Effective Arithmetic and Motives

Alexei Pantchichkine

We discuss the use of the motivic L-functions in arithmetic as a computational tool and as a mean of proving some fundamental theoretical results.

Classical L-functions. Dirichlet's class number formula. The Kubota and Leopoldt p-adic L-functions and the Iwasawa theory. L-functions of a global field of a positive characteristic. Examples of the use of L-functions.

2. Local and global methods in arithmetic. The Birch and Swinnerton-Dyer Conjecture. Constructing rational points from complex and p-adic L-functions. Fermat's Last Theorem and Modularity of Elliptic Curves (see William Stein, `An Explicit Approach to Number Theory'. Lecture notes available at:

http://modular.fas.harvard.edu/edu/Fall2001/124/lectures/lectures_all/lec
res.pdf

3. Motivic L functions and Galois representations. Weil conjectures. Complex and p-adic L-functions attached to motives and their use. Automorphic L-functions and Artin's Conjecture.

4. Modular forms and congruences. Deformation theory of Galois representations and families of modular forms. Applications to the proof of Serre's Modularity Conjectures (Khare-Wintenberger-Kisin et al., 2007).

References:

http://en.wikipedia.org/wiki/Serre_conjecture_%28number_theory%29

http://www.math.utah.edu/~shekhar/papers.html

The relation to the proof of the Sato-Tate Conjecture (2006). (see Manin Yu.I. and Panchishkin A.A., Introduction to Modern Number Theory, Encyclopaedia of Mathematical Sciences, vol. 49 (2nd ed.), Second printing Springer-Verlag, 2007, 514 p).

Constructive Analysis.

Douglas S. Bridges

These lectures will cover a wide range of topics in constructive real and functional analysis; they will be based primarily on the reference [1]. Beginning with a brief introduction to constructive thinking and informal intuitionistic logic, I will present an interval-arithmetic development of the real line, together with some basic real analysis. We will then deal with the constructive theory of metric, normed, and Hilbert spaces. This will lead us to the separation and Hahn-Banach theorems, which will be use to characterise linear functionals on the dual space and on the space of bounded operators on a Hilbert space. The final part of the analysis proper will deal with the existence of adjoints, locating the kernel and range of an operator, and applications of Baire theorem in functional analysis. If time permits, I will discuss recent work on the role of Brouwer fan theorem in analysis, part of a project on constructive reverse mathematics.

Reference:

[1] D.S. Bridges and L.S. Vita, Techniques of Constructive Analysis, Universitext, Springer-New-York, 2006.

Point-free Topology

Erik Palmgren

Locale theory and formal topology are two largely equivalent ways of studying spaces in terms of their complete lattice of open sets rather than in terms of their points. This has many advantages from a constructive point of view. More information about a continuous function's behaviour can be encoded than if merely considering its action on points. Point-free topology seems so far to be the most successful way of doing general topology constructively in the sense of Bishop's constructive mathematics. A main goal of our lectures is to relate formal topology to the standard theory of metric spaces, and show how it also in this case improves the notions. We present some recent developments.

Topics covered in the lectures:

1. Locales and formal topology.

2. Inductively generated covers. Compactness. Separation properties.

3. Limits and colimits in formal topology. Subspaces. Overt formal spaces.

4. Predicativity problems in formal topology. Issues of formalisation.

5. Localic completion of metric spaces.

Introduction to Combinatorial Homotopy

Francis Sergeraert

Kadeishvilis's lecture of this Summer School is devoted to Operadic Algebraic Topology, the modern style of Algebraic Topology, which simultaneously encompasses the most important current homological theories and also satisfies the constructiveness requirement. Operads are very rich algebraic or topological structures describing the deep homological and homotopical natures of some object. A striking result due to Michael Mandell explains such a description is complete, in a natural sense.

Combinatorial homotopy is a necessary prerequisite to understand the nature of the topological operadic theories and is the subject of this lecture. Ordinary homotopy theory seems having a quite topological nature but it is not so hard to describe it in a combinatorial framework, and this illustrates the power of simplicial techniques, the main tool for efficient operadic methods, the main tool also for combinatorial topology in general. It is essential to understand the difference between the notion of simplicial *complex*, elementary but too weak for actual applications, and the notion of simplicial *set*, much more complex (!), but very powerful. The audience of this lecture could usefully use Sections 3 and 4 of the lecture notes [1] as an introduction to this... introductory lecture.

