_{12}v_1)^2 - 4(n_{12}v_1)(n_{12}v_2) - 6(n_{12}v_2)^2 - 8(v_1v_2) + 4v_2^2) \left. \right] n_{12}^i \\
& \quad + \left[ \frac{G^2m_2^2}{r_{12}^3} (-2(n_{12}v_1) - 2(n_{12}v_2)) + \frac{G^2m_1m_2}{r_{12}^3} \left( -\frac{63}{4}(n_{12}v_1) + \frac{55}{4}(n_{12}v_2) \right) \right. \\
& \quad + \frac{Gm_2}{r_{12}^2} \left( -6(n_{12}v_1)(n_{12}v_2)^2 + \frac{9}{2}(n_{12}v_2)^3 + (n_{12}v_2)v_1^2 - 4(n_{12}v_1)(v_1v_2) \right. \\
& \quad \quad \left. \left. + 4(n_{12}v_2)(v_1v_2) + 4(n_{12}v_1)v_2^2 - 5(n_{12}v_2)v_2^2 \right) \right] v_{12}^i \left. \right\} \\
& + \frac{1}{c^5} \left\{ \left[ \frac{208G^3m_1m_2^2}{15r_{12}^4}(n_{12}v_{12}) - \frac{24G^3m_1^2m_2}{5r_{12}^4}(n_{12}v_{12}) + \frac{12G^2m_1m_2}{5r_{12}^3}(n_{12}v_{12})v_{12}^2 \right] n_{12}^i \right. \\
& \quad \left. + \left[ \frac{8G^3m_1^2m_2}{5r_{12}^4} - \frac{32G^3m_1m_2^2}{5r_{12}^4} - \frac{4G^2m_1m_2}{5r_{12}^3}v_{12}^2 \right] v_{12}^i \right\}
\end{aligned}$$

... and a few more.

