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Supersolids, vortices, and glasses

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Supersolids, Dislocations & Vortices

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Photo: Emmons Glacier on Mt Rainier (Aug 3, 2008)

Ingredients

- Effective model of glasses, including dislocations and defects
- Continuous gauge model of glasses with static disorder
- ‘Supersolid’ model: add superflow confined to dislocations and introduce dynamics to these 
- Must first understand dislocation dynamics 

Legend:  = Philip Stamp  = Lara Thompson

This work was done by: L Thompson (Physics, UBC)
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We acknowledge discussions with: M Schechter (Physics, UBC)
NV Prokof'ev (Physics, UMass Amherst)

RATIONALE

At the moment theorists are struggling to understand the variety of experiments on supersolidity in He-4, which lead to seemingly conflicting interpretations. The purpose of the present study is therefore not to give a theory to be tested against experiment, but instead to look at a few questions which apparently underlie this field, and which may need to be answered first. The 2 questions we will address here are

(i) What are the important interactions in a Bose glass at low T, and how must a theory of these incorporate superflow along dislocations?

(ii) How do line singularities like dislocations and vortices move in a Bose system?

It will be seen that we are opening a few skeleton-packed closets here. However, we do give partial answers to these questions.

INTERACTIONS in Low-T GLASSES

M Schecter, PCE Stamp,
J Phys CM 20, 244136 (2008)
& to be published
L Thompson, PCE Stamp
To be published

First we recall some basic notions of the theory of elastic solids. We assume that a set of reference points x_α is shifted by a distortion of the system, caused by either the presence of external forces or internal strains, to give a set of points $\tilde{x}_\alpha$. Then, defining $X_\alpha = \tilde{x}_\alpha - x_\alpha$ and the gradient fields:

$$\frac{\partial X_\alpha}{\partial x_\beta} = U_{\alpha\beta} + \Omega_{\alpha\beta}$$

$$U_{\alpha\beta} = \frac{1}{2}\left[\frac{\partial X_\alpha}{\partial x_\beta} + \frac{\partial X_\beta}{\partial x_\alpha}\right]$$

Strain field

$$\Omega_{\alpha\beta} = \frac{1}{2}\left[\frac{\partial X_\alpha}{\partial x_\beta} - \frac{\partial X_\beta}{\partial x_\alpha}\right]$$

Torsion field

In what follows we will assume that on the length scales of interest, the disordered system can be divided into a homogeneous medium (in which local crystal structure is unimportant) and a distribution of dislocations and defects

We can then write down the following effective Hamiltonian

$$\begin{aligned} H[X_j(\mathbf{x})] &= \int d^3r H(\mathbf{x}) \\ &= \int d^3x \; [H_{ph}(\mathbf{x}) + \delta H_{ph}(\mathbf{x}) + H_{def}(\mathbf{x}) + H_{def}^{int}(\mathbf{x})] \end{aligned}$$

(a) BARE PHONONS:

In real space we have

$$H_{ph} = H_o + \frac{\lambda}{2} U_{\alpha\alpha}^2 + \mu\, U_{\alpha\beta}^2$$
$$= H_o + \mu[U_{\alpha\beta} - \frac{1}{3}\delta_{\alpha\beta} U_{\alpha\alpha}]^2 + \frac{K}{2}\, U_{\alpha\alpha}^2$$

In terms of the usual Lame coefficients. In momentum space one has

$$H_{\rm ph} = \sum_{q,\mu} \left(\frac{|P_{q,\mu}|^2}{2M} + M\omega_{q,\mu}^2 \frac{|X_{q,\mu}|^2}{2} \right)$$

Using phonon operators:

$$X_{1\alpha}(x) = \frac{1}{\sqrt{N}} \sum_{q,\mu} X_{q,\mu} \mathbf{e}_{q,\mu,\alpha} \mathrm{e}^{iqx}$$

(b) GRADIENT PHONON TERMS:

The bare phonon terms do not capture all of the physics – we need to add terms which describe the way the system accommodates torsional stress from dislocations. For a start this means incorporating higher gradient terms involving torsion, as follows:

$$\delta H = (\mu + \frac{\lambda}{2})\Lambda_1^2 (\partial_\alpha U_{\gamma\gamma})^2 + \mu \Lambda_2^2 (\partial_\alpha \Omega_\beta)^2$$

(where Λ_1, Λ_2 are 2 length scales to be determined by expt) & then assuming that the source of the torsion is a network of STATIC dislocations– the locus of these sources is $\mathbf{x}_d$

Then we have $$\epsilon_{\alpha\beta\gamma}\partial_\beta \Omega_{\gamma\kappa}(\mathbf{x}) = n_\alpha^{||}(\mathbf{x}_d)\, b_\kappa \delta(\mathbf{x} - \mathbf{x}_d)$$

where $\mathbf{n}(\mathbf{x}_d)$ is the local tangent vector to $\mathbf{x}_d$, & $\mathbf{b}$ is the Burgers vector.

(c) DEFECT TERMS

In a typical disordered quantum solid there will be defects and possibly impurities. If we only include on-site dynamics then a simple TLS-style Hamiltonian describes them:

$$H_{\mathrm{def}}^{(\tau)} = \sum_j \epsilon_j \hat{\tau}_j^z - \Delta_j \hat{\tau}_j^x$$

