The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization
IAEA
International Atomic Energy Agency

1954-15

Summer School in Cosmology

21 July - 1 August, 2008

Gravitational waves
Lecture 4
additional material

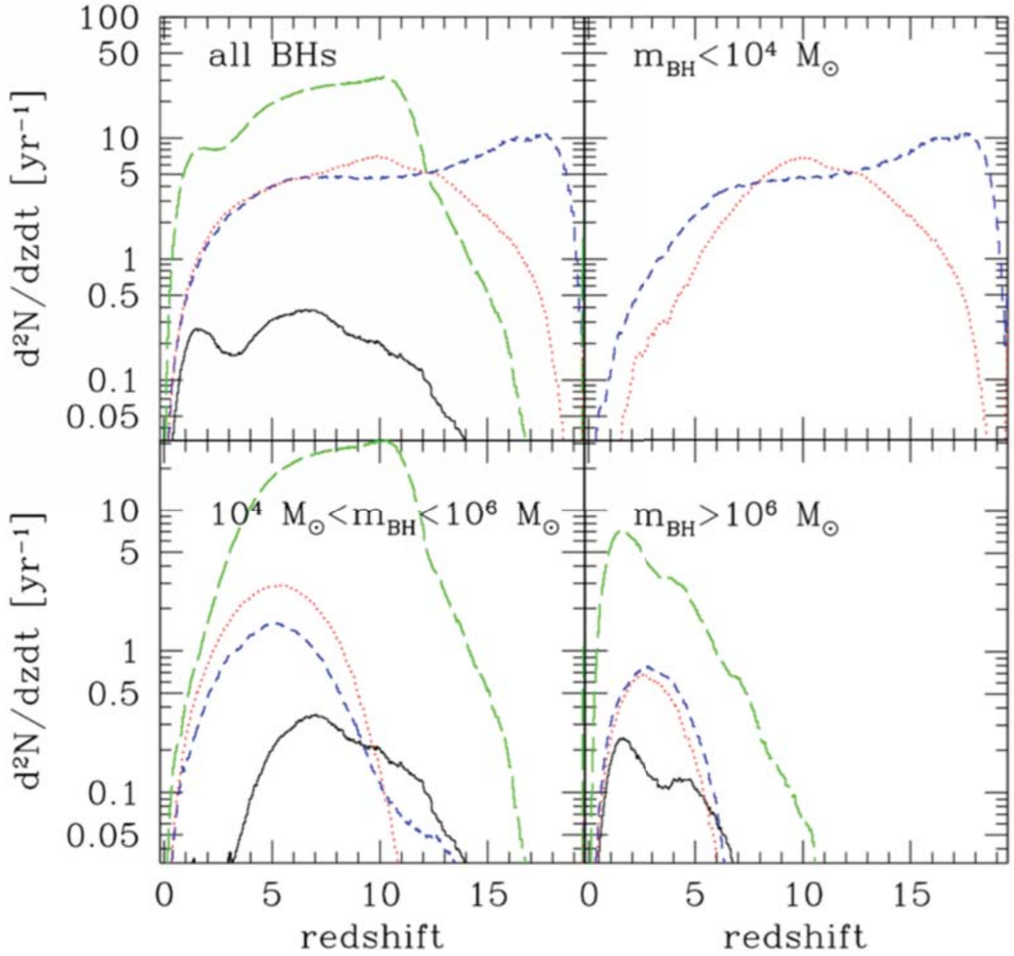
S.A. Hughes
MIT, USA

Gravitational waves and MBBH

Sources we are interested in: High redshift merger of seeds from hierarchical structure growth.

Model: Black hole growth tracking merger of dark matter halos via merger tree.