$$\begin{aligned}
& + \frac{1}{c^6} \left\{ \left[ \frac{Gm_2}{r_{12}^3} \left( \frac{35}{16} (n_{12}v_2)^6 - \frac{15}{8} (n_{12}v_2)^4 v_1^2 + \frac{15}{2} (n_{12}v_2)^4 (v_1v_2) + 3(n_{12}v_2)^2 (v_1v_2)^2 \right. \right. \right. \\
& \quad - \frac{15}{2} (n_{12}v_2)^4 v_2^2 + \frac{3}{2} (n_{12}v_2)^2 v_1^2 v_2^2 - 12(n_{12}v_2)^2 (v_1v_2)v_2^2 - 2(v_1v_2)^2 v_2^2 \\
& \quad \left. \left. + \frac{15}{2} (n_{12}v_2)^2 v_2^2 + 4(v_1v_2)v_2^4 - 2v_2^6 \right) \right. \\
& \quad + \frac{G^2 m_1 m_2}{r_{12}^3} \left( -\frac{171}{8} (n_{12}v_1)^4 + \frac{171}{2} (n_{12}v_1)^3 (n_{12}v_2) - \frac{723}{4} (n_{12}v_1)^2 (n_{12}v_2)^2 \right. \\
& \quad + \frac{383}{2} (n_{12}v_1)(n_{12}v_2)^3 - \frac{455}{8} (n_{12}v_2)^4 + \frac{229}{4} (n_{12}v_1)^2 v_1^2 \\
& \quad - \frac{205}{2} (n_{12}v_1)(n_{12}v_2)v_1^2 + \frac{191}{4} (n_{12}v_2)^2 v_1^2 - \frac{91}{8} v_1^4 - \frac{229}{2} (n_{12}v_1)^2 (v_1v_2) \\
& \quad + 244(n_{12}v_1)(n_{12}v_2)(v_1v_2) - \frac{225}{2} (n_{12}v_2)^2 (v_1v_2) + \frac{91}{2} v_1^2 (v_1v_2) \\
& \quad - \frac{177}{4} (v_1v_2)^2 + \frac{229}{4} (n_{12}v_1)^2 v_2^2 - \frac{283}{2} (n_{12}v_1)(n_{12}v_2)v_2^2 \\
& \quad \left. \left. + \frac{259}{4} (n_{12}v_2)^2 v_2^2 - \frac{91}{4} v_1^2 v_2^2 + 43(v_1v_2)v_2^2 - \frac{81}{8} v_2^4 \right) \right. \\
& \quad + \frac{G^2 m_2^2}{r_{12}^3} \left( -6(n_{12}v_1)^2 (n_{12}v_2)^2 + 12(n_{12}v_1)(n_{12}v_2)^3 + 6(n_{12}v_2)^4 \right. \\
& \quad + 4(n_{12}v_1)(n_{12}v_2)(v_1v_2) + 12(n_{12}v_2)^2 (v_1v_2) + 4(v_1v_2)^2 \\
& \quad \left. \left. - 4(n_{12}v_1)(n_{12}v_2)v_2^2 - 12(n_{12}v_2)^2 v_2^2 - 8(v_1v_2)v_2^2 + 4v_2^4 \right) \right. \\
& \quad + \frac{G^3 m_2^3}{r_{12}^4} \left( -(n_{12}v_1)^2 + 2(n_{12}v_1)(n_{12}v_2) + \frac{43}{2} (n_{12}v_2)^2 + 18(v_1v_2) - 9v_2^2 \right) \\
& \quad + \frac{G^3 m_1 m_2^2}{r_{12}^4} \left( \frac{415}{8} (n_{12}v_1)^2 - \frac{375}{4} (n_{12}v_1)(n_{12}v_2) + \frac{1113}{8} (n_{12}v_2)^2 - \frac{615}{64} (n_{12}v_{12})^2 \pi^2 \right. \\
& \quad \left. + 18v_1^2 + \frac{123}{64} \pi^2 v_{12}^2 + 33(v_1v_2) - \frac{33}{2} v_2^2 \right) \\
& \quad + \frac{G^3 m_1^2 m_2}{r_{12}^4} \left( -\frac{45887}{168} (n_{12}v_1)^2 + \frac{24025}{42} (n_{12}v_1)(n_{12}v_2) - \frac{10469}{42} (n_{12}v_2)^2 + \frac{48197}{840} v_1^2 \right. \\
& \quad \left. - \frac{36227}{420} (v_1v_2) + \frac{36227}{840} v_2^2 + 110(n_{12}v_{12})^2 \ln\left(\frac{r_{12}}{r_1}\right) - 22v_{12}^2 \ln\left(\frac{r_{12}}{r_1}\right) \right) \\
& \quad + \frac{16G^4 m_2^4}{r_{12}^5} + \frac{G^4 m_1^2 m_2^2}{r_{12}^5} \left( 175 - \frac{41}{16} \pi^2 \right) + \frac{G^4 m_1^2 m_2}{r_{12}^5} \left( \frac{3187}{1260} + \frac{44}{3} \ln\left(\frac{r_{12}}{r_1}\right) \right) \\
& \quad + \frac{G^4 m_1 m_2^3}{r_{12}^5} \left( \frac{110741}{630} - \frac{41}{16} \pi^2 - \frac{44}{3} \ln\left(\frac{r_{12}}{r_2}\right) \right) n_{12}^2 \\
& \quad + \left[ \frac{Gm_2}{r_{12}^3} \left( \frac{15}{2} (n_{12}v_1)(n_{12}v_2)^4 - \frac{45}{8} (n_{12}v_2)^5 - \frac{3}{2} (n_{12}v_2)^3 v_1^2 + 6(n_{12}v_1)(n_{12}v_2)^2 (v_1v_2) \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \quad \left. - 6(n_{12}v_2)^3 (v_1v_2) - 2(n_{12}v_2)(v_1v_2)^2 - 12(n_{12}v_1)(n_{12}v_2)^2 v_2^2 + 12(n_{12}v_2)^3 v_2^2 \right. \\
& \quad \left. + (n_{12}v_2)v_1^2 v_2^2 - 4(n_{12}v_1)(v_1v_2)v_2^2 + 8(n_{12}v_2)(v_1v_2)v_2^2 + 4(n_{12}v_1)v_2^2 \right. \\
& \quad \left. - 7(n_{12}v_2)v_2^2 \right) \\
& \quad + \frac{G^2 m_2^2}{r_{12}^3} \left( -2(n_{12}v_1)^2 (n_{12}v_2) + 8(n_{12}v_1)(n_{12}v_2)^2 + 2(n_{12}v_2)^3 + 2(n_{12}v_1)(v_1v_2) \right. \\
& \quad \left. + 4(n_{12}v_2)(v_1v_2) - 2(n_{12}v_1)v_2^2 - 4(n_{12}v_2)v_2^2 \right) \\
& \quad + \frac{G^2 m_1 m_2}{r_{12}^3} \left( -\frac{243}{4} (n_{12}v_1)^3 + \frac{565}{4} (n_{12}v_1)^2 (n_{12}v_2) - \frac{269}{4} (n_{12}v_1)(n_{12}v_2)^2 \right. \\