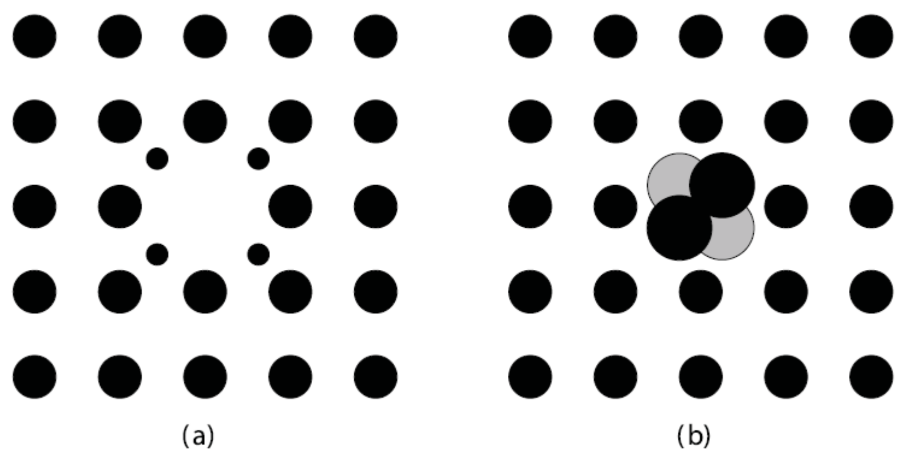

Figure 1. (a) Off-center impurity with four equivalent sites and (b) orientational impurity in the dumbbell approximation with two equivalent states. To first order in the displacement, the two opposite sites in (a) are also equivalent, leading to an effective two-state impurity.

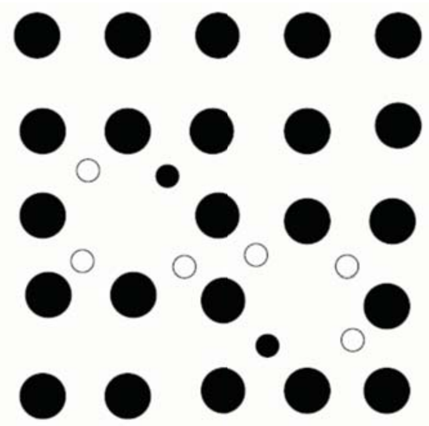

Figure 2. Two nearby impurities, with distorted plaquettes. Filled small circles denoted occupied impurity sites.

However we also need to include the interaction with the background medium. There is a local interaction with degrees of freedom $\hat{S}_j$ describing defects that are not inversion-symmetric:

$$H_1^{\mathrm{int}} = -\sum_{\alpha,\beta}\sum_j \left(\eta(\mathbf{r}_j)\delta^{\alpha\beta}\frac{\partial X_{j\alpha}}{\partial x_{j\alpha}} + \gamma_{\alpha\beta}(\mathbf{r}_j)\frac{\partial X_{j\alpha}}{\partial x_{j\beta}} S_j^z \right)$$

However defects that are Inversion-symmetric in their local environment only interact with phonon gradient terms – which again are crucial:

$$H_2^{\mathrm{int}} = -\sum_{\alpha,\beta,\delta}\sum_j \zeta_{\alpha\beta\delta}(\mathbf{r}_j)\frac{\partial^2 X_{j\alpha}}{\partial x_{j\beta}\partial x_{j\delta}}\tau_j^z$$

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Gauge models for spin-glasses

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(Received 12 June 1978)

$$H_{\text{eff}} = \frac{1}{2} \int d^d x [r_0 |\phi(x)|^2 + \tfrac{1}{4} u |\phi(x)|^4 + |(\vec{\nabla} - i\vec{Q})\phi(x)|^2]. \tag{2.1}$$

The quantity $\vec{Q}$ is the wave vector of the lowest-energy spin-density wave, into which the spins condense below the Curie or Néel temperature. Thus, $\vec{Q}=0$ represents a ferromagnet and $\vec{Q} \neq 0$ an antiferromagnet.

$$F_{\mu\nu}(x) = \partial_\mu Q_\nu(x) - \partial_\nu Q_\mu(x), \tag{2.5}$$

$$P[\vec{Q}] \propto \exp\left(-\frac{1}{2f} \int d^d x \sum_{\mu\nu} F^2_{\mu\nu}(x)\right). \tag{2.6}$$

Gauge field **Q** is a static realization of disorder

Can alternatively derive **Q** from a continuum deformation model (in the spirit of the previous slides) with localized dislocations etc.

f (the mean square vorticity in **Q**) is the *'degree of frustration'*

MODIFICATIONS for a SUPERSOLID

The picture so far applies to a system where the defects (dislocations, vacancies, interstitials) are frozen in space. However in a supersolid (and indeed even in the normal 'quantum solid' phase) in He-4, this is not the case – we expect all of these to be quite mobile. Thus, to the Hamiltonian given above we need to add these kinetic terms.

Moreover, from the work of Prokof'ev & Svistunov (and increasingly from experiment) we have evidence that superflow (ie., dissociation and coherent superflow of vacancies & interstitials) occurs along the dislocations – the superfluid density being strongly confined around the dislocation core. Thus in the above we need to add a superfluid field, confined to the dislocations.

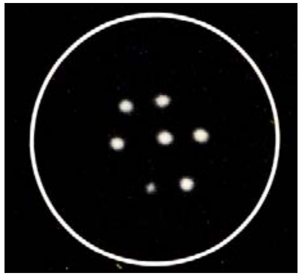
One can write down a field theory which takes all of this physics into account. But before one can do anything with it, it is necessary to solve a very important question, viz.,

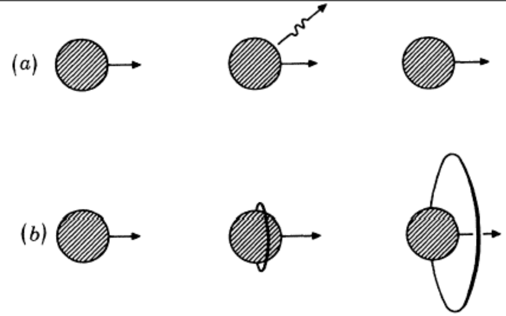
WHAT IS THE CORRECT DYNAMICS FOR A QUANTIZED LINE DEFECT ?