Curves: Different assumptions about seed holes and their growth. *All* models consistent with AGN optical luminosity for $1 < z < 6$



Sesana, Volonteri, & Haardt 2007 MNRAS **377, 1711**

Equations of motion from post-Newtonian spacetime of a binary

Looking at motion of binary members in post-Newtonian spacetime lets us see how zeroth order Newtonian rule is modified by GR.

$$
\begin{aligned}
a_1^i &= -\frac{Gm_2 n_{12}^i}{r_{12}^2} \\
&+\frac{1}{c^2}\Bigg\{\left[\frac{5G^2 m_1 m_2}{r_{12}^3}+\frac{4G^2 m_2^2}{r_{12}^3}+\frac{Gm_2}{r_{12}^2}\left(\frac{3}{2}(n_{12}v_2)^2 - v_1^2 + 4(v_1 v_2) - 2v_2^2\right)\right] n_{12}^i \\
&\quad + \frac{Gm_2}{r_{12}^2}\left(4(n_{12}v_1) - 3(n_{12}v_2)\right) v_{12}^i\Bigg\}
\end{aligned}
$$

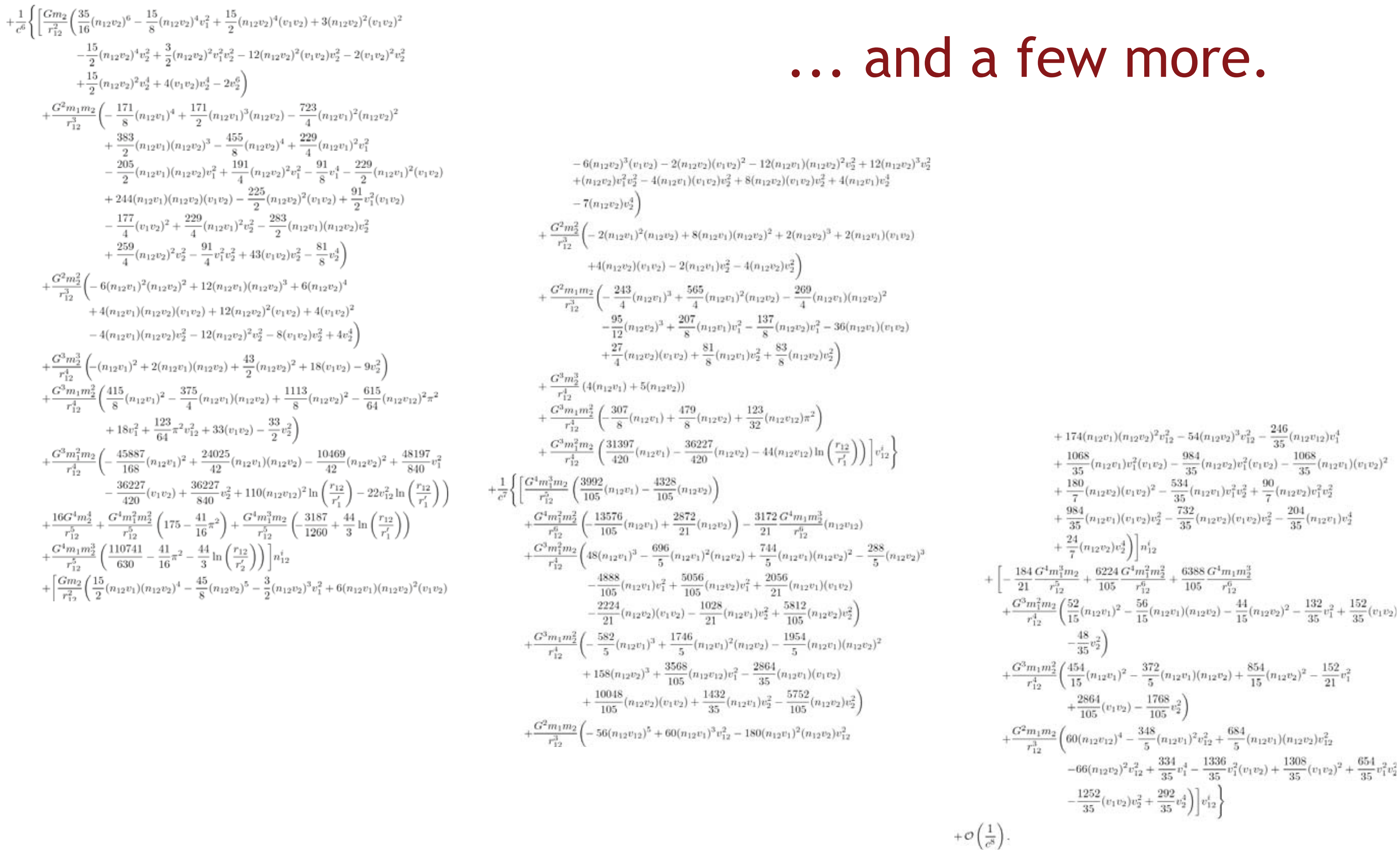
Lowest order term duplicates Newton.