& \quad - \frac{95}{12} (n_{12}v_2)^3 + \frac{207}{8} (n_{12}v_1)v_1^2 - \frac{137}{8} (n_{12}v_2)v_1^2 - 36(n_{12}v_1)(v_1v_2) \\
& \quad \left. + \frac{27}{4} (n_{12}v_2)(v_1v_2) + \frac{81}{8} (n_{12}v_1)v_2^2 + \frac{83}{8} (n_{12}v_2)v_2^2 \right) \\
& \quad + \frac{G^3 m_2^3}{r_{12}^4} (4(n_{12}v_1) + 5(n_{12}v_2)) \\
& \quad + \frac{G^3 m_1 m_2^2}{r_{12}^4} \left( -\frac{307}{8} (n_{12}v_1) + \frac{479}{8} (n_{12}v_2) + \frac{123}{32} (n_{12}v_{12}) \pi^2 \right) \\
& \quad + \frac{G^3 m_1^2 m_2}{r_{12}^4} \left( \frac{31397}{420} (n_{12}v_1) - \frac{36227}{420} (n_{12}v_2) - 44(n_{12}v_{12}) \ln\left(\frac{r_{12}}{r_1}\right) \right) v_{12}^i \Bigg\} \\
& + \frac{1}{c^2} \left\{ \left[ \frac{G^4 m_1^3 m_2}{r_{12}^5} \left( \frac{3992}{105} (n_{12}v_1) - \frac{4328}{105} (n_{12}v_2) \right) \right. \right. \\
& \quad + \frac{G^4 m_1^2 m_2^2}{r_{12}^5} \left( -\frac{13576}{105} (n_{12}v_1) + \frac{2872}{21} (n_{12}v_2) \right) - \frac{3172}{21} \frac{G^4 m_1 m_2^2}{r_{12}^5} (n_{12}v_{12}) \\
& \quad + \frac{G^3 m_1^2 m_2}{r_{12}^5} \left( 48(n_{12}v_1)^3 - \frac{696}{5} (n_{12}v_1)^2 (n_{12}v_2) + \frac{744}{5} (n_{12}v_1)(n_{12}v_2)^2 - \frac{288}{5} (n_{12}v_2)^3 \right. \\
& \quad \left. - \frac{4888}{105} (n_{12}v_1)v_1^2 + \frac{5056}{105} (n_{12}v_2)v_1^2 + \frac{2056}{21} (n_{12}v_1)(v_1v_2) \right. \\
& \quad \left. - \frac{2224}{21} (n_{12}v_2)(v_1v_2) - \frac{1028}{21} (n_{12}v_1)v_2^2 + \frac{5812}{105} (n_{12}v_2)v_2^2 \right) \\
& \quad + \frac{G^3 m_1 m_2^2}{r_{12}^5} \left( -\frac{582}{5} (n_{12}v_1)^3 + \frac{1746}{5} (n_{12}v_1)^2 (n_{12}v_2) - \frac{1954}{5} (n_{12}v_1)(n_{12}v_2)^2 \right. \\
& \quad + 158(n_{12}v_2)^3 + \frac{3568}{105} (n_{12}v_{12})v_1^2 - \frac{2864}{35} (n_{12}v_1)(v_1v_2) \\
& \quad + \frac{10048}{105} (n_{12}v_2)(v_1v_2) + \frac{1432}{35} (n_{12}v_1)v_2^2 - \frac{5752}{105} (n_{12}v_2)v_2^2 \Bigg) \\
& \quad \left. + \frac{G^2 m_1 m_2}{r_{12}^5} \left( -56(n_{12}v_2)^5 + 60(n_{12}v_1)^3 v_{12}^2 - 180(n_{12}v_1)^2 (n_{12}v_2)v_{12}^2 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \quad \left. + 174(n_{12}v_1)(n_{12}v_2)^2 v_{12}^2 - 54(n_{12}v_2)^3 v_{12}^2 - \frac{246}{35} (n_{12}v_{12})v_1^4 \right. \\
& \quad + \frac{1068}{35} (n_{12}v_1)v_1^2 (v_1v_2) - \frac{984}{35} (n_{12}v_2)v_1^2 (v_1v_2) - \frac{1068}{35} (n_{12}v_1)(v_1v_2)^2 \\
& \quad + \frac{180}{7} (n_{12}v_2)(v_1v_2)^2 - \frac{534}{35} (n_{12}v_1)v_1^2 v_2^2 + \frac{90}{7} (n_{12}v_2)v_1^2 v_2^2 \\
& \quad + \frac{984}{35} (n_{12}v_1)(v_1v_2)v_2^2 - \frac{732}{35} (n_{12}v_2)(v_1v_2)v_2^2 - \frac{204}{35} (n_{12}v_1)v_2^4 \\
& \quad \left. + \frac{24}{7} (n_{12}v_2)v_2^4 \right) n_{12}^i \\
& + \left[ -\frac{184}{21} \frac{G^4 m_1^3 m_2}{r_{12}^5} + \frac{6224}{105} \frac{G^4 m_1^2 m_2^2}{r_{12}^5} + \frac{6388}{105} \frac{G^4 m_1 m_2^3}{r_{12}^5} \right. \\
& \quad + \frac{G^3 m_1^2 m_2}{r_{12}^5} \left( \frac{52}{15} (n_{12}v_1)^2 - \frac{56}{15} (n_{12}v_1)(n_{12}v_2) - \frac{44}{15} (n_{12}v_2)^2 - \frac{132}{35} v_1^2 + \frac{152}{35} (v_1v_2) \right. \\
& \quad \left. - \frac{48}{35} v_2^2 \right) \\
& \quad + \frac{G^3 m_1 m_2^2}{r_{12}^5} \left( \frac{454}{15} (n_{12}v_1)^2 - \frac{372}{5} (n_{12}v_1)(n_{12}v_2) + \frac{854}{15} (n_{12}v_2)^2 - \frac{152}{21} v_1^2 \right. \\
& \quad \left. + \frac{2864}{105} (v_1v_2) - \frac{1768}{105} v_2^2 \right) \\
& \quad + \frac{G^2 m_1 m_2}{r_{12}^5} \left( 60(n_{12}v_{12})^4 - \frac{348}{5} (n_{12}v_1)^2 v_{12}^2 + \frac{684}{5} (n_{12}v_1)(n_{12}v_2)v_{12}^2 \right. \\
& \quad - 66(n_{12}v_2)^2 v_{12}^2 + \frac{334}{35} v_1^4 - \frac{1336}{35} v_1^2 (v_1v_2) + \frac{1308}{35} (v_1v_2)^2 + \frac{654}{35} v_1^2 v_2^2 \\
& \quad \left. - \frac{1252}{35} (v_1v_2)v_2^2 + \frac{292}{35} v_2^4 \right) v_{12}^i \Bigg\}
\end{aligned}$$