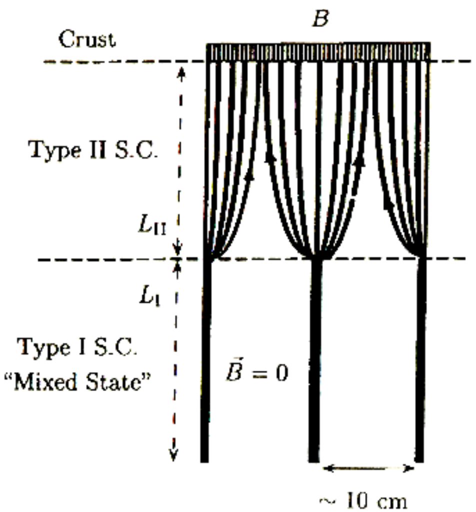
We need to answer this question before we can look at the dynamics of the supersolid at all, because it is clear that the superfluid flow, and the response of the supersolid to shear stresses or to rotation fields, must be bound up with the dislocation dynamics.

Q. VORTICES ARE EVERYWHERE

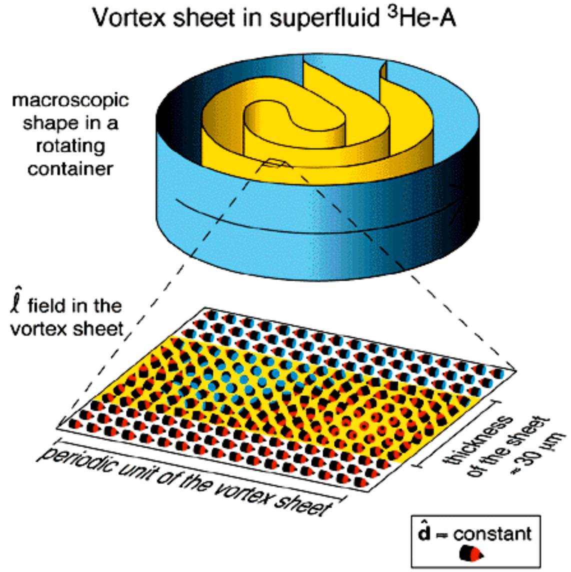
RIGHT: Vortices & vortex rings in He-4
BELOW: pulsar, & structure of its vortex lattice



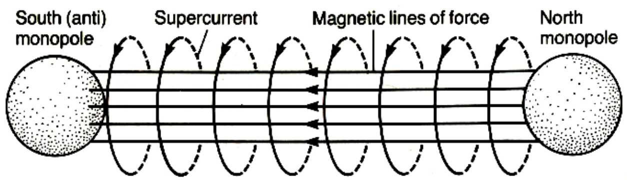


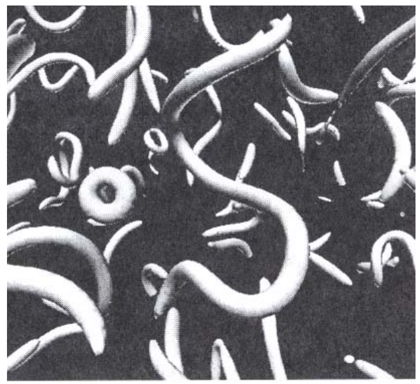


ABOVE: vortices penetrating a superconductor
RIGHT: Vortices in He-3 A

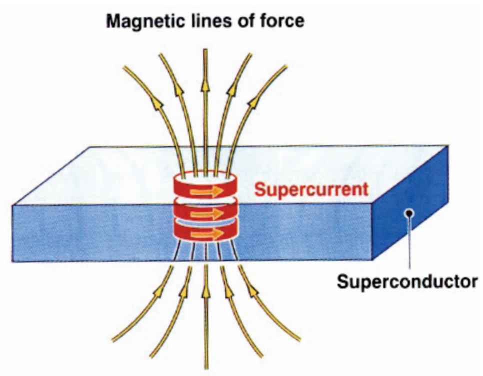


Conjectured structure of cosmic string, & of a 'cosmic tangle' of These in early universe

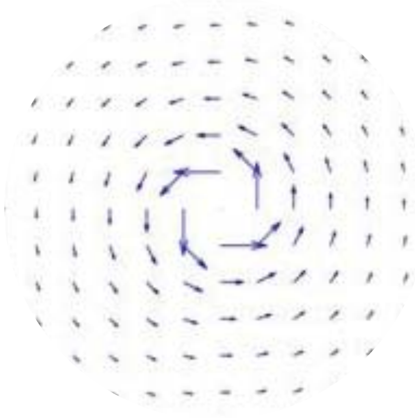




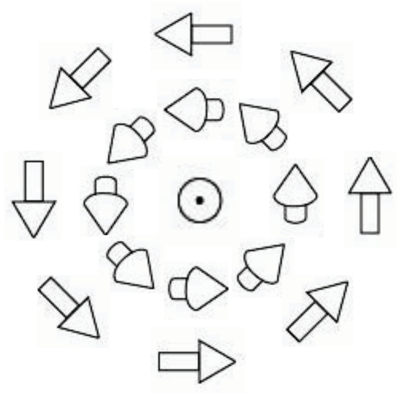
Generality of results



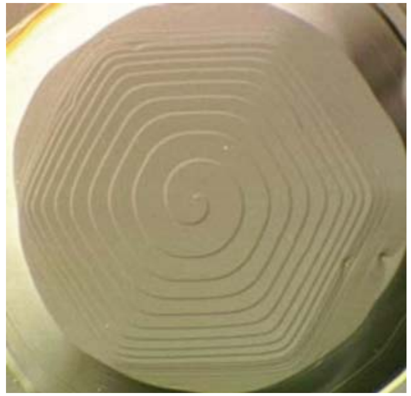
superconducting (sc) vortex



superfluid (sf) vortex



magnetic spin vortex



crystal dislocation

- In all 3 systems, the **vortex** is formed from a **2π twist in a U(1) symmetry** (the pairing phase in sc or sf and the easy plane angle in the magnetic system)
- Despite microscopic differences, the resulting effective forces in the vortex equation of motion are the **same**
- However, in sc's there are additional pinning forces; and sf vortices are hard to image. In spin systems: no pinning, easy to image!

A dislocation is likewise a 1D line singularity with the same effective low energy effective description.

Vortex as an effective particle

For the last 40 years there has been a very strenuous debate going on about the form of the equation of motion for a quantum vortex, focusing in particular on

(i) what are the dissipative forces acting on it?
(ii) what is its effective mass?