Post-Newton gives corrections in *v*/*c*.

... and more corrections ...

$$
\begin{aligned}
&+\frac{1}{c^4}\Bigg\{\Bigg[-\frac{57G^3m_1^2m_2}{4r_{12}^4}-\frac{69G^3m_1m_2^2}{2r_{12}^4}-\frac{9G^3m_2^3}{r_{12}^4}\\
&\quad+\frac{Gm_2}{r_{12}^2}\Big(-\frac{15}{8}(n_{12}v_2)^4+\frac{3}{2}(n_{12}v_2)^2v_1^2-6(n_{12}v_2)^2(v_1v_2)-2(v_1v_2)^2+\frac{9}{2}(n_{12}v_2)^2v_2^2\\
&\qquad\quad+4(v_1v_2)v_2^2-2v_2^4\Big)\\
&\quad+\frac{G^2m_1m_2}{r_{12}^3}\left(\frac{39}{2}(n_{12}v_1)^2-39(n_{12}v_1)(n_{12}v_2)+\frac{17}{2}(n_{12}v_2)^2-\frac{15}{4}v_1^2-\frac{5}{2}(v_1v_2)+\frac{5}{4}v_2^2\right)\\
&\quad+\frac{G^2m_2^2}{r_{12}^3}\left(2(n_{12}v_1)^2-4(n_{12}v_1)(n_{12}v_2)-6(n_{12}v_2)^2-8(v_1v_2)+4v_2^2\right)\Bigg]n_{12}^i\\
&\quad+\Bigg[\frac{G^2m_2^2}{r_{12}^3}\left(-2(n_{12}v_1)-2(n_{12}v_2)\right)+\frac{G^2m_1m_2}{r_{12}^3}\left(-\frac{63}{4}(n_{12}v_1)+\frac{55}{4}(n_{12}v_2)\right)\\
&\quad\;+\frac{Gm_2}{r_{12}^2}\Big(-6(n_{12}v_1)(n_{12}v_2)^2+\frac{9}{2}(n_{12}v_2)^3+(n_{12}v_2)v_1^2-4(n_{12}v_1)(v_1v_2)\\
&\qquad\quad+4(n_{12}v_2)(v_1v_2)+4(n_{12}v_1)v_2^2-5(n_{12}v_2)v_2^2\Big)\Bigg]v_{12}^i\Bigg\}\\
&+\frac{1}{c^5}\Bigg\{\left[\frac{208G^3m_1m_2^2}{15r_{12}^4}(n_{12}v_{12})-\frac{24G^3m_1^2m_2}{5r_{12}^4}(n_{12}v_{12})+\frac{12G^2m_1m_2}{5r_{12}^3}(n_{12}v_{12})v_{12}^2\right]n_{12}^i\\
&\quad+\left[\frac{8G^3m_1^2m_2}{5r_{12}^4}-\frac{32G^3m_1m_2^2}{5r_{12}^4}-\frac{4G^2m_1m_2}{5r_{12}^3}v_{12}^2\right]v_{12}^i\Bigg\}
\end{aligned}
$$

... and a few more.



[Blanchet 2006, Liv Rev Rel 9, 4, Eq. (168)]

Evolution of spins

The *motions* of masses (orbits and spins) drives a "magnetic-type" coupling of mass currents to spacetime.

