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**9th Workshop on Three-Dimensional Modelling of Seismic Waves
Generation, Propagation and their Inversion**

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Seismic waves physics: a (partially) guided tour

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(Seismic) Wave physics

Basic physical concepts

What is a wave?

Discrete and continuous models

Born of wave equation

Dispersion

Basic physical concepts 2

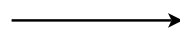PDE: Poisson, diffusion and wave equation
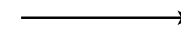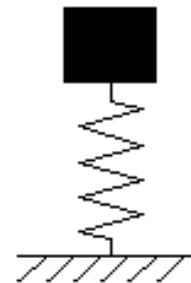
Scattering and diffusion

What is a **wave?**
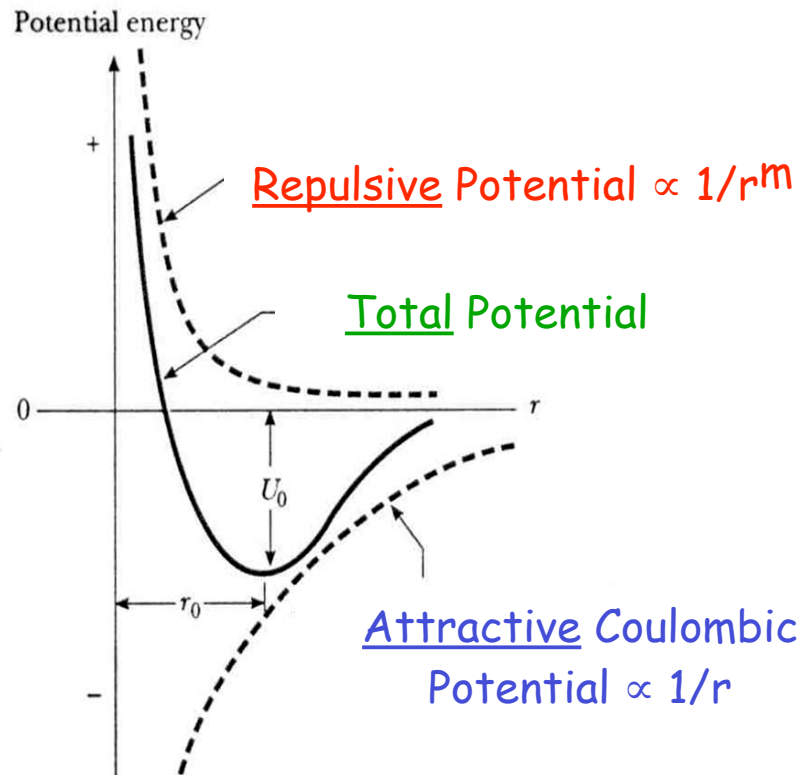
Small perturbations of a → **Linear** → **Harmonic**
stable equilibrium point   **restoring force**   **Oscillation**

Potential energy

Repulsive Potential $\propto 1/r^m$

Total Potential

Attractive Coulombic
Potential $\propto 1/r$

© Victor W. Sparrow, 1996

kinetic energy

potential energy

kinetic + potential
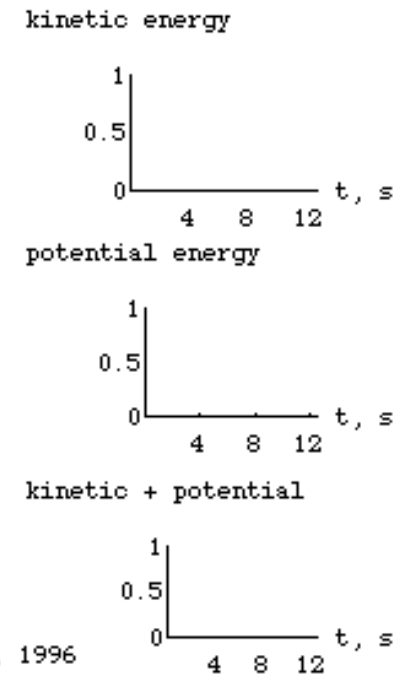
What is a **wave?** - 2

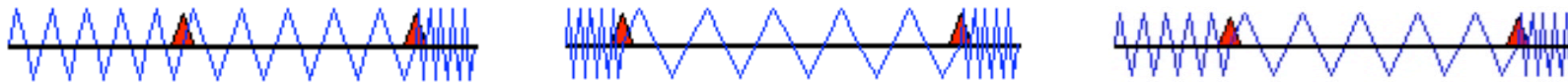**Small** perturbations of a **stable** equilibrium point $\longrightarrow$ **Linear** restoring force $\longrightarrow$ **Harmonic** Oscillation

Coupling of harmonic oscillators $\longrightarrow$ the disturbances can **propagate**, **superpose** and **stand**

Normal modes of the system
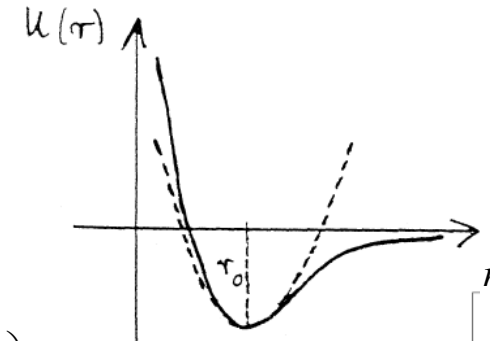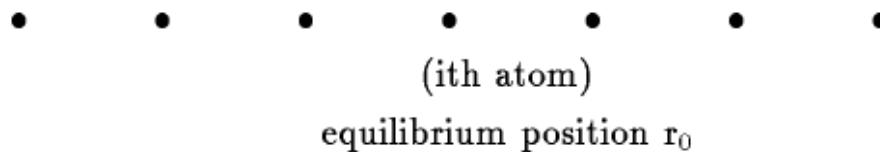
Interatomic potential

Now we consider a monatomic 1-D lattice in the x-direction. The lattice atoms are very close to eqilibrium. Let us examine a single i-th atom and find the r_i potential as a function of displacement from equilibrium, $U(r_i)$.

•   •   •   •   •   •   •

(ith atom)

equilibrium position r_0

We expand this potential into a Taylor's series:

$$U(r_i) = U(r_0) + (r_i - r_0)\left(\frac{dU}{dr_i}\right)_{r_0} + \frac{1}{2}(r_i - r_0)^2\left(\frac{d^2U}{dr_i^2}\right)_{r_0} + \frac{1}{6}(r_i - r_0)^3\left(\frac{d^3U}{dr_i^3}\right)_{r_0} + \ldots$$

The first term of this expansion is just the equilibrium binding energy ($\equiv$ const). The second term is the slope of the potential at its minimum (= 0). The fourth and higher terms become increasingly smaller. We are therefore left with the third term as the only significant change in the potential energy for a small displacement $u = r_i$-r_o. This has the form

$$\Delta U = \tfrac{1}{2}Cu^2 \quad (C = d^2U/dr_i^2 \text{ at } r_i = r_0)$$

representing the *harmonic approximation*, since it is the same as the energy stored in a spring, or the potential energy of a harmonic oscillator. Our simple model of the dynamic crystal structure should therefore be a "ball and spring" model, with the lengths of the springs equivalent to the equilibrium separations of the ion cores.

Monatomic 1D lattice

Let us examine the simplest periodic system within the context of harmonic approximation ($F = dU/du = Cu$) - a one-dimensional crystal lattice, which is a sequence of masses m connected with springs of force constant C and separation a.

The collective motion of these springs will correspond to solutions of a wave equation.
Note: by construction we can see that 3 types of wave motion are possible,
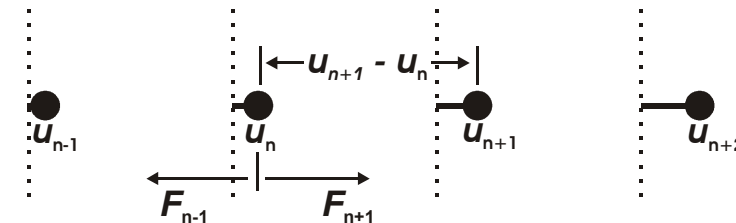2 transverse, 1 longitudinal (or compressional)

Mass M
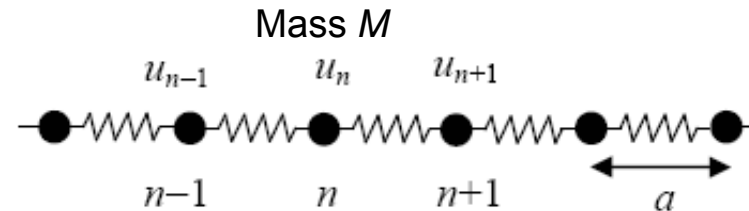
How does the system appear with a longitudinal wave?:

The force exerted on the n-th atom in the lattice is given by

$F_n = F_{n+1,n} - F_{n-1,n} = C[(u_{n+1} - u_n) - (u_n - u_{n-1})]$.

Applying Newton's second law to the motion of the n-th atom we obtain

$$M \frac{d^2 u_n}{dt^2} = F_n = -C(2u_n - u_{n+1} - u_{n-1})$$

Note that we neglected hereby the interaction of the n-th atom with all but its nearest neighbors. A similar equation should be written for each atom in the lattice, resulting in N coupled differential equations, which should be solved simultaneously (N - total number of atoms in the lattice). In addition the boundary conditions applied to end atoms in the lattice should be taken into account.

(Seismic) Wave physics

Basic physical concepts

What is a wave?

Discrete and continuous models

Born of wave equation

Continuum mechanics

Sound waves

☑ Consider a source causing a perturbation in the gas medium rapid enough to cause a pressure variation and not a simple molecular flux.

☑ The regions where compression (or rarefaction), and thus the density variation of the gas, occurs are big compared to the mean free path (average distance that gas molcules travel without collisions).

☑ The perturbation fronts are planes and the displacement induced in the gas, X, depends only on x & t (and not on y, z).

The conventional unit for pressure is bar=10^5N/m^2 and the pressure at the equilibrium is: 1atm=1.0133bar

The pressure perturbations associated to the sound wave passage are typically of the order of 10^{-7}bar, thus very small if compared to the value of pressure at the equlibrium.
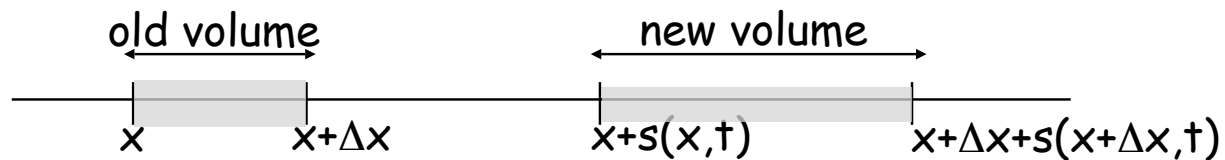
One can thus assume that:

$P=P_0+\Delta P$ $\rho=\rho_0+\Delta\rho$

where ΔP and $\Delta\rho$ are the values of the (small) perturbations of the pressure and density from the equlibrium.