Quite incredibly, the fundamental question of quantum vortex dynamics is still highly controversial.

The discussion is typically framed in terms of the forces acting on a vortex; the following terms are discussed:

Magnus force:	$\mathbf{F}_M^\perp = \rho_s \boldsymbol{\kappa} \times (\mathbf{v}_v - \mathbf{v}_s)$	**Arises from Berry phase**
Iordanskii force:	$\mathbf{F}_I^\perp = \rho_N \boldsymbol{\kappa} \times (\mathbf{v}_v - \mathbf{v}_N)$	**Transverse force from quasiparticles scattering off vortex**
Drag force:	$\mathbf{F}^\parallel \propto (\mathbf{v}_v - \mathbf{v}_N)$	**Longitudinal force from quasiparticles scattering off vortex**

Most recently: vortex mass is frequency dependent? (Thouless, condmat 2007)

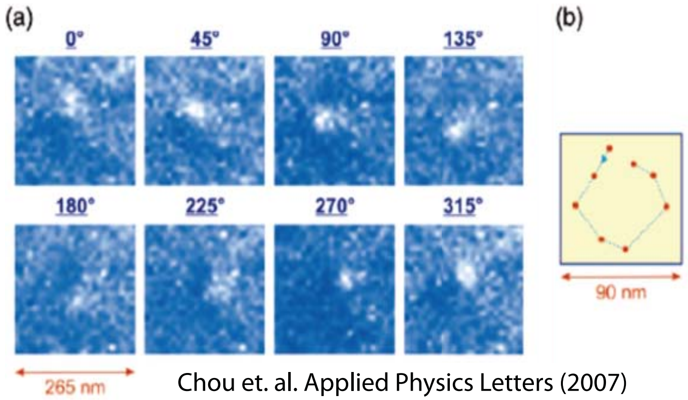
Vortex as a particle

Describe the vortex completely by its central coordinate, $\boldsymbol{X}(t)$

$$M\,\ddot{\boldsymbol{X}}(t) = \rho_S \hat{\kappa} \times (\dot{\boldsymbol{X}}(t) - \boldsymbol{U}_S) + D\,(\boldsymbol{U}_N - \dot{\boldsymbol{X}}(t)) + D'\,\hat{\kappa} \times (\boldsymbol{U}_N - \dot{\boldsymbol{X}}(t))$$

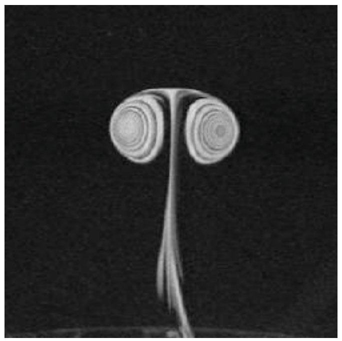
Mass estimates range from 0 to ∞	bare Magnus force	longitudinal damping force	transverse damping force

Experimental observation of vortex motion and pair-wise motion:



Chou et. al. Applied Physics Letters (2007)

Kerr imaging of magnetic vortex motion



pq = -1

pq = +1

pair-wise motion

The vortex + phonon system

Consider a simple superfluid system described by the action

$$\mathcal{S} = -\int dt d^2 r \left[\frac{\rho}{m_0} (\hbar \dot{\Phi} + \frac{|\hbar \nabla \Phi|^2}{2m_0}) + \frac{1}{2\chi\rho^2} \eta^2 \right]$$

This is a T = 0 K form: $\rho = \rho_s$ is the superfluid density (no normal fraction).

χ is the fluid compressibility.

To 0th order, we let $\chi \longrightarrow \infty$: the perturbing effect of a finite compressibility adds a log-divergent energy shift and additional damping forces (that can be shown numerically to be much smaller than the main result presenting in this talk).

phonon spectrum is linear with speed of sound, $c_0 = (\chi \rho_s)^{1/2}$

vortex profile: $\Phi_0 = \tilde{q}\theta(\mathbf{r} - \mathbf{R}(t))$ $\tilde{q} = \pm 1$ is the vortex topological charge

Alternative methods

- Integrate directly:

$$\Phi(\boldsymbol{r} - \boldsymbol{x}(t)) \longrightarrow \mathscr{L}[\Phi] \longrightarrow \delta_X \mathscr{L} = 0$$

For instance, to derive the bare Magnus force, inter-vortex forces, and forces due to boundary constraints.

- Include vortex profile perturbations (eg. due to motion): yields a log-divergent vortex effective mass and damping forces (no memory effects)

- Include phonons by averaging over their realizations directly:

$$\Phi(\boldsymbol{r} - \boldsymbol{x}(t)) + \varphi(\boldsymbol{r}) \longrightarrow \delta_X \langle \mathscr{L} \rangle_\varphi = 0$$

Still cannot predict memory effects

- <u>OR</u> by tracing out phonons from entangled vortex + phonon system:

$$\rho[\Phi(\boldsymbol{r} - \boldsymbol{x}(t)) + \varphi(\boldsymbol{r}),\ \Phi(\boldsymbol{r}' - \boldsymbol{y}(s)) + \varphi(\boldsymbol{r}')] \longrightarrow \rho_V(\boldsymbol{x}(t), \boldsymbol{y}(s))$$

Quantum Brownian motion

$$m\ddot{x} + \eta\dot{x} + V'(x) = F(t)$$

damping coeff — fluctuating force

$$\begin{aligned} \langle F(t) \rangle &= 0 \\ \langle F(t)F(t') \rangle &= 2\eta kT\delta(t-t') \longrightarrow \frac{1}{2\pi}\int e^{-i\omega(t-t')}\eta\hbar\omega \coth\left(\frac{\hbar\omega}{2kT}\right) d\omega \end{aligned}$$

classical Ohmic dissipation — **quantum Ohmic dissipation**

Specify quantum system by the density matrix ρ*(x,y)* as a path integral.
Average over the fluctuating force (assuming a Gaussian distribution):

$$\mathcal{J} = \int \mathcal{D}[x,y] \exp \frac{i}{\hbar}(S[x]-S[y]) \exp -\frac{1}{\hbar^2}\int_0^t \int_0^\tau d\tau ds (x(\tau)-y(\tau)) < F(\tau)F(s) > (x(s)-y(s))$$