1. The simplicial Lego.

2. Simplicial homology.

2. Constructing sophisticated objects in the simplicial framework.

3. Functional spaces in the simplicial framework.

[1] http://www-fourier.ujf-grenoble.fr/~sergerar/Papers/CWSS.pdf

Algorithms and Algebraic Geometry

Gert-Martin Greuel

1. Standard Bases and Operations on Ideals and Modules.

This includes an overview on standard bases in local, global and mixed rings, for ideals and modules. We show algorithms for solving the ideal and submodule membership problem and as well as for elimination of variables which amounts to compute the (closure) of the image of a morphism in algebraic geometry.

2. Integral Closure and Constructive Normalization.

We give an algorithm to compute the normalization of a reduced affine ring (its integral closure in the total ring of fractions) and for computing an ideal defining the non-normal locus.

3. Computation in Local Rings.

We show how local monomial orderings in a polynomial ring allow to effectively compute in the localization at a maximal ideal by the general theory of standard bases. We do also address the relation to standard bases in power series rings.

4. Algorithms in Singularity Theory.

Computations in local rings can be used to effectively compute important invariants of singularities, such as the Milnor number and the Tjurina number. In more detail we shall treat singular plane algebraic curves.

We assume that the participants do have basic knowledge in commutative algebra and algebraic geometry. We shall use the free computer algebra system SINGULAR (www.singular.uni-kl.de), where all the presented algorithms have been implemented, for additional exercises and tutorials.

Constructive Algebra

Ihsen Yengui

In my course I will try to approach constructively the problem of projective modules over polynomial rings originally raised by J.-P. Serre in 1955.

Serre remarked that it was not known whether there exist finitely generated projective modules over multivariate polynomial rings with coeffcients in a field, which are not free. This remark turned into the "Serre's conjecture" or "Serre's problem", stating that indeed there were no such modules. Proven independently by D. Quillen and A. A. Suslin in 1976, it became subsequently known as the Quillen-Suslin theorem. In my course I will give constructive proofs of Quillen and Suslin proofs of Serre's conjecture, simple and constructive proofs of some subsequent developments in the theory of projective modules over polynomial rings, and also I will cast light on a new progress very recently obtained concerning the Hermite ring Conjecture.

My course will be divided into four parts.

1. Constructive theory of projective modules, Quillen's proof of Serre's conjecture, notably Quillen's patching theorem, Horrocks' theorem, Quillen's induction theorem.

2. Suslin's proof of Serre's conjecture, a general method for making the use of maximal ideals constructive, the theorem of Traverso-Swan- Coquand about seminormality.

3. Constructive theory of arithmetical rings, constructive comparison between the rings R(X) and R<X> and application to the Lequain-Simis induction theorem, a constructive proof of the Lequain-Simis-Vasconcelos theorem asserting that for any arithmetical ring R, all finitely generated projective R[X1,...,Xn]-modules are extended from R.

4. A new progress concerning the long-standing Hermite ring Conjecture (1972) asserting that if R is an Hermite ring (that is, all finitely generated stably free R-modules are free) then so is R[X]. Of course, a positive solution to this conjecture will imply a positive solution to the famous Bass-Quillen Conjecture (1976) saying that for any local regular ring R, all finitely generated projective R[X]-modules are free. We will prove (constructively) that for any ring R with Krull dimension < 2, R[X] is an Hermite ring. Moreover, the corresponding completion of unimodular rows can be done using elementary matrices (instead of invertible matrices). In other words, we will prove that for any ring R with Krull dimension < 2 and n > 2, any unimodular vector in R[X]^n can be completed into an elementary matrice in R[X]^(nxn). We will also discuss possible generalizations of this result and some new open questions that it raises.

Groebner bases
Jean-Charles Faugäre

TBA.