$$+ \mathcal{O}\left(\frac{1}{c^8}\right).$$

[Blanchet 2006, Liv Rev Rel 9, 4, Eq. (168)]

# Evolution of spins

The *motions* of masses (orbits and spins) drives a “magnetic-type” coupling of mass currents to spacetime.

Creates new “forces”, modifying orbit acceleration; also causes spins of binary’s members to precess.

$$\begin{aligned}\frac{d\mathbf{S}_1}{dt} &= \frac{1}{r^3} \left[ \left( 2 + \frac{3}{2} \frac{m_2}{m_1} \right) \mu \sqrt{Mr} \hat{\mathbf{L}} \right] \times \mathbf{S}_1 + \frac{1}{r^3} \left[ \frac{1}{2} \mathbf{S}_2 - \frac{3}{2} (\mathbf{S}_2 \cdot \hat{\mathbf{L}}) \hat{\mathbf{L}} \right] \times \mathbf{S}_1 \\ \frac{d\mathbf{S}_2}{dt} &= \frac{1}{r^3} \left[ \left( 2 + \frac{3}{2} \frac{m_1}{m_2} \right) \mu \sqrt{Mr} \hat{\mathbf{L}} \right] \times \mathbf{S}_2 + \frac{1}{r^3} \left[ \frac{1}{2} \mathbf{S}_1 - \frac{3}{2} (\mathbf{S}_1 \cdot \hat{\mathbf{L}}) \hat{\mathbf{L}} \right] \times \mathbf{S}_2\end{aligned}$$

“Gravitomagnetic”  
field due to  
orbital motion

Form is

$$d\mathbf{S}/dt = \mathbf{S} \times \mathbf{B}_g$$

“Gravitomagnetic”  
field due to other  
body’s spin

# Evolution of spins

The *motions* of masses (orbits and spins) drives a “magnetic-type” coupling of mass currents to spacetime.