Sound wave equation - 1

The gas moves and causes density variations

Let us consider the displacement field, s(x,t) induced by sound

old volume   new volume

$$x \qquad x+\Delta x \qquad x+s(x,t) \qquad x+\Delta x+s(x+\Delta x,t)$$

and considering a unitary area perpendicular to x, direction of propagation, one has that the quantity of gas enclosed in the old and new volume is the same

$$\rho_0 \Delta x = \rho \left[x + \Delta x + s(x + \Delta x) - x - s(x) \right]$$

where, since Δx is small, $s(x + \Delta x) \approx s(x) + \dfrac{\partial s}{\partial x} \Delta x$

$$\rho_0 \Delta x = (\rho_0 + \Delta \rho) \left[\Delta x + \frac{\partial s}{\partial x} \Delta x \right] = \rho_0 \Delta x + \rho_0 \frac{\partial s}{\partial x} \Delta x + \Delta \rho \Delta x + ...$$

thus, neglecting the second-order term, one has:

$$\Delta\rho = -\rho_0 \frac{\partial s}{\partial x}$$

relation between the variation of displacement along x with the density variation. The minus sign is due to the fact that, if the variation is positive the volume increases and the density decreases.

If the displacement field is constant the gas is simply translated without perturbation.

Density variations cause pressure variations

The pressure in the medium is related to density with a relationship of the kind **P=f(ρ)**,
that at the equilibrium is **P_0=f(ρ_0)**.

$$P = P_0 + \Delta P = f(\rho) = f(\rho_0 + \Delta\rho) \approx f(\rho_0) + \Delta\rho f'(\rho_0) = P_0 + \Delta\rho\kappa$$

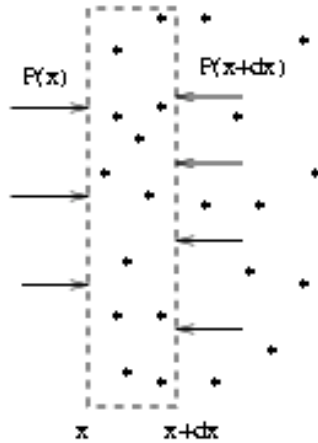and neglecting second-order terms:

$$\Delta P = \kappa\Delta\rho$$

$$\kappa = f'(\rho_0) = \left(\frac{dP}{d\rho}\right)_0$$

Pressure variations generate gas motion

The gas in the volume is accelerated by the different pressure exerted on the two sides...

$$P(x,t) - P(x + \Delta x, t) \approx -\frac{\partial P}{\partial x} \Delta x = -\frac{\partial (P_0 + \Delta P)}{\partial x} \Delta x = -\frac{\partial \Delta P}{\partial x} \Delta x$$

$$= \rho_0 \Delta x \frac{\partial^2 s}{\partial t^2} \quad \text{for Newton's 2nd law}$$

thus:

$$\rho_0 \frac{\partial^2 s}{\partial t^2} = -\frac{\partial \Delta P}{\partial x}$$

Using 1, 2 and 3 we have

$$\rho_0 \frac{\partial^2 s}{\partial t^2} = -\frac{\partial \Delta P}{\partial x} = -\frac{\partial \left(\kappa \Delta \rho \right)}{\partial x} = -\frac{\partial \left[\kappa \left(-\rho_0 \frac{\partial s}{\partial x} \right) \right]}{\partial x}$$

thus:

$$\frac{1}{\kappa} \frac{\partial^2 s}{\partial t^2} = \frac{\partial^2 s}{\partial x^2}$$

i.e. the typical wave equation, describing a perturbation traveling with velocity

$$v = \sqrt{\kappa}$$

From the sound wave equation

$$v = \sqrt{\kappa} = \sqrt{\left(\frac{dP}{d\rho}\right)_0}$$

Newton computed the derivative of the pressure assuming that the heat is moving from one to another region in a such rapid way that the temperature cannot vary - isotherm PV=constant i.e. P/ρ=constant, thus

$$v = \sqrt{\left(\frac{dP}{d\rho}\right)_0} = \sqrt{\left(\text{cons}\tan t\right)_0} = \sqrt{\left(\frac{P}{\rho}\right)_0}$$

called **isothermal sound velocity**

Laplace correctly assumed that the heat flux between a compressed gas region to a rarefied one was negligible, and, thus, that the process of the wave passage was adiabatic PV^γ=constant, P/ρ^γ=constant, with γ ratio of the specific heats: C_p/C_v

$$v = \sqrt{\left(\frac{dP}{d\rho}\right)_0} = \sqrt{\left(\frac{\gamma}{\rho}\cos\tan t\rho^\gamma\right)_0} = \sqrt{\gamma\left(\frac{P}{\rho}\right)_0}$$

called **adiabatic sound velocity**

Sound wave velocity in the air

Using the ideal gas law
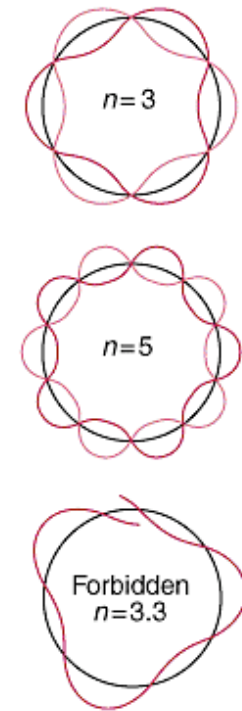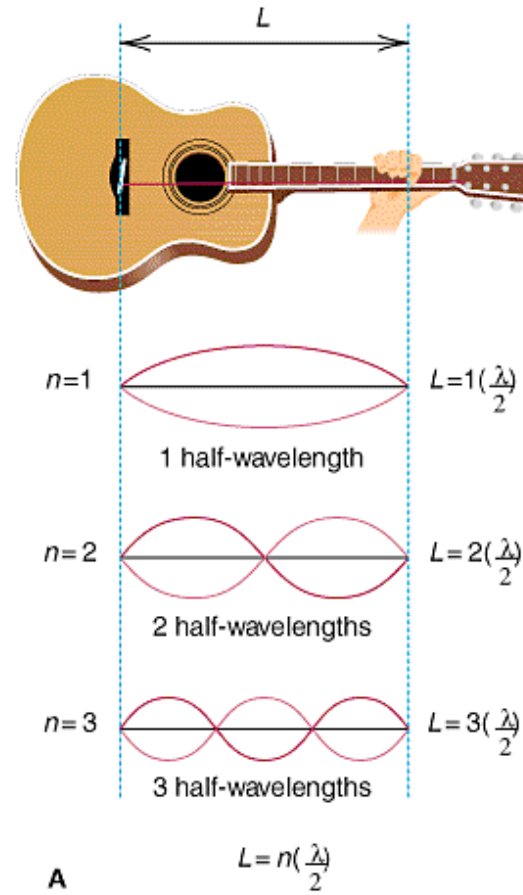
PV=nRT=NkT

one can write the velocity on many ways:
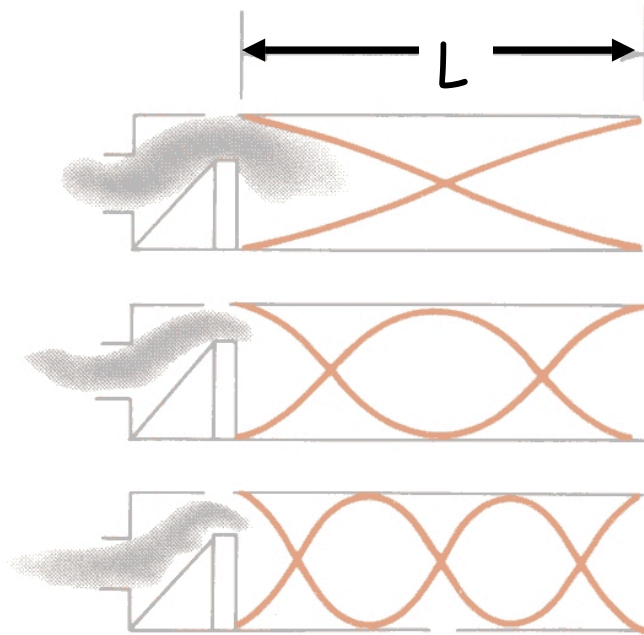
$$v = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma PV}{\rho V}} = \sqrt{\frac{\gamma nRT}{m}} = \sqrt{\frac{\gamma NkT}{Nm_{mol}}} = \sqrt{\frac{\gamma KT}{m_{mol}}} = \sqrt{\frac{\gamma RT}{weight_{mol}}}$$

showing that it depends on **temperature only**. If the "dry" air is considered (biatomic gas γ=7/5) one has:

v=331.4+0.6T$_c$ m/s   (temperature measured in Celsius)

Sound speed

Sound velocity depends on the compressibility of the medium.

If the medium has a bulk modulus B and density at the equilibrium is ρ, the sound speed is: $v = (B/\rho)^{1/2}$

that can be compared with the velocity of transversal waves on a string:

$$v = (F/\mu)^{1/2}$$

Thus, velocity depends on the elastic of the medium (B or F) and on inertial (ρ or μ) properties

What is a **wave?** - 3

Small perturbations of a **stable** equilibrium point $\longrightarrow$ **Linear restoring force** $\longrightarrow$ **Harmonic Oscillation**

Coupling of harmonic oscillators $\longrightarrow$ the disturbances can **propagate**, superpose and stand

WAVE: organized **propagating imbalance**, satisfying differential equations of motion

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

General form of LWE

(Seismic) wave propagation

Basic physical concepts

What is a wave?

Discrete and continuous models

Born of wave equation

Dispersion

discreteness

stiffness

geometry

boundaries

Dispersion relation

☑ In classical mechanics, the Hamilton's principle the perturbation scheme applied to an averaged Lagrangian for an harmonic wave field gives a characteristic equation: $\Delta(\omega, k_i) = 0$
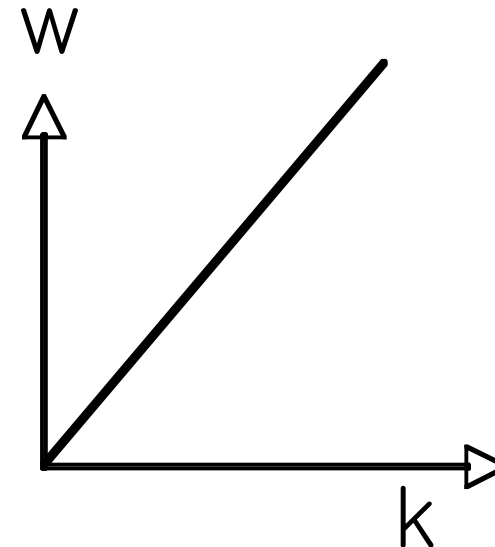
Transverse wave in a string

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\mu}{F} \frac{\partial^2}{\partial t^2} \right)\phi = 0 \Rightarrow \omega = \pm kc$$

Acoustic wave

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\rho}{B} \frac{\partial^2}{\partial t^2} \right)\phi = 0 \Rightarrow \omega = \pm kc$$

Longitudinal wave in a rod

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\rho}{E} \frac{\partial^2}{\partial t^2} \right)\phi = 0 \Rightarrow \omega = \pm kc$$

Dispersion relation

☑ In physics, the dispersion relation is the relation between the energy of a system and its corresponding momentum. For example, for massive particles in free space, the dispersion relation can easily be calculated from the definition of kinetic energy:

$$E = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

☑ For electromagnetic waves, the energy is proportional to the frequency of the wave and the momentum to the wavenumber. In this case, Maxwell's equations tell us that the dispersion relation for vacuum is linear: $\omega = ck$.