Symbolic Summation (A = B)

Peter Paule

The lectures give an introduction to computer algebra methods for handling summation problems arising in areas like combinatorics or special functions. After an introductory survey, topics treated include: rational summation, hypergeometric summation (Gosper, Zeilberger), and algorithms for simplifying (multiple) binomial sums. Special emphasis is put in concrete examples. Packages developed at the Research Institute for Symbolic Computation (RISC) will be shown in action.

Computational Algebra

Teo Mora

Macaulay introducing the notion of inverse systems proposed duality as a strong combinatorial tool for performing ideal theory investigation. This series of lectures surveys the recent combinatorial algorithmical results stemmed from Macaulay's ideas and the description they offer on the structure of a 0-dimensional ideal:

1. Moeller Algorthm and variations (FGLM, Fitzpatrick).

The algorithm applies elementary linear algebra to solve, with good complexity, the following problem: given a finite set of functionals which defines a 0-dim. ideal, return its Groebner basis. The main applications are point evaluation, computing lex Groebner bases (FGLM), decoding BCM codes (Fitzpatrick).

2. Cerlienco-Mureddu Algorithm, Lazard Theorem, Gianni--Kalkbrener Theorem.

The lecture gives a survey on the main results which shed light on the combinatorial structure of (radical) 0-dimensional ideals. The lecture gives also a guideline to the readings required by the fourth lecture.

3. Lasker--Noether Decomposition and Macaulay's Algorithm.

Lasker--Noether First Theorem states that each ideal is decomposed into irreducible components and a lemma proves that irreducible components are primaries. Lasker--Noether Second Theorem packs together the irreducible components associated to a specific prime in order to produce a decomposition into primaries notwithstanding that, when the theorem was stated, there was already an explicit procedure, given by Macaualy, which allowed to compute the irreducible components of any given primary. The aim of this lecture is to explain why the notion of irreducible decomposition has been removed from booktexts.

4. Variations on point interpolation.

The students are required to read the suggested papers and to test the corresponding algorithms on some easy and illuminating examples. The aim of the last lecture is a common discussion on these algorithms and on their practicability.

Constructive Logic

Thierry Coquand

The goal of these lectures is to be an introduction to the other lectures on constructive algebra and formal topology. Also, I would like to introduce the "Kronecker-Duval philosophy" (used in computer algebra) from the logical point of view.

Lectures 1 and 2: Hilbert's program, completeness theorems, cut-elimination, negative translation, coherent logic.

Lecture 3: Point-free presentation of spectral spaces, Joyal's representation of Zariski spectrum, boundary, Krull dimension, local-global principles.

Lecture 4: Example of Prufer ring, as a constructive approximation of Dedekind rings.

References: most of them are available at:

hlombardi.free.fr/publis/PublisRecherche.html

-Dynamical method in algebra: effective Nullstellensatze

-An elementary characterization of Krull dimension

-A logical approach to abstract algebra

-Theorie algorithmique des anneaux arithmetiques, de Prufer et de Dedekind

Operadic Algebraic Topology

Tornike Kadeishvili

The method of Algebraic Topology is to assign to a topological space certain algebraic models. But stepping from complex geometrical object to its algebraic model, one loses part of information. That is why most models are not complete invariants: the isomorphism of invariants does not guarantee the equivalence of spaces. Models with richer algebraic structures contain more information about the space. For example the invariant "cohomology algebra" distinguishes spaces, which can not be distinguished by the invariant "cohomology groups". During the development of Algebraic Topology many new cohomology and cochain operations were invented in order to get reacher algebraic models. The language of operads is a good tool to handle these algebraic objects with huge and complex algebraic structures. The aim of these lectures is to present these structures using operadic technics.

<p>The following topics are planned to discuss: Differential graded algebras, bialgebras, Lie algebras, cup product and Steenrod operations, Bar and Cobar constructions, A_\infty-, B_\infty-, C_\infty-, L_\infty-algebras, differential graded operads, operads defining the above mentioned algebras, E_\infty operads. Topological applications: C_\infty-algebra structure in cohomology and the rational homotopy type, E_\infty-algebra structure and the homotopy type.