Creates new “forces”, modifying orbit acceleration; also causes spins of binary’s members to precess.

$$\begin{aligned}\frac{d\mathbf{S}_1}{dt} &= \frac{1}{r^3} \left[ \left( 2 + \frac{3}{2} \frac{m_2}{m_1} \right) \mu \sqrt{Mr} \hat{\mathbf{L}} \right] \times \mathbf{S}_1 + \frac{1}{r^3} \left[ \frac{1}{2} \mathbf{S}_2 - \frac{3}{2} (\mathbf{S}_2 \cdot \hat{\mathbf{L}}) \hat{\mathbf{L}} \right] \times \mathbf{S}_1 \\ \frac{d\mathbf{S}_2}{dt} &= \frac{1}{r^3} \left[ \left( 2 + \frac{3}{2} \frac{m_1}{m_2} \right) \mu \sqrt{Mr} \hat{\mathbf{L}} \right] \times \mathbf{S}_2 + \frac{1}{r^3} \left[ \frac{1}{2} \mathbf{S}_1 - \frac{3}{2} (\mathbf{S}_1 \cdot \hat{\mathbf{L}}) \hat{\mathbf{L}} \right] \times \mathbf{S}_2\end{aligned}$$

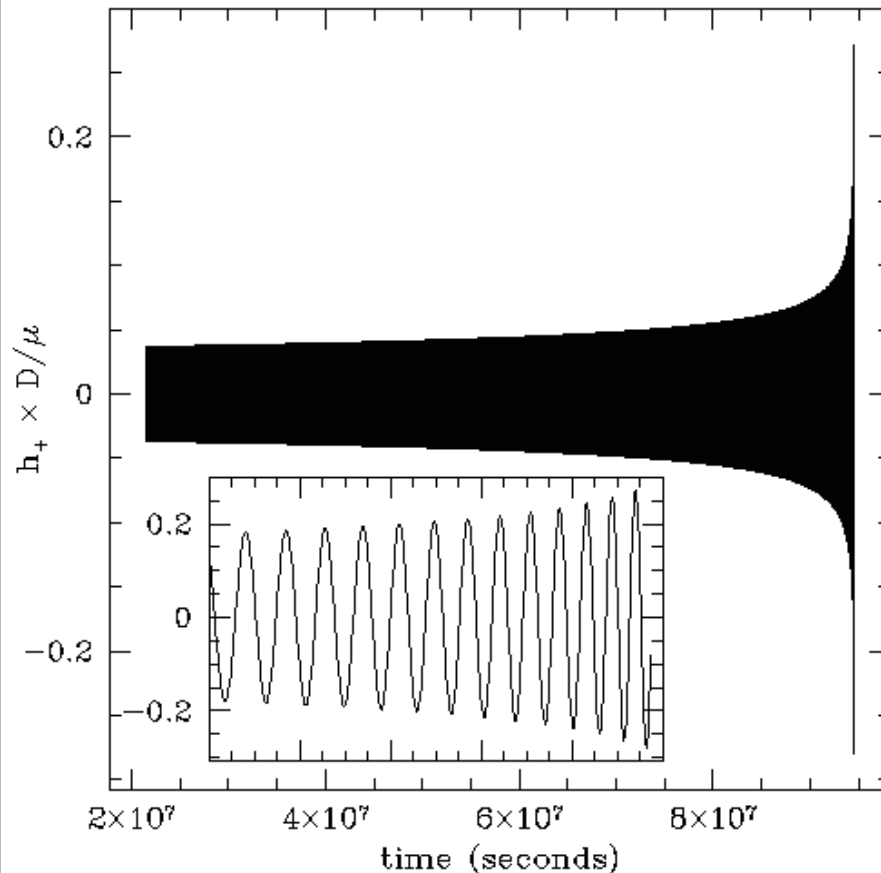
Angular momentum is *globally* conserved:

$$\mathbf{J} = \mathbf{L} + \mathbf{S}_1 + \mathbf{S}_2 = \text{constant}$$

Means that the *orbital plane* precesses to compensate.  
(Known as Lense-Thirring precession in weak-field.)

# Inspiral measurements

$$h_+ = \frac{[G(1+z)\mathcal{M}/c^2]^{5/3} [\pi f(t)/c]^{2/3}}{D_L} \mathcal{F}(\text{“angles”}) \cos [\Phi(t)]$$