☑ The name "**dispersion relation**" originally comes from optics. It is possible to make the effective speed of light dependent on wavelength by making light pass through a material which has a non-constant index of refraction, or by using light in a non-uniform medium such as a waveguide. In this case, the waveform will spread over time, such that a narrow pulse will become an extended pulse, i.e. be dispersed.

Dispersion...

✓ In optics, dispersion is a phenomenon that causes the separation of a wave into spectral components with different wavelengths, due to a dependence of the wave's speed on its wavelength. It is most often described in light waves, but it may happen to any kind of wave that interacts with a medium or can be confined to a waveguide, such as sound waves. There are generally two sources of dispersion: **material dispersion**, which comes from a frequency-dependent response of a material to waves; and **waveguide dispersion**, which occurs when the speed of a wave in a waveguide depends on its frequency.

✓ In optics, the phase velocity of a wave v in a given uniform medium is given by: **v**=**c/n**, where c is the speed of light in a vacuum and n is the refractive index of the medium. In general, the refractive index is some function of the frequency of the light, thus n = n(f), or alternately, with respect to the wave's wavelength n = n(λ). For visible light, most transparent materials (e.g. glasses) have a refractive index n decreases with increasing wavelength λ (**dn/dλ<0**, i.e. **dv/dλ>0**). In this case, the medium is said to have **normal dispersion** and if the index increases with increasing wavelength the medium has **anomalous dispersion**.

Group velocity

✔ Another consequence of dispersion manifests itself as a temporal effect. The phase velocity is the velocity at which the phase of any one frequency component of the wave will propagate. This is not the same as the **group velocity of the wave, which is the rate that changes in amplitude** (known as the envelope of the wave) will propagate. The group velocity v_g is related to the phase velocity by, for a homogeneous medium (here λ is the wavelength in vacuum, not in the medium):

$$v_g = c\left(n - \lambda \frac{dn}{d\lambda}\right)^{-1} = v - \lambda \frac{dv}{d\lambda}$$

and thus in the normal dispersion case
v_g is always < v !

Dispersion...

☑ The group velocity itself is usually a function of the wave's frequency. This results in group velocity dispersion (GVD),that is often quantified as the group delay dispersion parameter (again, this formula is for a uniform medium only): If D is less than zero, the medium is said to have **positive dispersion**. If D is greater than zero, the medium has **negative dispersion**.

$$D = -\frac{\lambda}{c}\left(\frac{d^2 n}{d\lambda^2}\right)$$
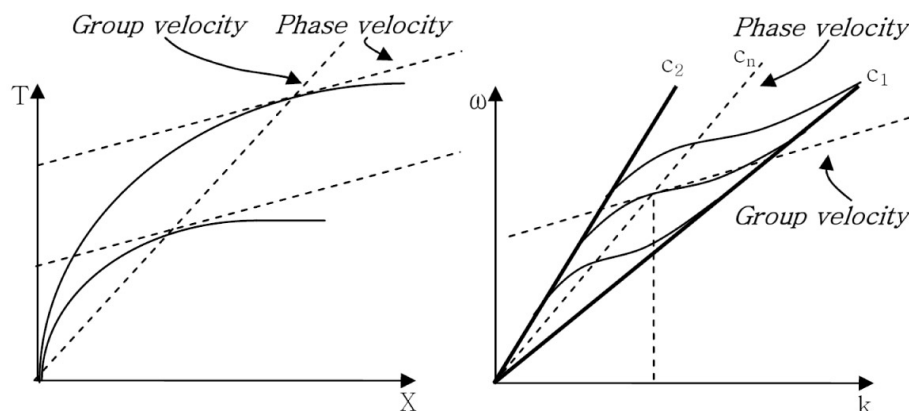
Airy Phase –
 wave that arises if the phase and the change in group velocity are stationary and gives the highest amplitude in terms of group velocity and are prominent on the seismogram.

Dispersion examples

☑ **Discrete systems: lattices**

☑ Boundary waves: plates and rods

Discontinuity interfaces are intrinsic in their propagation since they allow to store energy (not like body waves)!

☑ Stiff systems: rods and thin plates

Monatomic 1D lattice - continued

Now let us attempt a solution of the form: $\quad u_n = Ae^{i(kx_n - \omega t)}$,

where x_n is the equilibrium position of the n-th atom so that $x_n = na$. This equation represents a traveling wave, in which all atoms oscillate with the same frequency ω and the same amplitude A and have a wavevector k. Now substituting the guess solution into the equation and canceling the common quantities (the amplitude and the time-dependent factor) we obtain

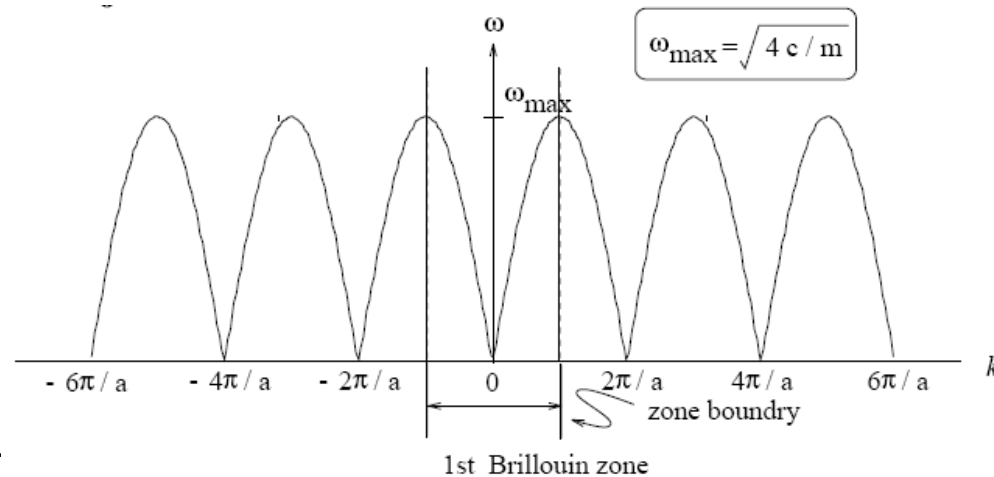$$M(-\omega^2)e^{ikna} = -C[2e^{ikna} - e^{ik(n+1)a} - e^{ik(n-1)a}].$$

This equation can be further simplified by canceling the common factor e^{ikna} , which leads to

$$M\omega^2 = C\left(2 - e^{ika} - e^{-ika}\right) = 2C(1 - \cos ka) = 4C\sin^2\frac{ka}{2}.$$

We find thus the dispersion relation for the frequency:

$$\omega = \sqrt{\frac{4C}{M}}\left|\sin\frac{ka}{2}\right|$$

which is the relationship between the frequency of vibrations and the wavevector k. The dispersion relation has a number of important properties.

$$\omega_{max} = \sqrt{4c/m}$$

ω_{max}

- 6π / a   - 4π / a   - 2π / a   0   2π / a   4π / a   6π / a k

zone boundry

1st Brillouin zone

Monatomic 1D lattice – continued

*Phase and group velocity*. The phase velocity is defined by

$$v_p = \frac{\omega}{k}$$ and the group velocity by $$v_g = \frac{d\omega}{dk}$$

The physical distinction between the two velocities is that v_p is the velocity of propagation of the plane wave, whereas the v_g is the velocity of the propagation of the wave packet. The latter is the velocity for the propagation of energy in the medium. For the particular

dispersion relation $$\omega = \sqrt{\frac{4C}{M}}\left|\sin\frac{ka}{2}\right|$$ the group velocity is given by $$v_g = \sqrt{\frac{Ca^2}{M}}\cos\frac{ka}{2}.$$

Apparently, the group velocity is zero at the edge of the zone where $k = \pm \pi/a$. Here the wave is standing and therefore the transmission velocity for the energy is zero.
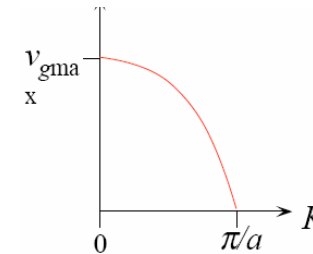
Long wavelength limit. The long wavelength limit implies that $\lambda \gg a$. In this limit $ka \ll 1$.

We can then expand the sine in 'ω' and obtain for the positive frequencies: $$\omega = \sqrt{\frac{C}{M}}ka.$$

We see that the frequency of vibration is proportional to the wavevector. This is equivalent to the statement that velocity is independent of frequency. In this case:

$$v_p = \frac{\omega}{k} = \sqrt{\frac{C}{M}}a.$$ This is the velocity of sound for the one dimensional lattice which is consistent with the expression we obtained earlier for elastic waves.

/

Monatomic 1D lattice – continued

Finite chain – Born – von Karman periodic boundary condition.

Unlike a continuum, there is only a finite number of distinguishable vibrational modes. But how many?