Creates new "forces", modifying orbit acceleration; also causes spins of binary's members to precess.

$$\frac{d\mathbf{S}_1}{dt} = \frac{1}{r^3}\left[\left(2+\frac{3}{2}\frac{m_2}{m_1}\right)\mu\sqrt{Mr}\hat{\mathbf{L}}\right]\times\mathbf{S}_1 + \frac{1}{r^3}\left[\frac{1}{2}\mathbf{S}_2 - \frac{3}{2}(\mathbf{S}_2\cdot\hat{\mathbf{L}})\hat{\mathbf{L}}\right]\times\mathbf{S}_1$$

$$\frac{d\mathbf{S}_2}{dt} = \frac{1}{r^3}\left[\left(2+\frac{3}{2}\frac{m_1}{m_2}\right)\mu\sqrt{Mr}\hat{\mathbf{L}}\right]\times\mathbf{S}_2 + \frac{1}{r^3}\left[\frac{1}{2}\mathbf{S}_1 - \frac{3}{2}(\mathbf{S}_1\cdot\hat{\mathbf{L}})\hat{\mathbf{L}}\right]\times\mathbf{S}_2$$

"Gravitomagnetic" field due to orbital motion

"Gravitomagnetic" field due to other body's spin

Form is $d\mathbf{S}/dt = \mathbf{S} \times \mathbf{B}_g$

Evolution of spins

The *motions* of masses (orbits and spins) drives a “magnetic-type” coupling of mass currents to spacetime.

Creates new “forces”, modifying orbit acceleration; also causes spins of binary’s members to precess.

$$\frac{d\mathbf{S}_1}{dt} = \frac{1}{r^3}\left[\left(2 + \frac{3}{2}\frac{m_2}{m_1}\right)\mu\sqrt{Mr}\hat{\mathbf{L}}\right] \times \mathbf{S}_1 + \frac{1}{r^3}\left[\frac{1}{2}\mathbf{S}_2 - \frac{3}{2}(\mathbf{S}_2 \cdot \hat{\mathbf{L}})\hat{\mathbf{L}}\right] \times \mathbf{S}_1$$

$$\frac{d\mathbf{S}_2}{dt} = \frac{1}{r^3}\left[\left(2 + \frac{3}{2}\frac{m_1}{m_2}\right)\mu\sqrt{Mr}\hat{\mathbf{L}}\right] \times \mathbf{S}_2 + \frac{1}{r^3}\left[\frac{1}{2}\mathbf{S}_1 - \frac{3}{2}(\mathbf{S}_1 \cdot \hat{\mathbf{L}})\hat{\mathbf{L}}\right] \times \mathbf{S}_2$$

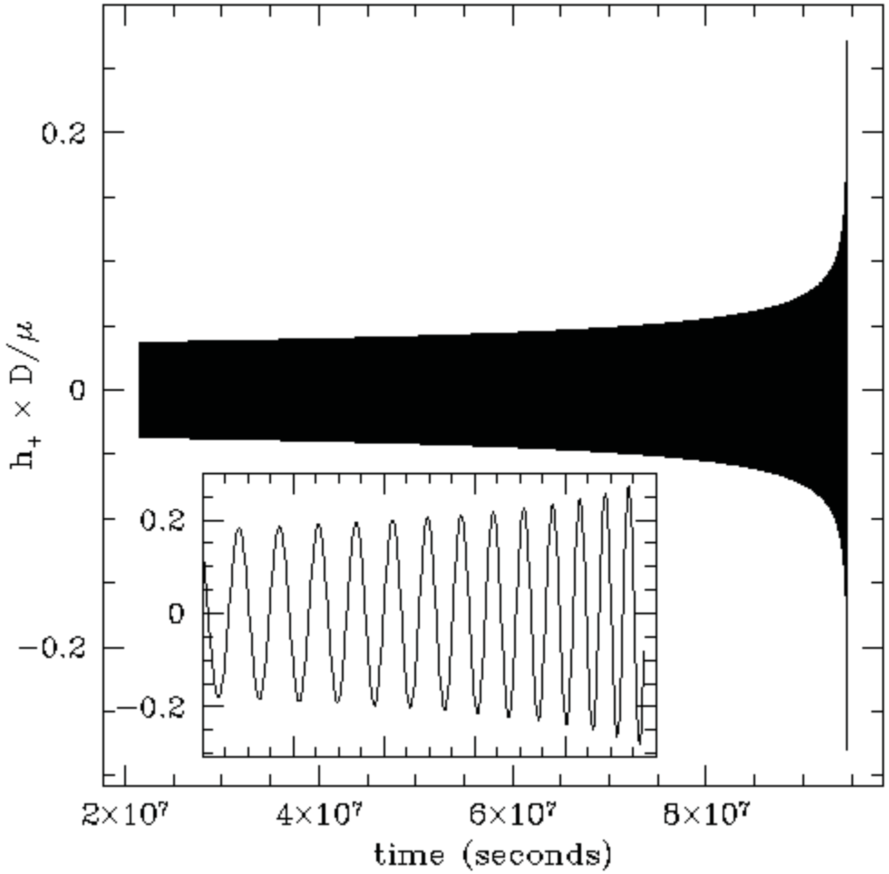
Angular momentum is *globally* conserved:

$$\mathbf{J} = \mathbf{L} + \mathbf{S}_1 + \mathbf{S}_2 = \text{constant}$$

Means that the *orbital plane* precesses to compensate. (Known as Lense-Thirring precession in weak-field.)

Inspiral measurements

$$h_+ = \frac{[G(1+z)\mathcal{M}/\,c^2]^{5/3}\,[\pi f(t)/c]^{2/3}}{D_L}\mathcal{F}(\text{``angles''})\cos[\Phi(t)]$$

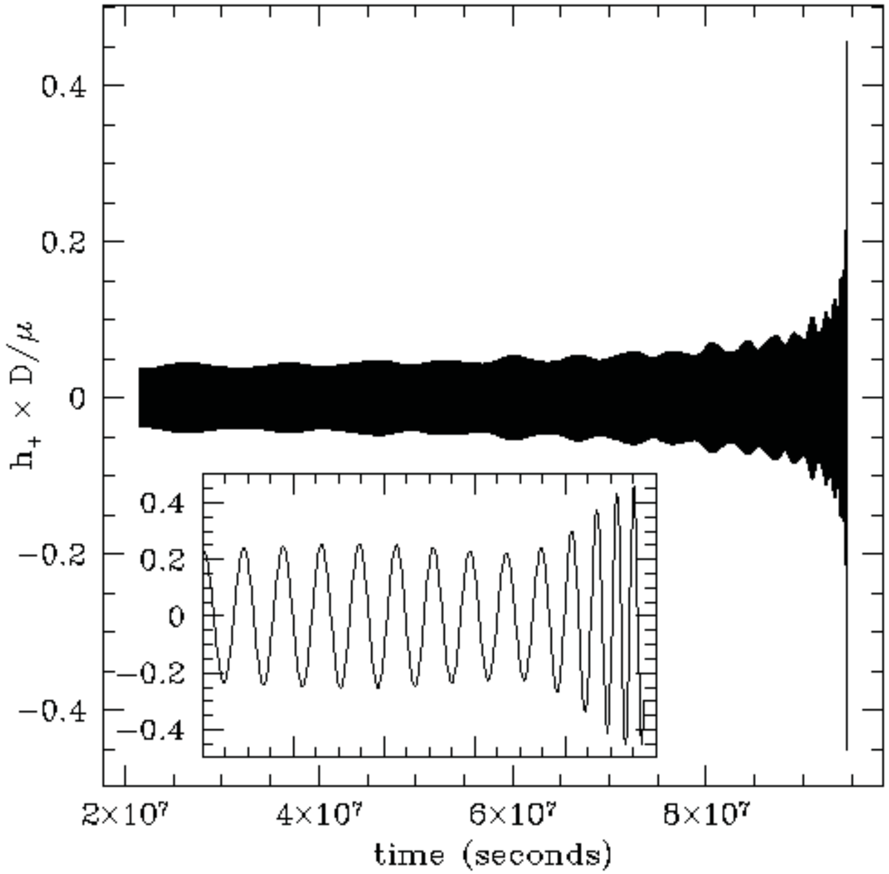


Example waveform: Both black holes non-spinning.

Very smooth evolution!