Feynman & Vernon, Ann. Phys. 24, 118 (1963);
Caldeira & Leggett, Physica A 121, 587 (1983)

more Quantum Brownian motion

Consider terms in the effective action coupling forward and backward paths in the path integral expression for $\rho(x,y)$:

$$\mathcal{S}_{eff}[x,y] = \mathcal{S}_0[x] - \mathcal{S}_0[y] - \frac{\eta}{2} x\dot{y} + \frac{\eta}{2} y\dot{x}$$

Then, defining new variables:

$$X(\tau) = x(\tau) + y(\tau) \text{ and } \xi(\tau) = x(\tau) - y(\tau)$$

Introduces damping forces, opposing $\dot{X}$ and along $\dot{\xi}$

→ **normal damping for classical motion along *X***
→ **spread in particle "width" <*(x-x_0)*^{2}>, x_0 ~ *X***

Such damping/fluctuating force correlator result from coupling particle *x* with an Ohmic bath of SHO's with ***linear*** coupling:

$$\sum_i c_i x q_i$$

Vortex velocity expansion

A vortex is a stable solution of the Bose system; phonons arise from the very same degrees of freedom as the vortex: hence, there can be no first order vortex-phonon coupling!

Recall that we can always formulate the dynamics for the reduced density matrix as

Density matrix propagator $$\tilde{\rho}(x; y; t = T) = \int dx' \int dy' J(x, y, T; x', y', 0)\rho_x(x', y', 0)$$

$$J(x, y, T; x', y', 0) = \int_x^{x'} \mathcal{D}[x(t)] \int_y^{y'} \mathcal{D}[y(t)] \exp \frac{i}{\hbar}(\mathcal{S}_x[x(t)] - \mathcal{S}_x[y(t)])\mathcal{F}[x(t), y(t)]$$

However we are NOT now going to do the usual Caldeira-Leggett trick of assuming a coupling between vortex and phonons which is linear in the phonon variables.
This is not even true for a soliton coupled to its environment.
What we need is another expansion parameter, and there is one – if the vortex moves slowly we can expand the coupling in powers of the VORTEX VELOCITY.

The vortex-phonon coupling is (in momentum space)

$$f_{\mathbf{q}} = -i \frac{\kappa}{\sqrt{\chi \rho_s}} \frac{e^{i\mathbf{q}\cdot\mathbf{R}}}{q} (\hat{q} \times \hat{z}) \cdot \frac{\dot{\mathbf{R}}}{c_o}$$

Velocity expansion (results)

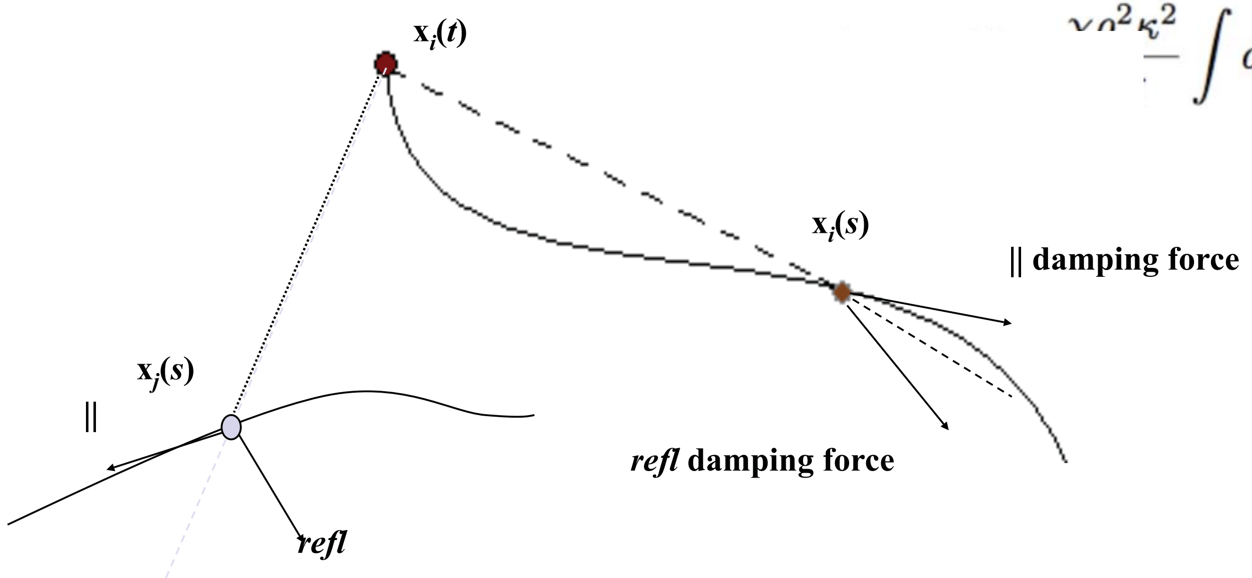
From the real and imaginary parts of the influence functional phase:

$$\hbar \ln \mathcal{F}[\mathbf{x},\mathbf{y}] = -\sum_k \int_0^T d\tau \int_0^\tau ds \Psi_f[\mathbf{x},\mathbf{y}]$$

Rewrite in terms of the 'center-of-mass' coordinate **R** = (**x** + **y**)/2 and the 'fluctuation' coordinate **r** = **x** - **y**.
(An inertial term in this basis appears as $M\dot{\mathbf{r}}\cdot\dot{\mathbf{R}}$ in the density matrix propagator).