Let us impose on the chain ends the Born – von Karman periodic boundary conditions specified as following: we simply join the two remote ends by one more spring in a ring or device in the figure below forcing atom N to interact with ion 1 via a spring with a spring constant C. If the atoms occupy sites a, $2a$, …, Na

The boundary condition is $u_{N+1} = u_1$ or $u_N = u_0$.
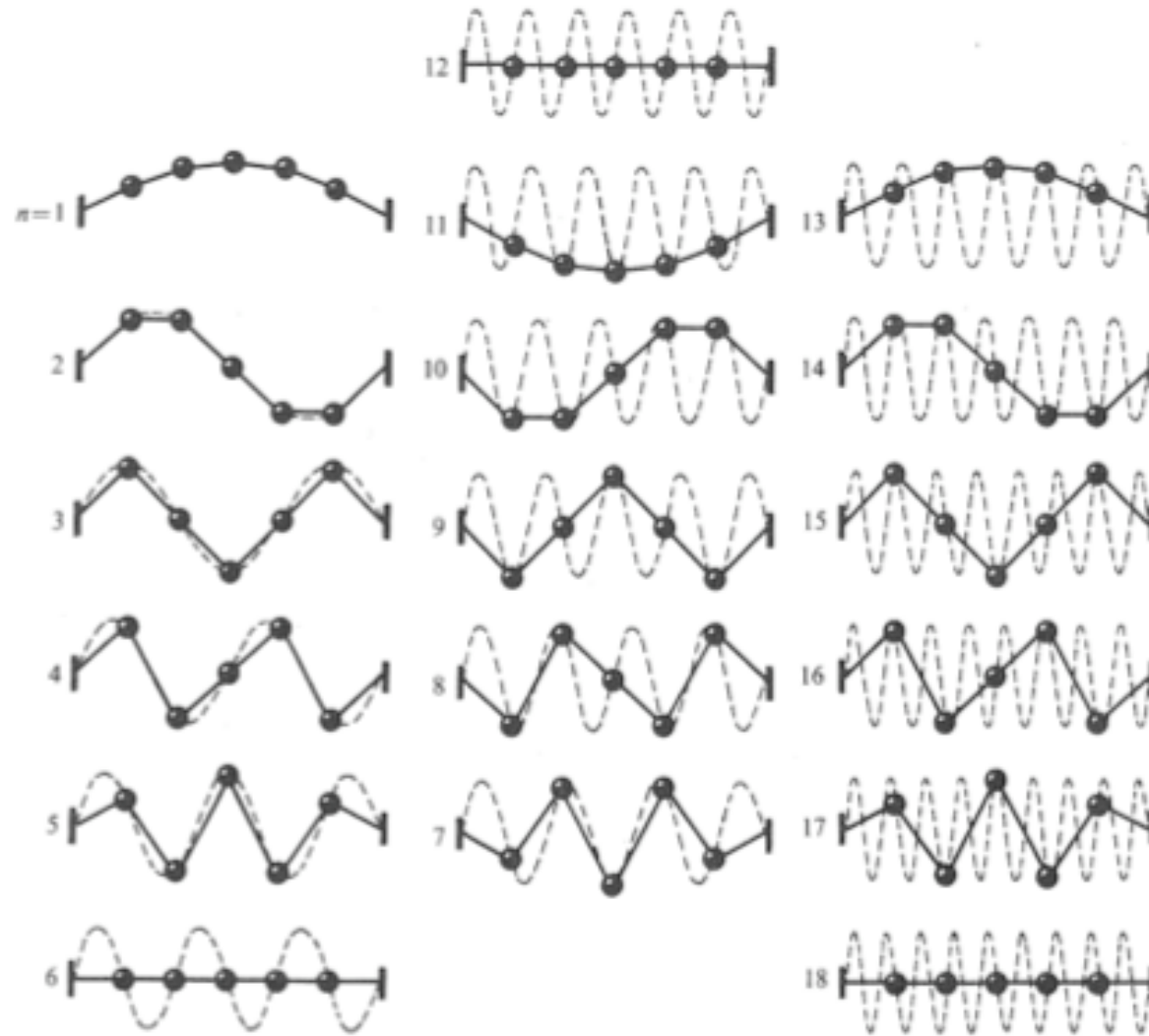

$$\longleftarrow Na \longrightarrow$$
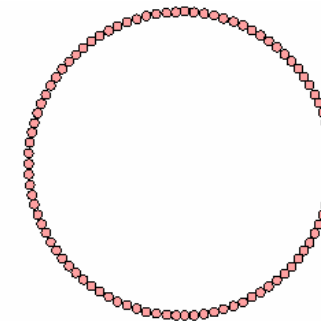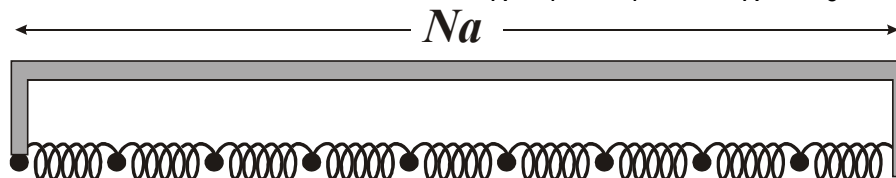
With the displacement solution of the form $u_n = A\exp[i(kna-wt)]$, the periodic boundary condition requires that $\exp(\pm ikNa) = 1$, which in turn requires 'k' to have the form:

$$k = \frac{2\pi}{a}\frac{n}{N} \quad (n - \text{an integer}), \quad \text{and} \quad -\frac{N}{2} \leq n \leq \frac{N}{2}, \quad \text{or}$$

$$k = \pm\frac{2\pi}{Na}, \ \pm\frac{4\pi}{Na}, \ \pm\frac{6\pi}{Na}, \ ..., \pm\frac{\pi}{a} \quad (N \text{ values of } k).$$

$n = 1$

$n = 2$

$n = 3$

$n = N/2$
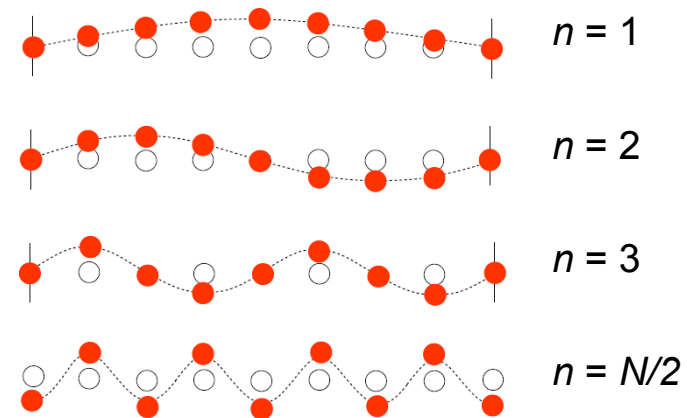
Diatomic 1D lattice

We can treat the motion of this lattice in a similar fashion as for the monatomic lattice. However, in this case, because we have two different kinds of atoms, we should write two equations of motion:

$$M_1 \frac{d^2 u_n}{dt^2} = -C(2u_n - u_{n+1} - u_{n-1})$$

$$M_2 \frac{d^2 u_{n+1}}{dt^2} = -C(2u_{n+1} - u_{n+2} - u_n)$$

In analogy with the monatomic lattice we are looking for the solution in the form of traveling mode for the two atoms:

$$\begin{bmatrix} u_n \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} A_1 e^{ikna} \\ A_2 e^{ik(n+1)a} \end{bmatrix} e^{-i\omega t} \qquad \text{in matrix form.}$$

Substituting this solution into the equations of the previous slide we obtain:

$$\begin{bmatrix} 2C - M_1 \omega^2 & -2C\cos ka \\ -2C\cos ka & 2C - M_2 \omega^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0.$$

This is a system of linear homogeneous equations for the unknowns A_1 and A_2. A nontrivial solution exists only if the determinant of the matrix is zero. This leads to the secular equation

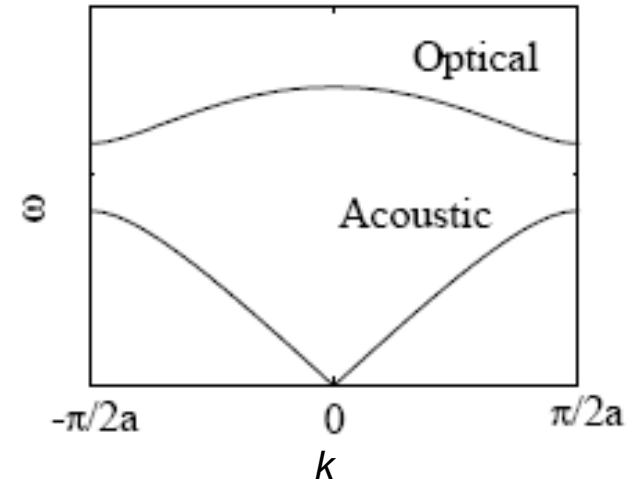$$\left(2C - M_1 \omega^2\right)\left(2C - M_2 \omega^2\right) - 4C\cos^2 ka = 0.$$

Optical

This is a quadratic equation, which can be readily solved:

$$\omega^2 = C\left(\frac{1}{M_1} + \frac{1}{M_2}\right) \pm C\sqrt{\left(\frac{1}{M_1} + \frac{1}{M_2}\right)^2 - \frac{4\sin^2 ka}{M_1 M_2}}$$

Depending on sign in this formula there are two different solutions corresponding to two different dispersion curves, as is shown in the figure:

The lower curve is called the **acoustic** branch, while the upper curve is called the **optical** branch.

The acoustic branch begins at $k = 0$ and $\omega = 0$, and as $k \Rightarrow 0$:

$$\omega_a(0) = \sqrt{\frac{C}{2(M_1 + M_2)}} \cdot ka$$

With increasing k the frequency increases in a linear fashion. This is why this branch is called *acoustic*: it corresponds to elastic waves, or sound. Eventually, this curve saturates at the edge of the Brillouin zone. On the other hand, the optical branch Has a nonzero frequency at zero k,

$$\omega_o = \sqrt{2C\left(\frac{1}{M_1} + \frac{1}{M_2}\right)}$$

and it does not change much with k.

Another feature of the dispersion curves is the existence of a forbidden gap between $\omega_a = (2C/M_1)^{1/2}$ and $\omega_o = (2C/M_2)^{1/2}$ at the zone boundaries ($k = \pm \pi/2a$).
The forbidden region corresponds to frequencies in which lattice waves cannot propagate through the linear chain without attenuation. It is interesting to note that a similar situation also exists in the energy band scheme of a solid to be discussed later.

The distinction between the acoustic and optical branches of lattice vibrations can be seen most clearly by comparing them at $k = 0$ (infinite wavelength). As follows from the equations of motion, for the acoustic branch $\omega = 0$ and $A_1 = A_2$. So, in this limit the two atoms in the cell have the same amplitude and phase. Therefore, the molecule oscillates as a rigid body, as shown in the left figure for the acoustic mode.