Inspiral measurements

$$h_+ = \frac{[G(1+z)\mathcal{M}/c^2]^{5/3}\,[\pi f(t)/c]^{2/3}}{D_L}\mathcal{F}(\text{“angles”})\cos[\Phi(t)]$$



Spins cranked up!
Spin 1 = Spin 2 =
99% maximum

Strong frequency and amplitude modulation gives spin precision.

Equations of numerical relativity

Need to rewrite the Einstein equations in a form that allows us to integrate with respect to a chosen time direction.

Do this by *projecting* components of Einstein parallel/perpendicular to this time:

t^μ Vector associated with time coordinate

$n^\mu = (t^\mu - \beta^\mu)/\alpha$ Vector normal to spatial slice

$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ Projector into a spatial slice

Spatial components of projection tensor are just the spatial metric components γ_{ij}.

Equations of numerical relativity

Contracting Einstein along normal gives us *constraints*: Relationships that geometry must satisfy at all times. Analogous to divergence equations of Maxwell's theory.

"Hamiltonian constraint:" $G_{\alpha\beta} n^\alpha n^\beta = 8\pi G T_{\alpha\beta} n^\alpha n^\beta$

$$\longrightarrow R + K^2 - K_{ij} K^{ij} = 16\pi G \rho$$

R : Ricci scalar of 3-metric γ_{ij} $\qquad \rho = T_{\alpha\beta} n^\alpha n^\beta$

$K_{ij} \equiv -\gamma_i{}^\alpha \gamma_j{}^\beta \nabla_\alpha n_\beta$ "Extrinsic curvature" of constant-time slice of geometry.

Equations of numerical relativity

Contracting Einstein along normal gives us *constraints*: Relationships that geometry must satisfy at all times. Analogous to divergence equations of Maxwell's theory.

"Momentum constraint:" $$G_{\alpha\beta} n^\alpha \gamma_i{}^\beta = 8\pi G T_{\alpha\beta} n^\alpha \gamma_i{}^\beta$$

$$\longrightarrow D_j K^j{}_i - D_i K = 8\pi G j_i$$

D_i : Covariant derivative of 3-metric γ_{ij}

$$j_i = -T_{\alpha\beta} n^\alpha \gamma_i{}^\beta$$

Equations of numerical relativity

Contracting twice with projectors gives *evolution equations*: Relations connecting geometry from moment to moment.

$$G_{\alpha\beta}\gamma_i{}^\alpha\gamma_j{}^\beta = 8\pi G T_{\alpha\beta}\gamma_i{}^\alpha\gamma_j{}^\beta \equiv 8\pi G S_{ij}$$

$$\longrightarrow \partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i$$

$$\partial_t K_{ij} = -D_i D_j \alpha + \alpha(R_{ij} - 2K_{ik}K^k{}_j + K K_{ij})$$
$$-8\pi G\alpha\left(S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)\right) + \beta^k D_k K_{ij}$$
$$+K_{ik}D_j\beta^k + K_{kj}D_i\beta^k$$

Lapse α, shift β^i?

Notice that these equations do not constrain the values of α and β^i: We are *completely* free to specify them as we wish!

This is how gauge invariance is manifested in a strong field calculation: The 4 functions α and β^i are equivalent to the 4 components of the weak-field gauge generator ξ^μ.

Wonderful!! Total freedom to pick them as convenient.

Terrible!! What's a good choice to use for interesting calculations??

Recipe

1. Pick a problem you want to solve. Most popular problem for numerical relativity: Final inspiral and merger of two black holes.

2. Find the 3-geometry γ_{ij} describing this system at an "initial" time. This geometry must satisfy the constraint equations!

3. Decide on a method to determine α and β^i. In principle, all are good. In practice, this can make or break your calculation.

4. Evolve. Compare to observations. Celebrate!

Result ...

For many years, all groups found a very consistent result: **The code crashes.**

Many issues impact this ... most important appears to be *constraint-violating* modes: Solutions to $(\partial_t \gamma_{ij}, \partial_t K_{ij})$ that violate the Hamiltonian and momentum constraints.

These modes can be unstable: Once excited, will grow and dominate. Instead of geometry that describes two black holes, quickly get a geometry that is nonsense.