$$\Im\Psi_f[\mathbf{x},\mathbf{y}] = \int\int dtds\dot{\mathbf{r}}(t)\left(-\gamma_0(t-s,Q(t,s))\dot{\mathbf{R}}(s) + \gamma_2(t-s,Q(t,s))\dot{\mathbf{R}}_{refl}(s)\right)$$

$$\frac{\gamma\rho^2\kappa^2}{\quad}\int d\omega\omega cos\omega(t-s)J_n(\frac{\omega Q}{c_0})$$



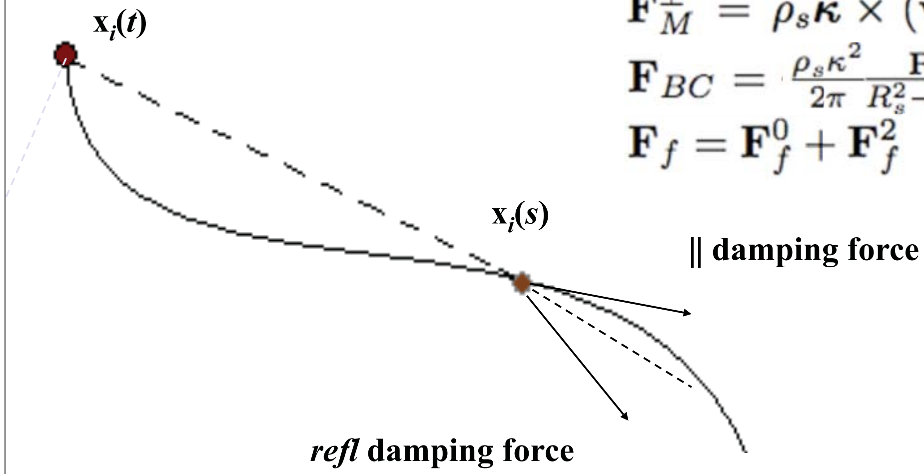
$$\boxed{Q = R(t) - R(s)}$$

Vortex eom

The vortex equation of motion:

$$\boldsymbol{F}_M + \boldsymbol{F}_f + \mathbf{F}_{BC} + \boldsymbol{\mathcal{F}} + \boldsymbol{F}_{ext} = 0$$

where $\boldsymbol{F}_M$ is the bare Magnus force, $\boldsymbol{F}_f$ is the new path dependent force, $\boldsymbol{F}_{BC}$ is any force due to boundary constraints, $\boldsymbol{\mathcal{F}}$ is the random corresponding fluctuating force, and $\boldsymbol{F}_{ext}$ is any externally applied force (eg. due to pinning or probing...)



$$\mathbf{F}_M^{\perp} = \rho_s \boldsymbol{\kappa} \times (\mathbf{v}_v - \mathbf{v}_s)$$
$$\mathbf{F}_{BC} = \frac{\rho_s \kappa^2}{2\pi} \frac{\mathbf{R}}{R_s^2 - R^2}$$
$$\mathbf{F}_f = \mathbf{F}_f^0 + \mathbf{F}_f^2$$

$$\mathbf{F}_f^0 = -\int_0^t ds \gamma_0(t-s, \mathrm{Q}) \dot{\mathbf{R}}(s), \text{ and}$$
$$\mathbf{F}_f^2 = \int_0^t ds \gamma_2(t-s, \mathrm{Q}) \dot{\mathbf{R}}_{refl}(s)$$

$$\gamma_n(t-s, Q) = \frac{\chi \rho_s^2 \kappa^2}{4\pi} \int d\omega \omega cos\omega(t-s) J_n(\frac{\omega Q}{c_0})$$
$$\boldsymbol{Q} = \boldsymbol{R}(t) - \boldsymbol{R}(s)$$

Forces are memory dependent! Not surprising: the vortex is an extended object. Memory effects are introduced by including the non-instantaneous propagation of phonons.

Results cont'd

$$\hbar\Re(\Psi_f[\boldsymbol{r},\boldsymbol{R}]) = -\int dt \int ds \boldsymbol{r}(t)_i A^f_{ij}(t-s, Q(t,s))\boldsymbol{r}(s)_j$$

$$A^f_{ij}(t-s, Q(t,s)) = \frac{\hbar\chi\rho_s^2\kappa^2}{4}\int_0^\infty d\omega\omega^2 \coth\frac{\hbar\omega\beta}{2}\cos\omega(t-s) J_0(\frac{\omega Q}{c_0})\delta_{ij} + J_2(\frac{\omega Q}{c_0})(2Q_iQ_j - \delta_{ij})$$

$$\boxed{\boldsymbol{Q} = \boldsymbol{R}(t) - \boldsymbol{R}(s)}$$

The real part is interpreted (as usual in Caldeira-Leggett formalism) as the remnant after statistically averaging over a fluctuating force.

$$A_{ij}(t-s) = \lim_{T\to\infty}\frac{1}{T}\int_0^T ds\langle\mathcal{F}_i(t)\mathcal{F}_j(s)\rangle$$

Must check that the fluctuation-dissipation theorem is satisfied.