Acoustic

optical made

Optical

acustical made
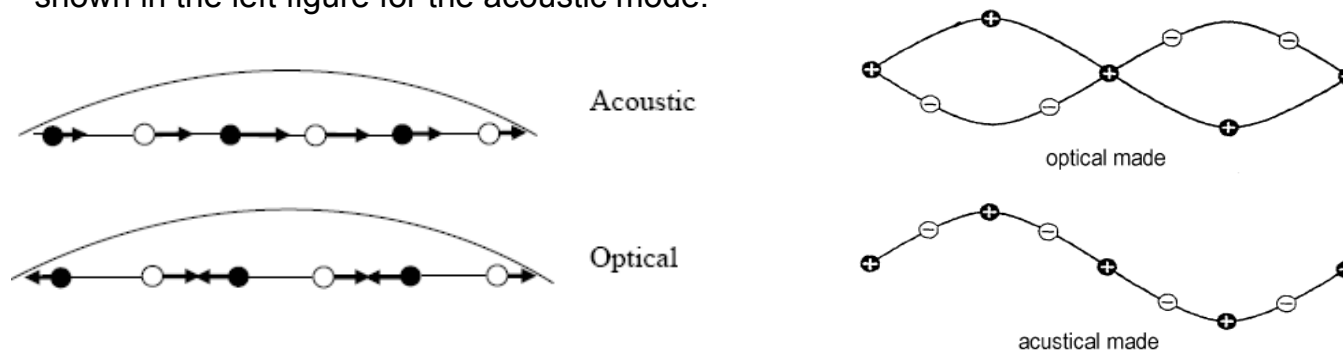
On the other hand, for the optical vibrations, by substituting ω_o we obtain for $k = 0$:
$$M_1 A_1 + M_2 A_2 = 0 \quad (M_1/M_2 = -A_2/A_1).$$
This implies that the optical oscillation takes place in such a way that the center of mass of a molecule remains fixed. The two atoms move in out of phase as shown. The frequency of these vibrations lies in the infrared region (10^{12} to 10^{14} Hz) which is the reason for referring to this branch as *optical*. If the two atoms carry opposite charges, we may excite a standing wave motion with the electric field of a light wave.
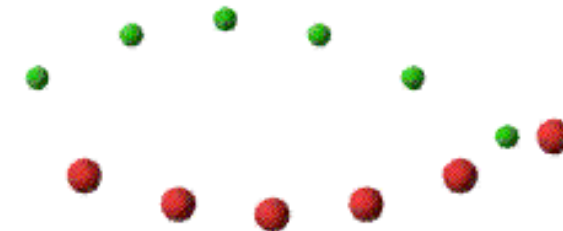
Acoustic and optical modes

Monoatomic chain
acoustic longitudinal mode

Monoatomic chain
acoustic transverse mode

Diatomic chain
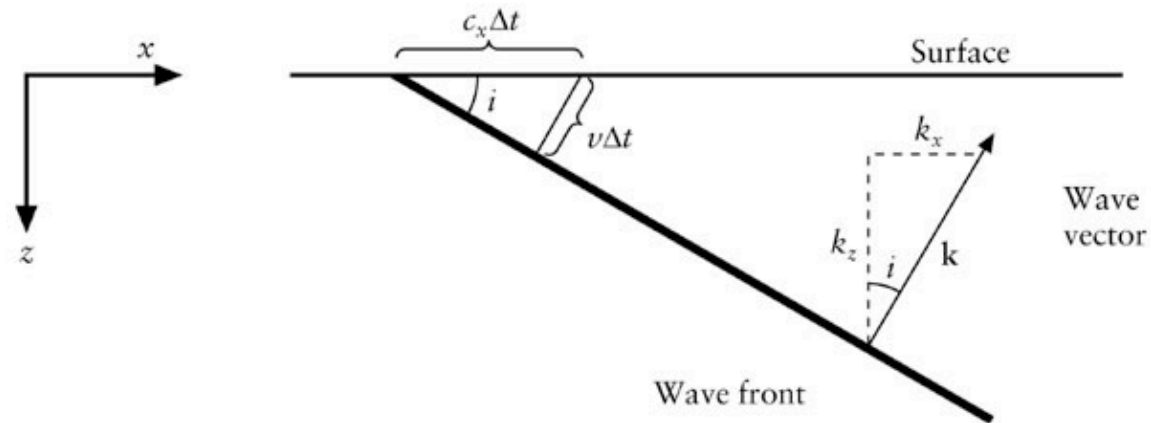**acoustic transverse** mode

Diatomic chain
optical transverse mode

☑ Discrete systems: lattices

☑ **Boundary waves: plates and rods**

Discontinuity interfaces are intrinsic in their propagation since they allow to store energy (not like body waves)!

☑ Stiff systems: rods and thin plates

Apparent horizontal velocity

$$k_x = k \sin(i) = \omega \frac{\sin(i)}{\alpha} = \frac{\omega}{c}$$

$$k_z = k \cos(i) = \sqrt{k^2 - k_x^2} = \omega \sqrt{\left(\frac{1}{\alpha}\right)^2 - \left(\frac{1}{c}\right)^2} = \frac{\omega}{c} \sqrt{\left(\frac{c}{\alpha}\right)^2 - 1} = k_x r_\alpha$$

In current terminology, k_x is k!

In an elastic half-space no SH type surface waves exist. Why?
Because there is total reflection and no interaction between an evanescent P wave and a phase shifted SV wave as in the case of Rayleigh waves. What happens if we have a layer delimited by two free boundaries, i.e. a homogeneous plate?
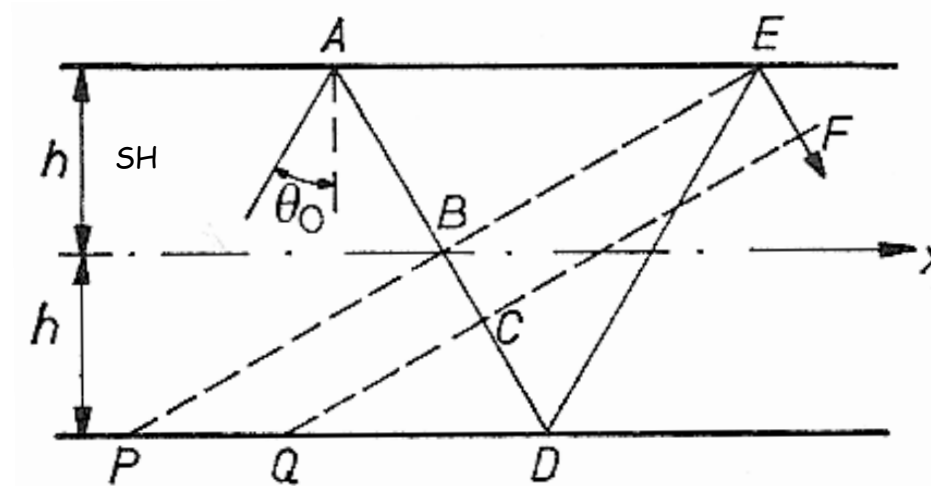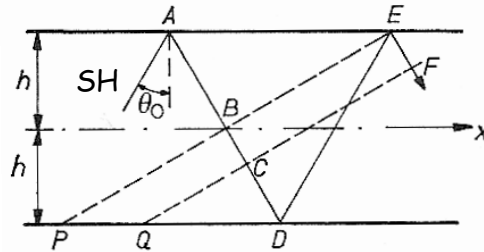

Repeated reflection in the layer allow interference between incident and reflected SH waves: SH reverberations can be totally trapped.

$$u_y = A\exp[i(\omega t + \omega\eta_\beta z - kx)] + B\exp[i(\omega t - \omega\eta_\beta z - kx)]$$

$$k = k_x = \frac{\omega}{c}; \quad \omega\eta_\beta = k_z = \frac{\omega}{c}\sqrt{\frac{c^2}{\beta^2} - 1} = kr_\beta$$

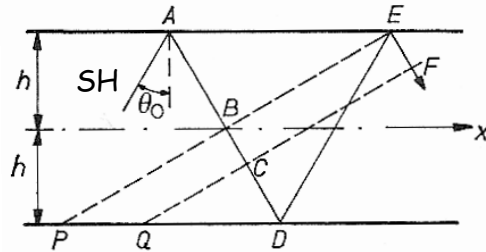$$u_y = A\exp[i(\omega t + kr_\beta z - kx)] + B\exp[i(\omega t - kr_\beta z - kx)]$$

The formal derivation is very similar to the derivation of the Rayleigh waves. The conditions to be fulfilled are: free surface conditions

$$\sigma_{zy}(0) = \mu\left.\frac{\partial u_y}{\partial z}\right|_0 = ikr_\beta\mu\left\{A\exp[i(\omega t - kx)] - B\exp[i(\omega t - kx)]\right\} = 0$$
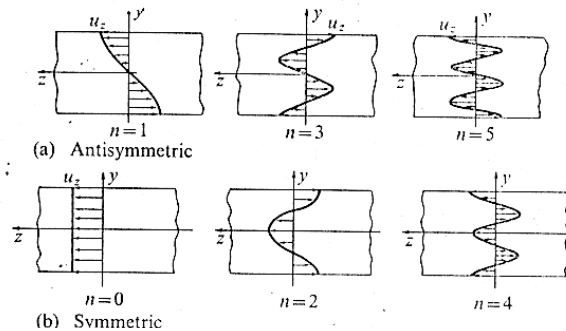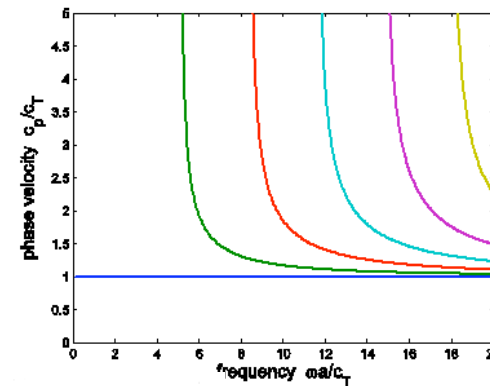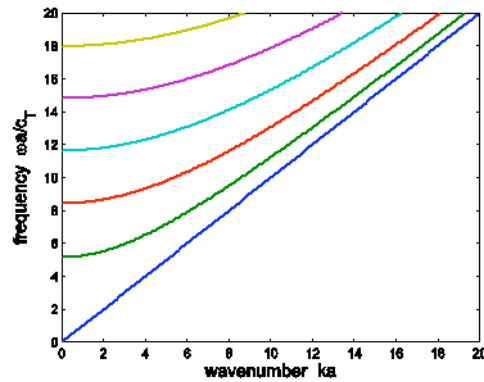
$$\sigma_{zy}(2h) = \mu\left.\frac{\partial u_y}{\partial z}\right|_{2h} = ikr_\beta\mu\left\{A\exp[i(\omega t + kr_\beta 2h - kx)] - B\exp[i(\omega t - kr_\beta 2h - kx)]\right\} = 0$$
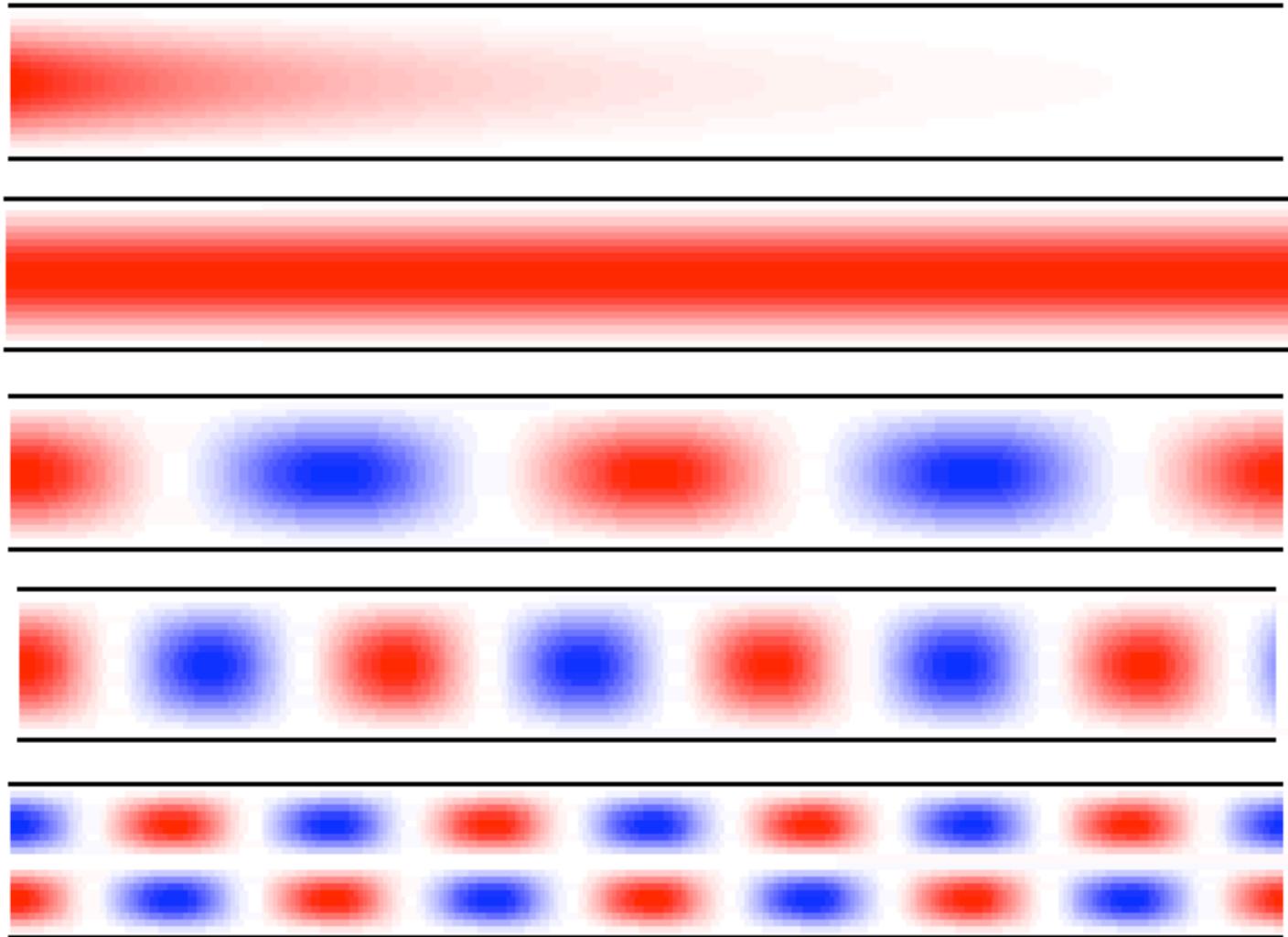
that leads to: $kr_\beta 2h = n\pi$ with n=0,1,2,...