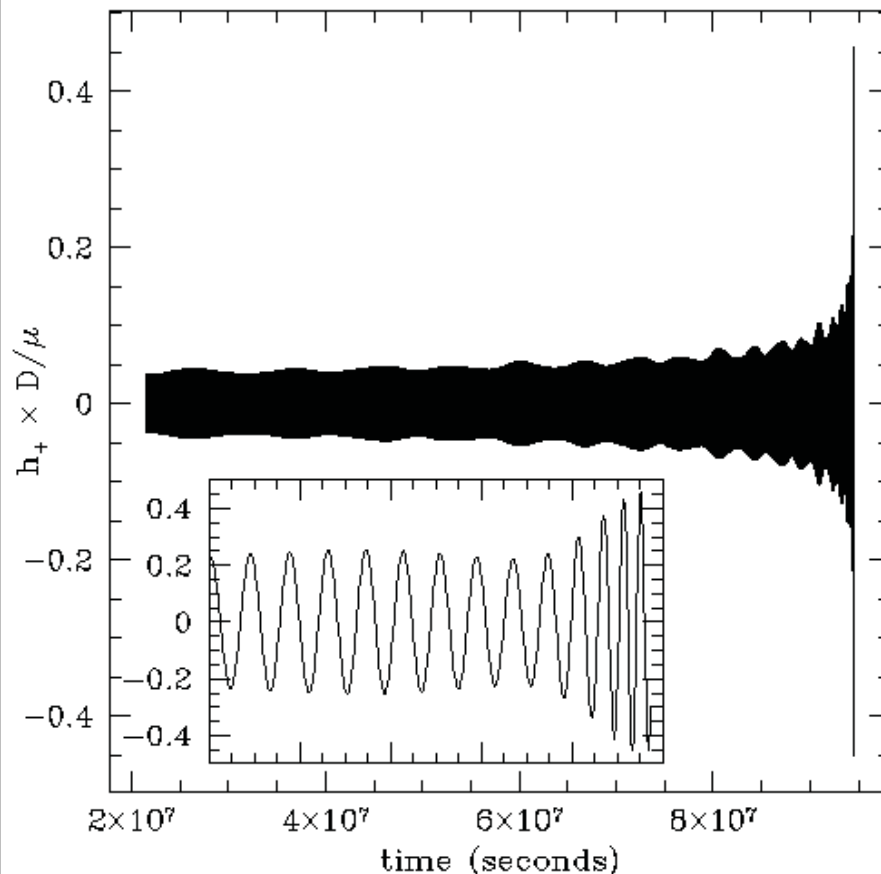
**Example waveform:  
Both black holes non-  
spinning.**

**Very smooth evolution!**



# Inspiral measurements

$$h_+ = \frac{[G(1+z)\mathcal{M}/c^2]^{5/3} [\pi f(t)/c]^{2/3}}{D_L} \mathcal{F}(\text{"angles"}) \cos [\Phi(t)]$$



Spins cranked up!  
Spin 1 = Spin 2 =  
99% maximum

Strong frequency and  
amplitude modulation  
gives spin precision.

# Equations of numerical relativity

Need to rewrite the Einstein equations in a form that allows us to integrate with respect to a chosen time direction.

Do this by *projecting* components of Einstein parallel/perpendicular to this time:

$t^\mu$  Vector associated with time coordinate

$n^\mu = (t^\mu - \beta^\mu)/\alpha$  Vector normal to spatial slice

$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$  Projector into a spatial slice

Spatial components of projection tensor are just the spatial metric components  $\gamma_{ij}$ .

# Equations of numerical relativity

Contracting Einstein along normal gives us *constraints*: Relationships that geometry must satisfy at all times. Analogous to divergence equations of Maxwell's theory.

“Hamiltonian constraint:”  $G_{\alpha\beta}n^\alpha n^\beta = 8\pi G T_{\alpha\beta}n^\alpha n^\beta$   
 $\longrightarrow R + K^2 - K_{ij}K^{ij} = 16\pi G\rho$

$R$  : Ricci scalar of 3-metric  $\gamma_{ij}$        $\rho = T_{\alpha\beta}n^\alpha n^\beta$

$K_{ij} \equiv -\gamma_i^\alpha \gamma_j^\beta \nabla_\alpha n_\beta$       “Extrinsic curvature” of  
constant-time slice of geometry.

# Equations of numerical relativity

Contracting Einstein along normal gives us *constraints*: Relationships that geometry must satisfy at all times. Analogous to divergence equations of Maxwell's theory.

“Momentum constraint:”  $G_{\alpha\beta}n^\alpha\gamma_i^\beta = 8\pi GT_{\alpha\beta}n^\alpha\gamma_i^\beta$   
 $\longrightarrow D_j K^j_i - D_i K = 8\pi G j_i$

$D_i$ : Covariant derivative  
of 3-metric  $\gamma_{ij}$

$$j_i = -T_{\alpha\beta}n^\alpha\gamma_i^\beta$$

# Equations of numerical relativity

Contracting twice with projectors gives *evolution equations*: Relations connecting geometry from moment to moment.