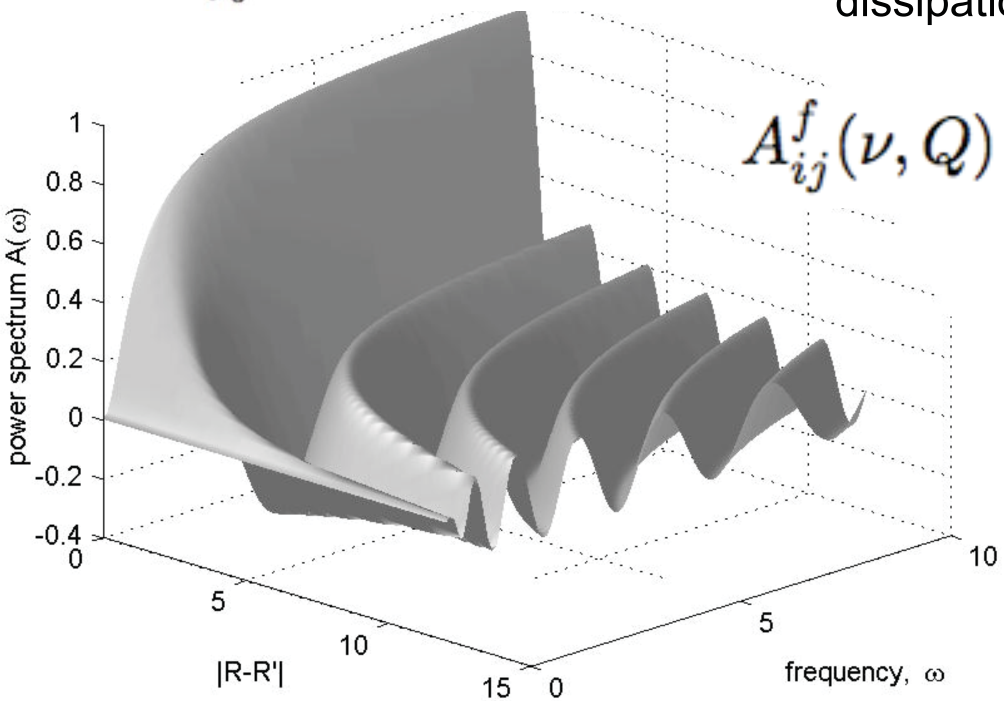
Fluctuation-dissipation theorem

In momentum space (without approximation to the Bessel functions, *however*, without including their implicit time-dependence via Q(t,s))

eg. $\gamma^f(t-s,Q) = 2\int_0^\infty d\nu\gamma^f(\nu,Q)\cos\nu(t-s)$

A quantum version of the generalized (frequency dependent) fluctuation-dissipation theorem is satisfied:

$$A^f_{ij}(\nu,Q) = \hbar\nu\coth\frac{\hbar\nu\beta}{2}\gamma^f_{ij}(\nu,Q)$$



We haven't properly Fourier transformed with the full time dependence, including the Bessel function arguments dependence on *Q(t,s)*.

Results in momentum space

In the limit $J_0 \approx 1$ and $J_2 \approx 1$ (which should correspond to the slow vortex limit $\mathbf{V} \ll c_0$):

$$\boldsymbol{F}_f^0 \rightarrow M_\nu \nu^2 \boldsymbol{R}_\nu$$

where $$M_\nu = \frac{\chi \rho_s^2 \kappa^2}{8\pi} \left(E_1(-\frac{\nu}{\Omega}) e^{-\frac{\nu}{\Omega}} + E_1(\frac{\nu}{\Omega}) e^{\frac{\nu}{\Omega}} \right)$$

$\Omega = 2\pi\, c_0/a_0$ ultraviolet cutoff

$$M_{\nu+i\delta} \rightarrow \frac{1}{2} \left(E_1(\frac{\nu}{\Omega}) e^{\frac{\nu}{\Omega}} + E_1(-\frac{\nu}{\Omega}) e^{-\frac{\nu}{\Omega}} + i \frac{\delta}{\Omega} \left(2\frac{\Omega}{\nu} - E_1(\frac{\nu}{\Omega}) e^{\frac{\nu}{\Omega}} + E_1(-\frac{\nu}{\Omega}) e^{-\frac{\nu}{\Omega}} \right) \right) \quad \delta \ll \Omega$$

$$\rightarrow \log \frac{\Omega}{\nu} - \gamma_{Eul} - \frac{\pi}{2} \frac{\delta}{\Omega} + i \left(\frac{\delta}{\nu} - \frac{\pi}{2} \right) \quad \delta, \nu \ll \Omega$$

A log-divergent mass is the ‘typical’ result for a vortex effective mass. With appropriate length scales in a finite system (size R_S): $\ln \Omega / \nu \approx \ln R_S/a_0$

What does this all mean??

What is the vortex mass? What do memory effects imply? or frequency dependence?

The vortex equation of motion:

$$\boldsymbol{F}_M + \boldsymbol{F}_f + \mathbf{F}_{BC} + \boldsymbol{\mathcal{F}} + \boldsymbol{F}_{ext} = 0$$

where $\boldsymbol{F}_M$ is the bare Magnus force, $\boldsymbol{F}_f$ is the new path dependent force, $\boldsymbol{F}_{BC}$ is any force due to boundary constraints, $\boldsymbol{\mathcal{F}}$ is the random corresponding fluctuating force, and $\boldsymbol{F}_{ext}$ is any externally applied force (eg. due to pinning or probing...)

The simple view of a particle with an effective mass acted upon by a set of forces is too naive: the ‘particle’ properties are governed by the forces present.
The mass and damping depend explicitly on all applied forces!

Example: single vortex in a dot

The equation of motion (dropping the fluctuating force for an ‘averaged’ motion)

$$\begin{pmatrix} m_\nu \nu^2 + \frac{\gamma_{BC}}{R_s^2 - \langle R^2 \rangle} & -i\rho_s \kappa \nu \\ i\rho_s \kappa \nu & m_\nu \nu^2 + \frac{\gamma_{BC}}{R_s^2 - \langle R^2 \rangle} \end{pmatrix} \begin{pmatrix} R_{\nu x} \\ R_{\nu y} \end{pmatrix} = 0$$

where the bare Magnus force is balanced by the path dependent force and the interaction force with the image vortex (due to hard boundary conditions).

To ensure non-trivial solutions, this yields the characteristic equation restricting allowed frequencies

$$m_\nu \nu^2 + \frac{\gamma_{BC}}{R_s^2 - \langle R^2 \rangle} = \pm \rho_s \kappa \nu$$

corresponding to eigensolutions with $R_{\nu x} = \pm i R_{\nu y}$ circular solutions oriented ccw/cw for κ positive and real(ν) > 0 (with opposite orientation for negative frequency solutions.

Only one (complex) frequency solution: motion is a decaying spiral with ensuing motion independent of initial velocity!