$$\omega^2 = k^2\beta^2 + \left(\frac{n\pi\beta}{2h}\right)^2 \qquad c = \beta\left[1 - \left(\frac{n\pi\beta}{2h\omega}\right)^2\right]^{-\frac{1}{2}}$$

(a) Antisymmetric

(b) Symmetric

EM waveguide animations

Created by Hsiu C. Han, 1996

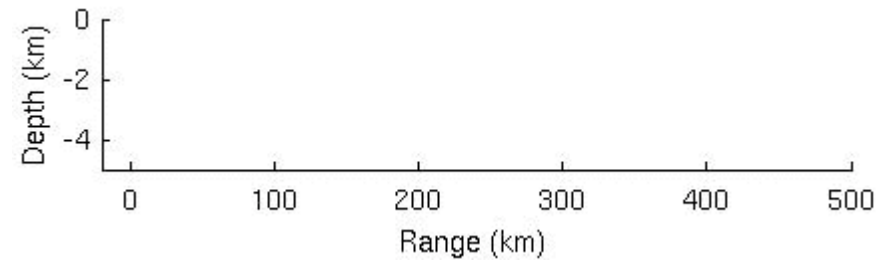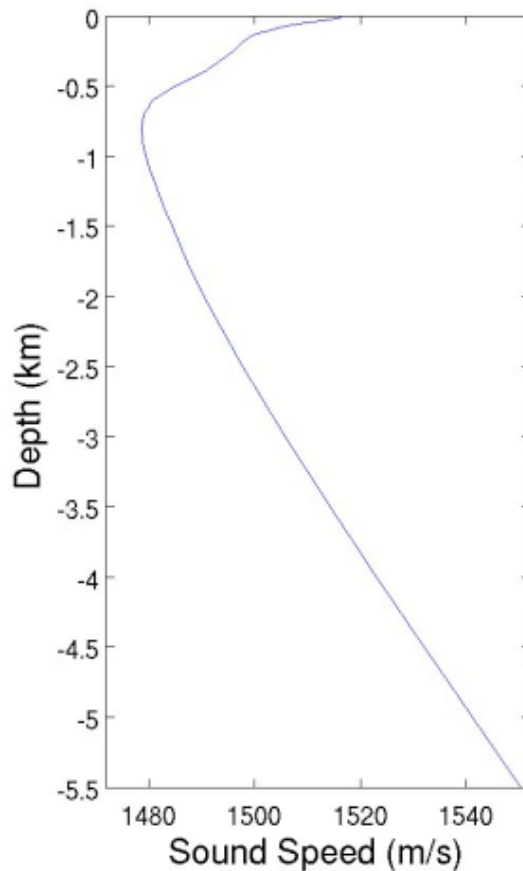http://www.ee.iastate.edu/~hsiu/descriptions/paral.html

Acoustic waveguides...

☑ **SOFAR channel** (Sound Fixing And Ranging channel)

Sound speed as a function of depth at a position
north of Hawaii in the Pacific Ocean derived from
the 2005 World Ocean Atlas. The SOFAR channel
axis is at ca. 750-m depth.

Dispersion examples

☑ Discrete systems: lattices

☑ Boundary waves: plates and rods

Discontinuity interfaces are intrinsic in their propagation since they allow to store energy (not like body waves)!

☑ **Stiff systems: rods and thin plates**

Stiffness...

☑ How "**stiff**" or "flexible" is a material? It depends on whether we pull on it, twist it, bend it, or simply compress it. In the simplest case the material is characterized by two independent "stiffness constants" and that different combinations of these constants determine the response to a pull, twist, bend, or pressure.

Euler Bernoulli equation

$$\left(\frac{\partial^4}{\partial x^4} - \frac{\rho A}{EI} \frac{\partial^2}{\partial t^2} \right) w = 0 \Rightarrow \omega = \pm k^2 \sqrt{\frac{EI}{\rho A}}$$
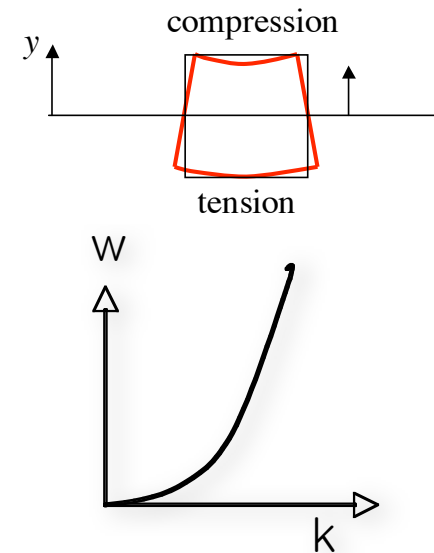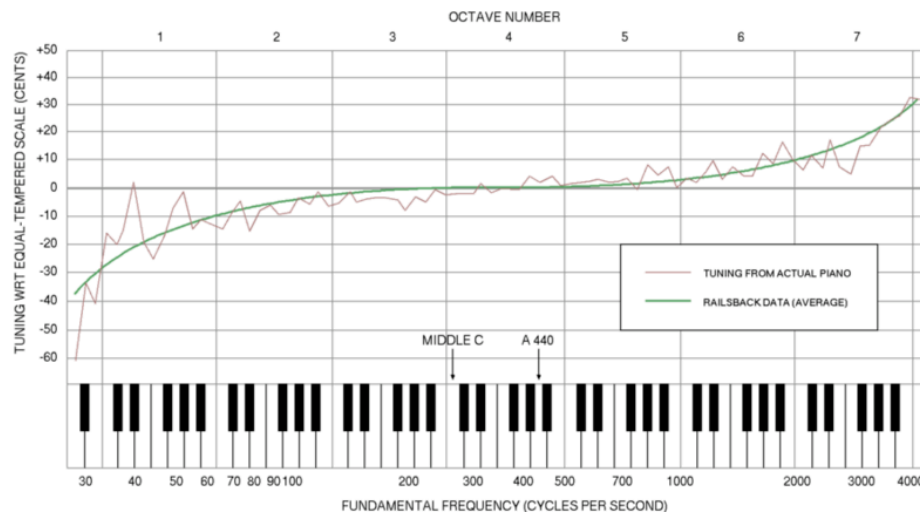
Stiffness...

✓ Stiffness in a vibrating string introduces a restoring force proportional to the bending angle of the string and the usual stiffness term added to the wave equation for the ideal string. Stiff-string models are commonly used in piano synthesis and they have to be included in tuning of piano strings due to inharmonic effects.

$$\left(\frac{\partial^4}{\partial x^4} + \frac{E}{\rho}\frac{\partial^2}{\partial x^2} - \frac{\rho A}{EI}\frac{\partial^2}{\partial t^2} \right) w = 0 \Rightarrow \quad \omega = \pm k\sqrt{\frac{E}{\rho}}\left(1 + k^2\sqrt{\frac{I}{A}} \right)^{1/2}$$

$$\Rightarrow \quad \omega \approx \pm k\sqrt{\frac{E}{\rho}}\left(1 + \frac{1}{2}k^2\sqrt{\frac{I}{A}} \right)$$
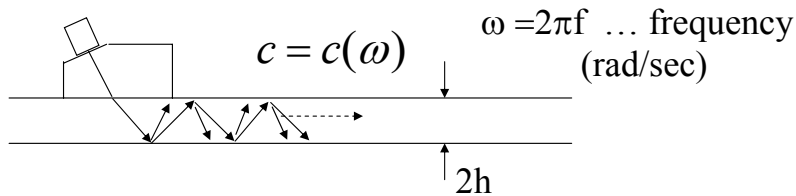
$= (\omega$

Waves in plates

In low frequency plate waves, there are two distinct type of harmonic motion. These are called symmetric or **extensional** waves and antisymmetric or **flexural** waves.