$$G_{\alpha\beta}\gamma_i^\alpha\gamma_j^\beta = 8\pi GT_{\alpha\beta}\gamma_i^\alpha\gamma_j^\beta \equiv 8\pi GS_{ij}$$

$$\longrightarrow \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i$$

$$\begin{aligned} \partial_t K_{ij} = & -D_i D_j \alpha + \alpha (R_{ij} - 2K_{ik} K^k_j + K K_{ij}) \\ & - 8\pi G \alpha \left( S_{ij} - \frac{1}{2} \gamma_{ij} (S - \rho) \right) + \beta^k D_k K_{ij} \\ & + K_{ik} D_j \beta^k + K_{kj} D_i \beta^k \end{aligned}$$

## Lapse $\alpha$ , shift $\beta^i$ ?

Notice that these equations do not constrain the values of  $\alpha$  and  $\beta^i$ : We are *completely* free to specify them as we wish!

This is how gauge invariance is manifested in a strong field calculation: The 4 functions  $\alpha$  and  $\beta^i$  are equivalent to the 4 components of the weak-field gauge generator  $\xi^\mu$ .

*Wonderful!!* Total freedom to pick them as convenient.

**Terrible!!** What's a good choice to use for interesting calculations??

# Recipe

1. Pick a problem you want to solve. Most popular problem for numerical relativity: Final inspiral and merger of two black holes.
2. Find the 3-geometry  $\gamma_{ij}$  describing this system at an “initial” time. This geometry must satisfy the constraint equations!
3. Decide on a method to determine  $\alpha$  and  $\beta^i$ . In principle, all are good. In practice, this can make or break your calculation.
4. Evolve. Compare to observations.  
Celebrate!

## Result ...

For many years, all groups found a very consistent result: **The code crashes.**

Many issues impact this ... most important appears to be *constraint-violating* modes: Solutions to  $(\partial_t \gamma_{ij}, \partial_t K_{ij})$  that violate the Hamiltonian and momentum constraints.

These modes can be unstable: Once excited, will grow and dominate. Instead of geometry that describes two black holes, quickly get a geometry that is nonsense.



# The breakthrough

Tremendous advance  
roughly 3 years ago: How  
we represent a black hole  
on our numerical grid.

$$\psi = u + \sum_i \frac{Gm_i}{|\vec{r} - \vec{r}_i|}$$

“Punctures”: smooth  
field plus singular bit at  
location of black hole  $i$ .

Old approach: Factor out singular piece,  
treat separately from “smooth” degrees of  
freedom. Mathematically “nice” ... but  
requires fixing coordinates of BHs.

End up with very twisted representation of  
geometry — seeds instabilities.

# The breakthrough

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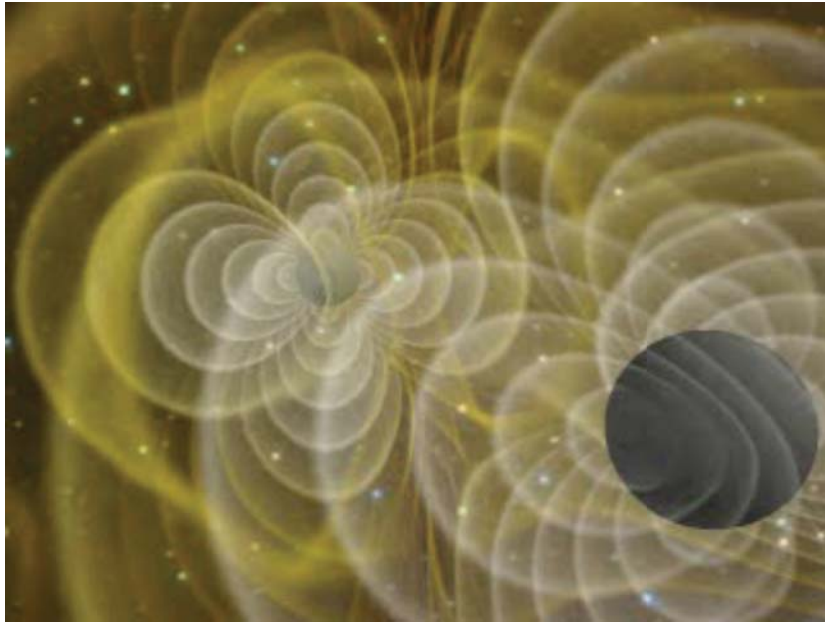
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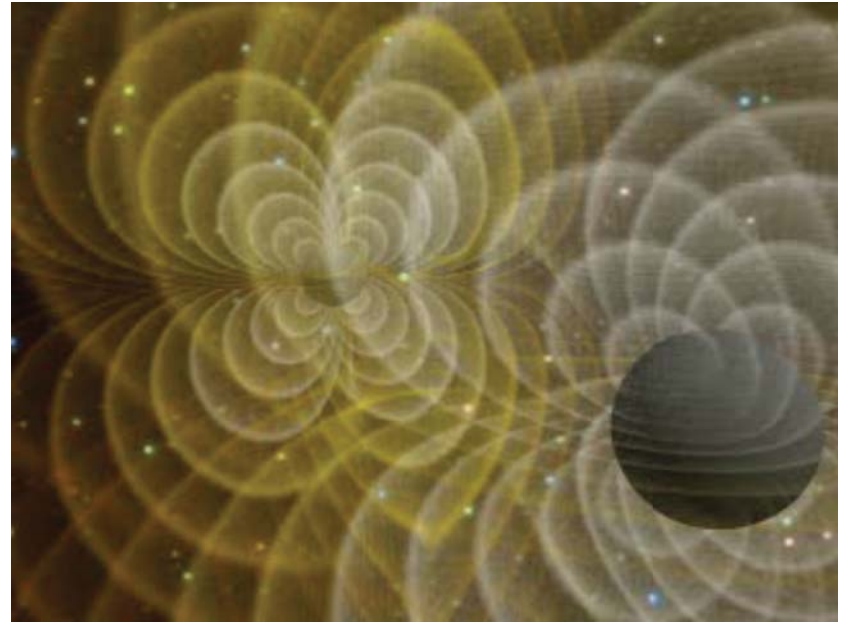
New approach: *Don't factor.*