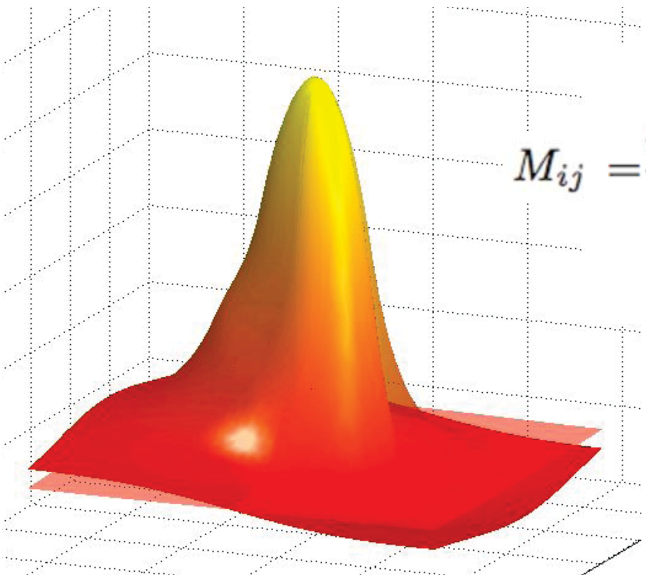
Many-vortex dynamics

As already realized in 1985 by Slonczewski (for a magnetic system), **the dynamics of a collection of vortices *do not* diagonalize into those of a set of interacting particles.**

For instance, in the magnetic system, the inertial energy is

$$E_{mass} = \sum_{i,j=1}^{n} \frac{1}{2} \dot{X}_i M_{ij} \dot{X}_j$$

$$M_{ij} = \frac{\pi q_i q_j r_v^2}{J a^4} \begin{cases} \ln \frac{R_S}{r_{ij}} + \frac{1}{2}\Big((\hat{\mathbf{X}}_i \cdot \hat{\mathbf{e}}_{ij})(\hat{\mathbf{X}}_j \cdot \hat{\mathbf{e}}_{ij}) \\ \quad -(\hat{\mathbf{X}}_i \times \hat{\mathbf{e}}_{ij}) \cdot (\hat{\mathbf{X}}_j \times \hat{\mathbf{e}}_{ij}) \Big), & i \neq j; \\ \ln \frac{R_S}{\sqrt{a^2 + r_v^2}}, & i = j. \end{cases}$$



A vortex under motion deforms (the system cannot respond instantaneously) while deforming in the presence of other vortices. The combinations entails off-diagonal (in vortex index) mass terms.

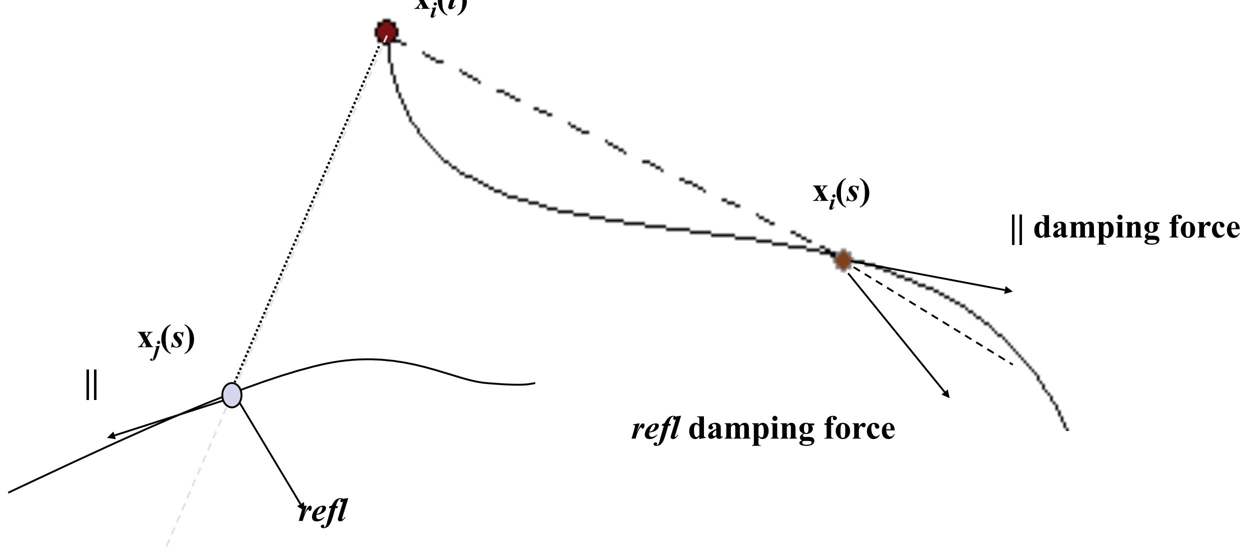
J. C. Slonczewski. Motions of magnetic vortex solitons. In J. Rauluszkiewicz, H. Szymczak, and H. K. Lachowicz, editors, *Physics of Magnetic Materials.* World Scientific, Singapore (1985).

Many-vortex dynamics cont'd

$$\boldsymbol{F}_M^{\alpha} + \boldsymbol{F}_f^{\alpha\beta} + \mathbf{F}_{BC}^{\alpha} + \boldsymbol{\mathcal{F}}^{\alpha} + \boldsymbol{F}_{ext}^{\alpha} = 0$$

$$\mathbf{F}_f^0 = -\int_0^t ds\,\gamma_0(t-s, Q^{\alpha\beta})\,\dot{\mathbf{R}}^{\beta}(s), \text{ and} \qquad \boldsymbol{Q}^{\alpha\beta} = \boldsymbol{R}^{\alpha}(t) - \boldsymbol{R}^{\beta}(s)$$

$$\mathbf{F}_f^2 = \int_0^t ds\,\gamma_2(t-s, Q^{\alpha\beta})\,\dot{\mathbf{R}}^{\beta}_{refl}(s)$$



Conclusions/Yet to come!

All results are in the ‘isolated’ vortex limit:

What happens when the vortex lines (or analogously, the dislocation lines) intersect?

Can we derive effective dynamics for the intersection points?

... toward a dynamic picture of an intersecting vortex network

... then add a superflow along this network: an effective model of a supersolid?!?