$c = c(\omega)$

$\omega = 2\pi f$ … frequency (rad/sec)

2h

$$\phi = f(y)\exp[ik(x - ct)]$$

$$\psi = g(y)\exp[ik(x - ct)]$$

$\phi = f($
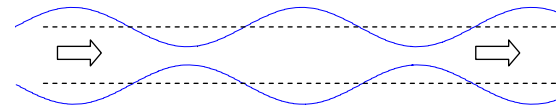
(a)

extensional waves

$$f = A\cosh(\alpha y)$$

$$g = B\sinh(\beta y)$$

2h

y

x

(b)

flexural waves

$$f = A'\sinh(\alpha y)$$

$$g = B'\cosh(\beta y)$$

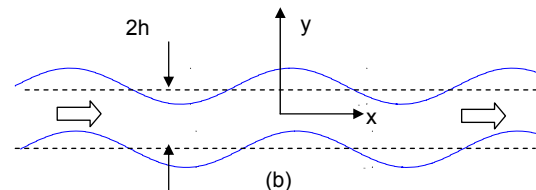satisfying the boundary conditions $\tau_{yy} = \tau_{xy} = 0$

on $y = \pm h$ gives the Rayleigh-Lamb equations:

$$\frac{\tanh(\beta h)}{\tanh(\alpha h)} = \left[\frac{4\omega^2 \alpha \beta}{c^2 \left(\omega^2 / c^2 + \beta^2 \right)^2} \right]^{\pm 1}$$

+ … extensional waves

- … flexural waves

$$\alpha = \left| \frac{\omega}{c} \right| \sqrt{1 - \frac{c^2}{c_p^2}} \quad , \quad \beta = \left| \frac{\omega}{c} \right| \sqrt{1 - \frac{c^2}{c_s^2}}$$

consider the extensional waves

$$\frac{\tanh\left[\ 2\pi fh\sqrt{1/c^2-1/c_s^2}\ \right]}{\tanh\left[\ 2\pi fh\sqrt{1/c^2-1/c_p^2}\ \right]} = \frac{4\sqrt{1-c^2/c_s^2}\ \sqrt{1-c^2/c_p^2}}{\left(2-c^2/c_s^2\right)^2}$$

If we let $\quad kh = \dfrac{2\pi fh}{c} \gg 1 \quad$ (high frequency)

then both tanh functions are $\ \cong 1$

and we find $\quad \left(2-c^2/c_s^2\right)^2 = 4\sqrt{1-c^2/c_s^2}\ \sqrt{1-c^2/c_p^2}$

so we just have Rayleigh waves on both stress-free surfaces:

In contrast for $kh << 1$ (low frequency)

we find
$$\tanh(\alpha h) \cong \alpha h$$
$$\tanh(\beta h) \cong \beta h$$

and the Rayleigh-Lamb equation reduces to

$$\left(2 - c^2 / c_s^2\right)^2 = 4\left(1 - c^2 / c_p^2\right)$$

which can be solved for c to give

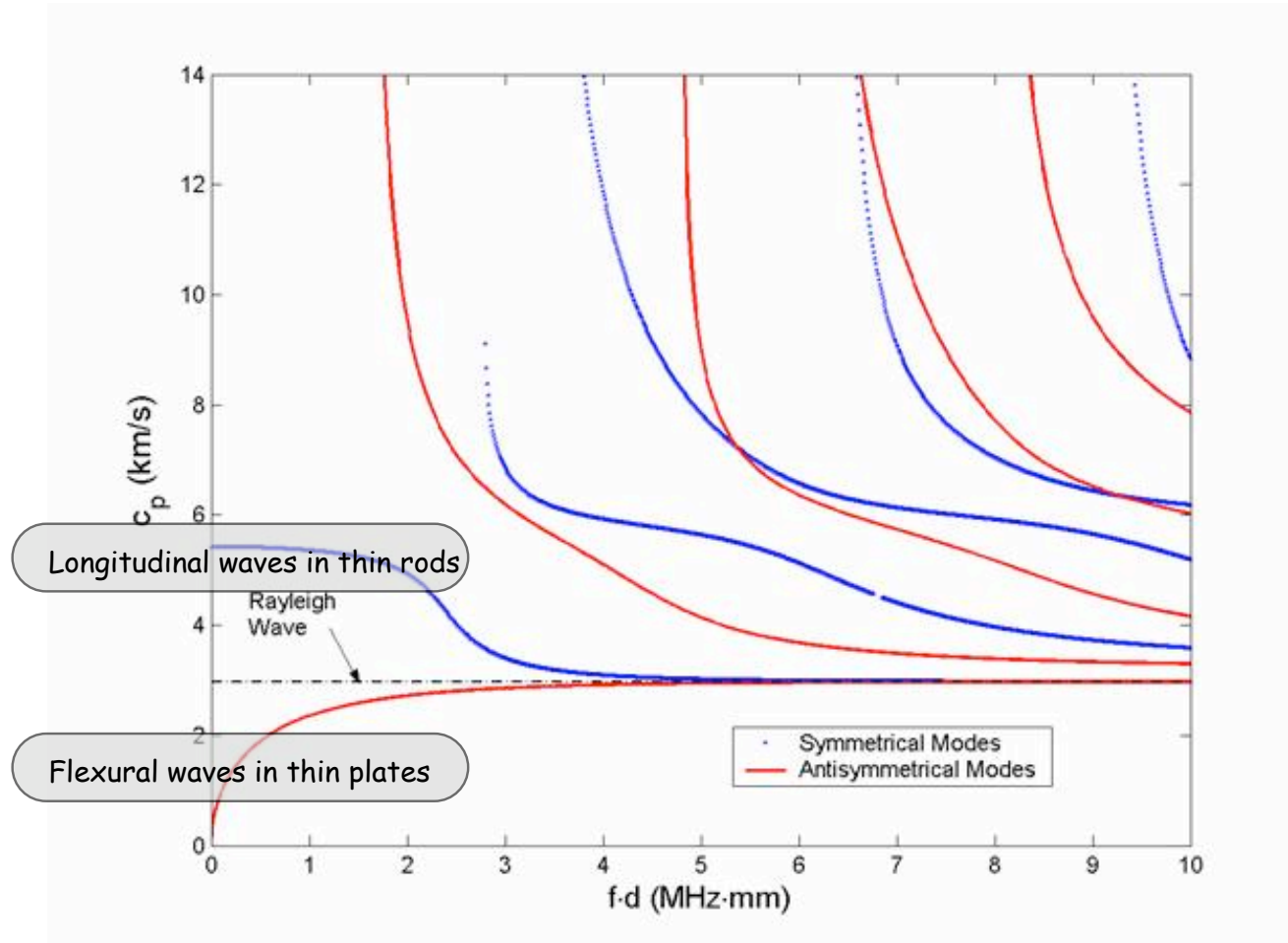$$c = c_{plate} = \sqrt{\frac{E}{\rho\left(1 - v^2\right)}}$$

Waves in plates

In low frequency plate waves, there are two distinct type of harmonic motion. These are called symmetric or **extensional** waves and antisymmetric or **flexural** waves.
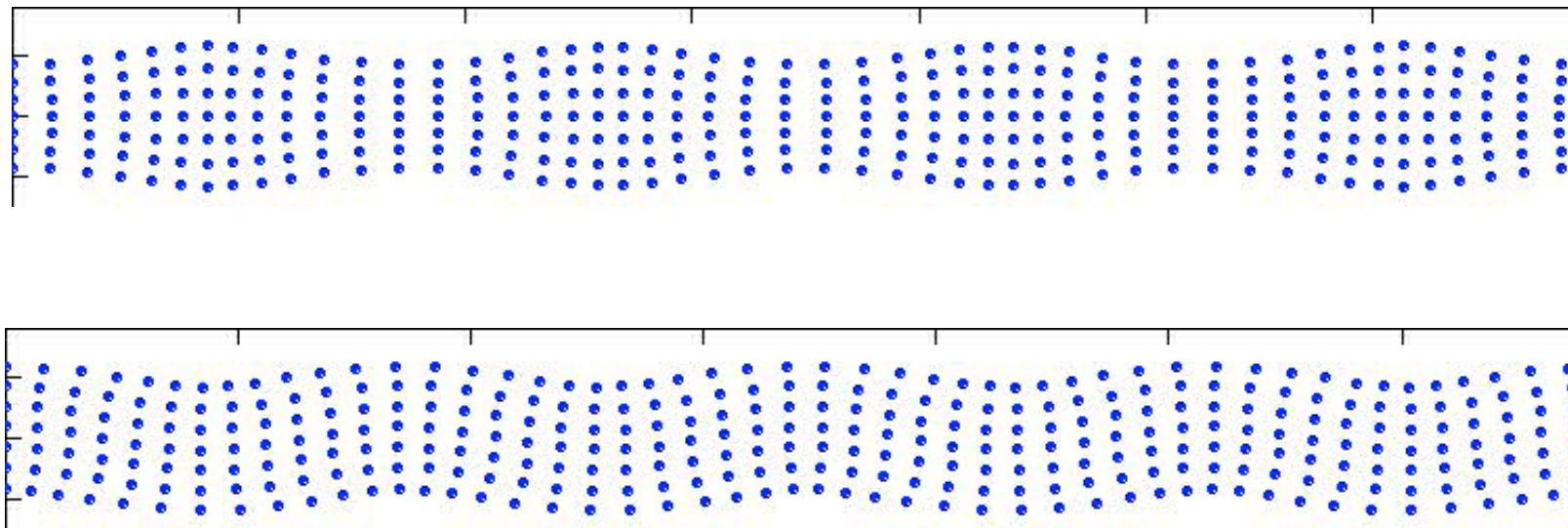
Lamb waves

Lamb waves are waves of plane strain that occur in a free plate, and the traction force must vanish on the upper and lower surface of the plate. In a free plate, a line source along y axis and all wave vectors must lie in the x-z plane. This requirement implies that response of the plate will be independent of the in-plane coordinate normal to the propagation direction.