Smooth and spiky pieces evolve together;  
let the numerics do what they will.  
BH coordinates slosh around as they like:  
Instabilities are suppressed.

# Last dance of two BHs



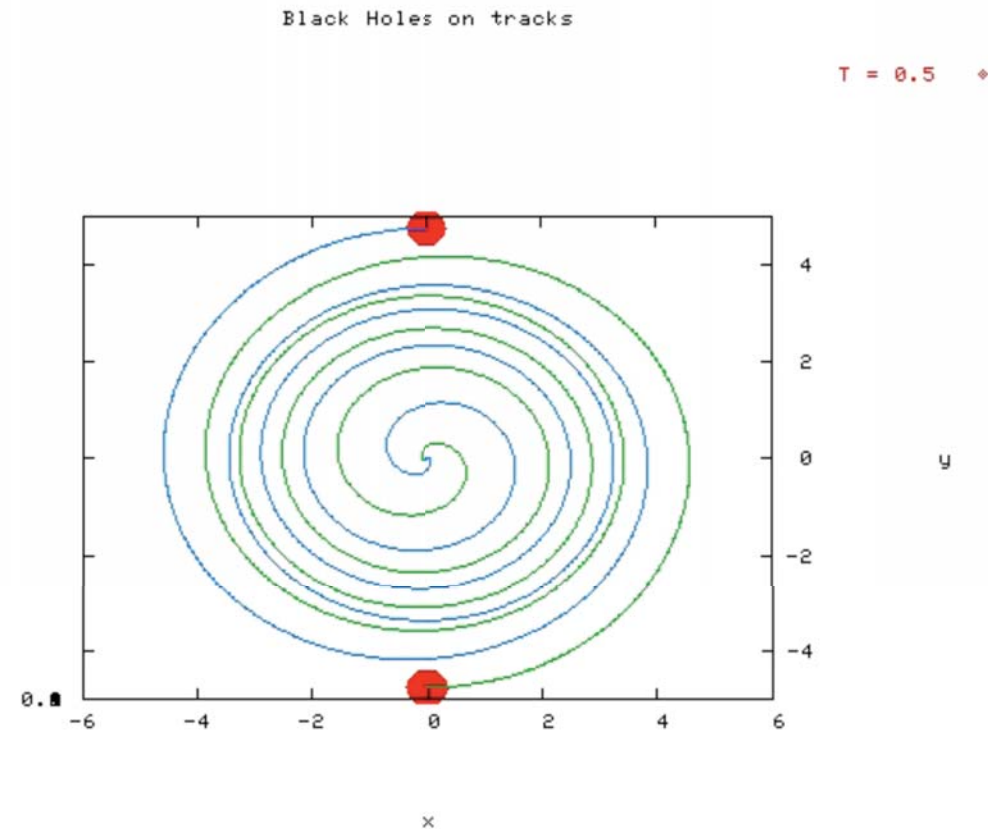
Contours: Curvature components corresponding to “+” polarization of GWs



Contours: Curvature components corresponding to “x” polarization of GWs

Movies courtesy GSFC Numerical Relativity Group

# Simpler view of dynamics



Animation courtesy GSFC Numerical Relativity Group