The breakthrough

Tremendous advance roughly 3 years ago: How we represent a black hole on our numerical grid.

$$\psi = u + \sum_i \frac{G m_i}{|\vec{r} - \vec{r}_i|}$$

"Punctures": smooth field plus singular bit at location of black hole *i*.

Old approach: Factor out singular piece, treat separately from "smooth" degrees of freedom. Mathematically "nice" ... but requires fixing coordinates of BHs.

End up with very twisted representation of geometry — seeds instabilities.

The breakthrough

Tremendous advance roughly 3 years ago: How we represent a black hole on our numerical grid.

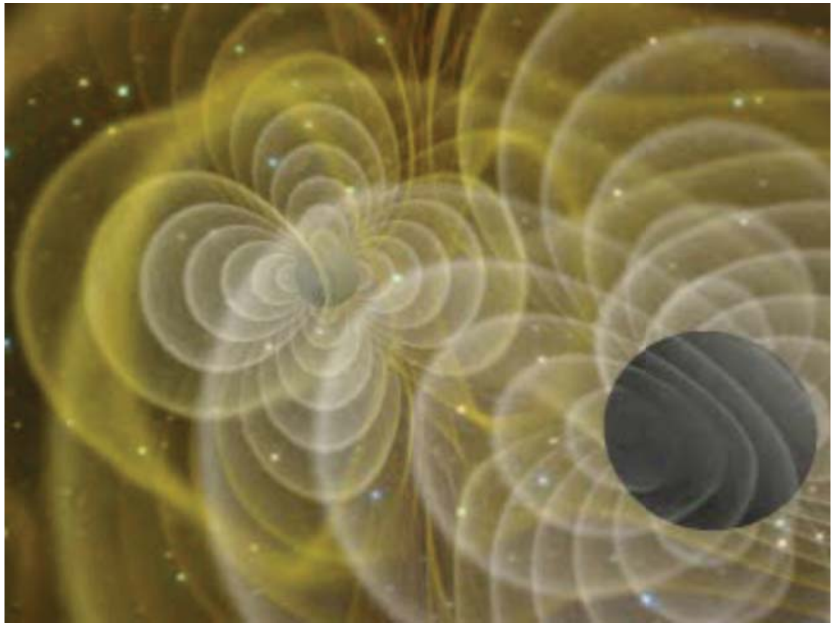
$$\psi = u + \sum_i \frac{Gm_i}{|\vec{r} - \vec{r}_i|}$$

"Punctures": smooth field plus singular bit at location of black hole *i*.

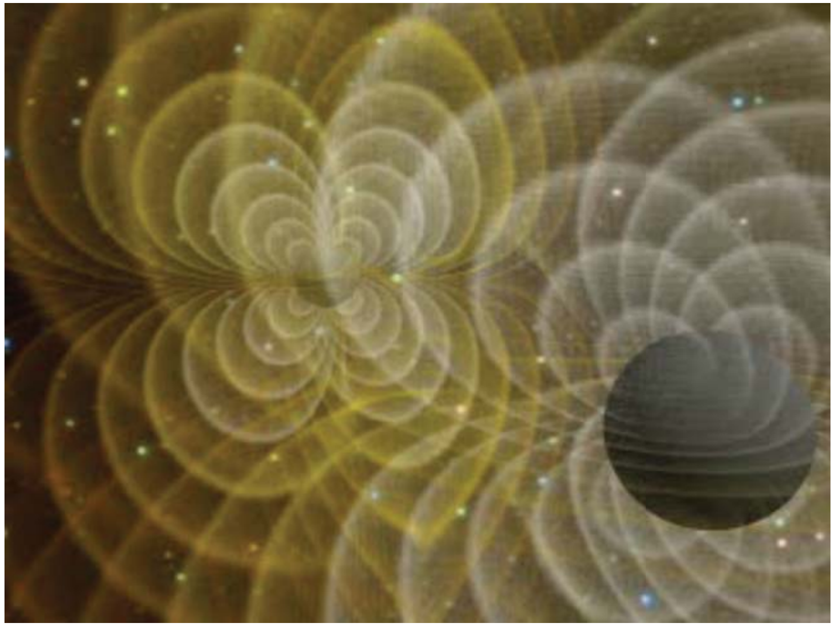
New approach: ***Don't factor.***

Smooth and spiky pieces evolve together; let the numerics do what they will.
BH coordinates slosh around as they like: Instabilities are suppressed.

Last dance of two BHs



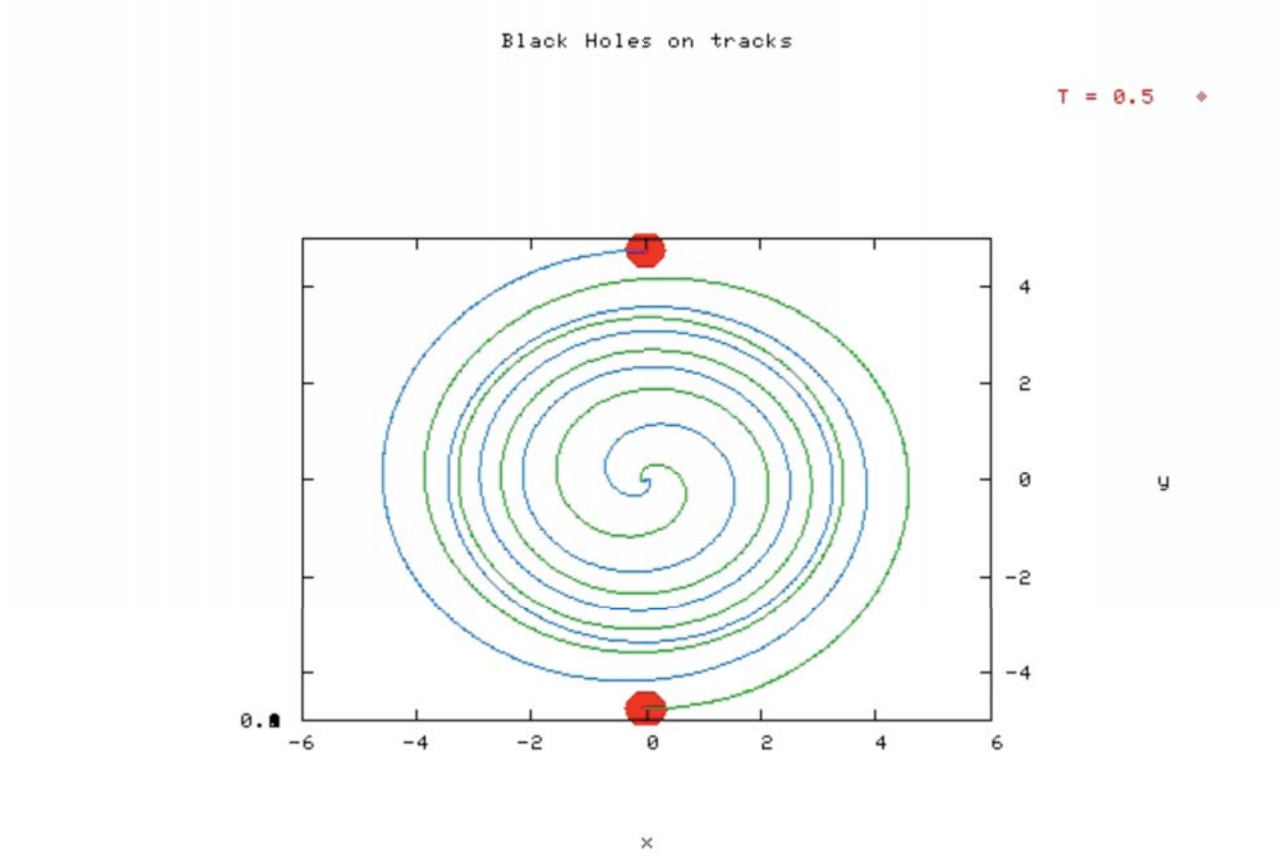
Contours: Curvature components corresponding to “+” polarization of GWs



Contours: Curvature components corresponding to “x” polarization of GWs

Movies courtesy GSFC Numerical Relativity Group

Simpler view of dynamics



Animation courtesy GSFC Numerical Relativity Group