Elastic waves in rods

Three types of elastic waves can propagate in rods: (1) **longitudinal waves**, (2) **flexural waves**, and (3) **torsional waves**. Longitudinal waves are similar to the symmetric Lamb waves, flexural waves are similar to antisymmetric Lamb waves, and torsional waves are similar to horizontal shear (SH) waves in plates.
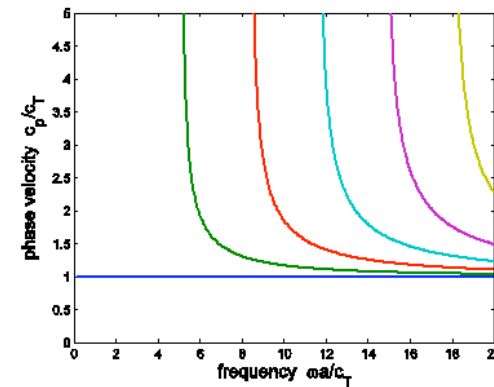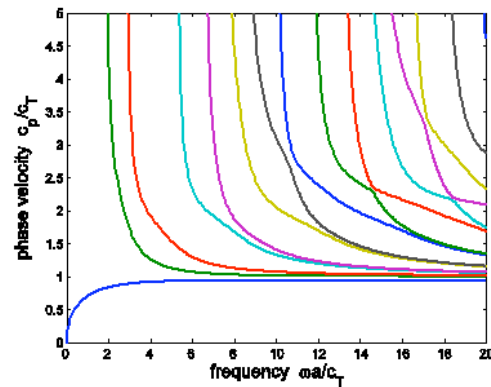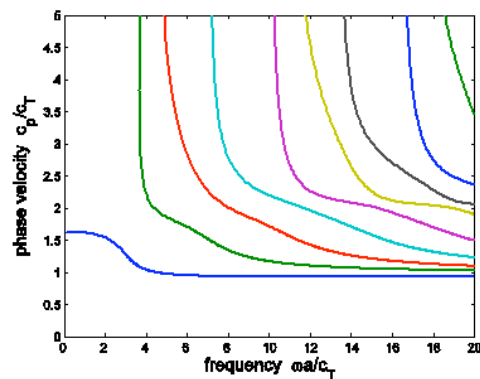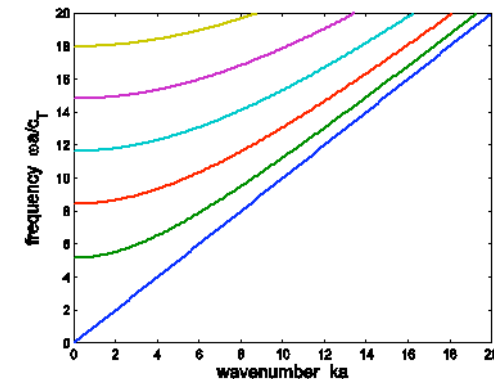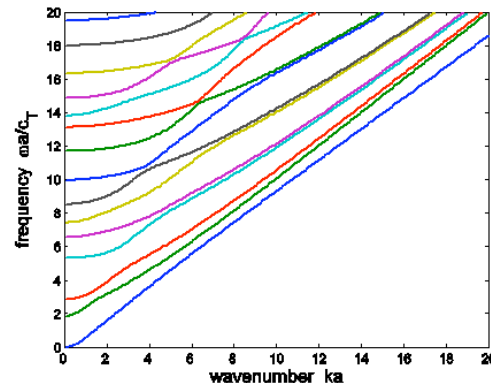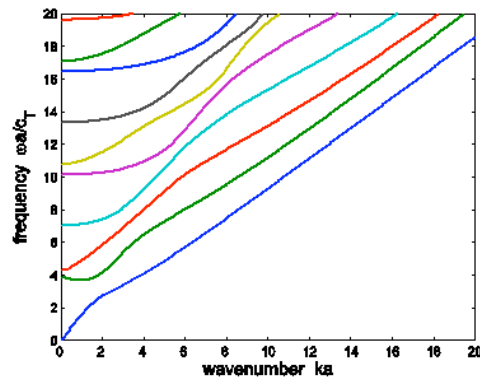
Torsional modes dispersion

Torsional modes

Small perturbations of a
stable equilibrium point $\longrightarrow$ **Linear restoring force** $\longrightarrow$ **Harmonic Oscillation**

Coupling of
harmonic oscillators $\longrightarrow$ the disturbances can propagate, superpose, stand, and **be dispersed**

WAVE: organized **propagating imbalance**, satisfying differential equations of motion

Organization can be destroyed,
when interference is destructive

non linearity

strong
scattering

Turbulence
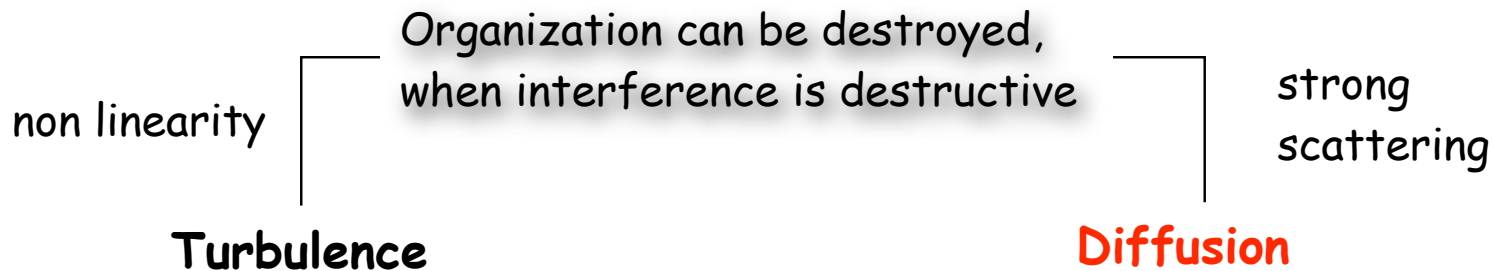
Diffusion

(Seismic) wave propagation

Basic physical concepts
- What is a wave?
- Discrete and continuous models
- Born of wave equation
- Dispersion

Basic physical concepts 2
- PDE: Poisson, diffusion and wave equation
- Navier-Stokes equation
- Scattering and diffusion

Classification of Partial Differential Equations (PDE)

Second-order PDEs of two variables are of the form:

$$a\frac{\partial^2 f(x,y)}{\partial x^2} + b\frac{\partial^2 f(x,y)}{\partial x \partial y} + c\frac{\partial^2 f(x,y)}{\partial y^2} + d\frac{\partial f(x,y)}{\partial x} + e\frac{\partial f(x,y)}{\partial y} = F(x,y)$$

$b^2 - 4ac < 0$	elliptic	LAPLACE equation
$b^2 - 4ac = 0$	parabolic	DIFFUSION equation
$b^2 - 4ac > 0$	hyperbolic	WAVE equation

Elliptic equations produce **stationary and energy-minimizing** solutions

Parabolic equations a **smooth-spreading flow** of an initial disturbance

Hyperbolic equations a **propagating disturbance**

Initial conditions: starting point for propagation problems

Boundary conditions: specified on domain boundaries to provide the interior solution in computational domain

∂R

R

s

n
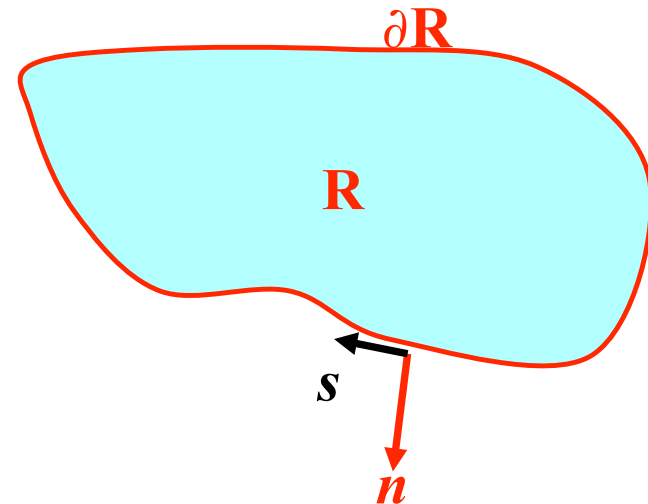
$$
\begin{cases}
\textbf{(i) Dirichlet condition}: u = f \text{ on } \partial R \\[2mm]
\textbf{(ii) Neumann condition}: \dfrac{\partial u}{\partial n} = f \text{ or } \dfrac{\partial u}{\partial s} = g \text{ on } \partial R \\[2mm]
\textbf{(iii) Robin (mixed) condition}: \dfrac{\partial u}{\partial n} + ku = f \text{ on } \partial R
\end{cases}
$$

Wave Equation

Hyperbolic Equation

$b^2 - 4ac = 0 - 4(1)(-c^2) > 0$: Hyperbolic

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq a, \quad 0 \leq t$$

I.C.s $\begin{cases} u(x,0) = f_1(x) \\ u_t(x,0) = f_2(x) \end{cases}$ $\quad 0 \leq x \leq a$

B.C.s $\begin{cases} u(0,t) = g_1(t) \\ u(a,t) = g_2(t) \end{cases}$ $\quad t > 0$

Elliptic PDEs

Steady-state two-dimensional heat conduction equation is prototypical elliptic PDE

Laplace equation - no heat generation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Poisson equation - with heat source

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = f(x,y)$$

Heat transfer in a one-dimensional rod

$x = 0$ $x = a$

$g_1(t)$ $g_2(t)$
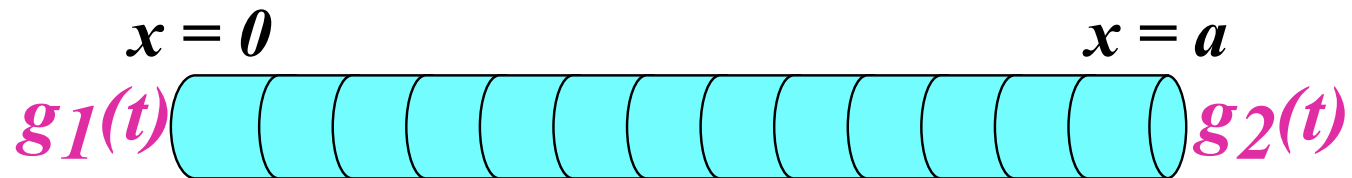
$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq a, \quad 0 \leq t \leq T$$

I.C.s $u(x,0) = f(x)$ $0 \leq x \leq a$

B.C.s $\begin{cases} u(0,t) = g_1(t) \\ u(a,t) = g_2(t) \end{cases}$ $0 \leq t \leq T$

Coupled PDE

Navier-Stokes Equations
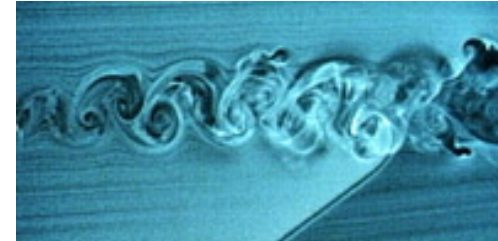
$$
\begin{cases}
\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\[2em]
\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) \\[2em]
\dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} + \nu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right)
\end{cases}
$$

(Seismic) wave propagation

Basic physical concepts
What is a wave?
Discrete and continuous models
Born of wave equation
Dispersion

Basic physical concepts 2
PDE: Poisson, diffusion and wave equation
Navier-Stokes equation
Scattering and diffusion

Fluids...

The space occupied by the material will be called the **domain**.

☑ **Solids** are materials that have a more or less intrinsic configuration or shape and do not conform to their domain under nominal conditions.

☑ **Fluids** do not have an intrinsic shape; **gases** are fluids that will completely fill their domain (or container) and **liquids** are fluids that form a free surface in the presence of gravity.

Fluid mechanics assumptions

☑ Classical fluid mechanics, like classical thermodynamics, is concerned with **macroscopic phenomena** (bulk properties) rather than microscopic (molecular-scale) phenomena.

☑ The molecular makeup of a fluid will be ignored in all that follows, and the crucially important **physical properties** of a fluid, e.g., its mass density, ρ, and specific heat, C_p, among others, must be provided from outside of this theory. It is assumed that these physical properties, along with **flow properties**, e.g., the pressure, P, velocity, $\mathbf{v}$, temperature, T, etc., are in principle definable at every point in space, as if the fluid was a smoothly varying continuum, rather than a swarm of very fine, discrete particles (molecules).

Continuity equation

Conservation of matter

The total mass of fluid flowing, in unit of time, through a surface S, has to be equal to the decrease, in unit time, in the mass of fluid in the volume V:

$$\oint \rho \mathbf{v} dS = -\frac{\partial}{\partial t} \int \rho dV$$

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) = \frac{\partial \rho}{\partial t} + \rho \, \text{div}(\mathbf{v}) + \mathbf{v} \cdot \text{grad}(\rho) = 0$$

that can be compared with what we obtained considering 1D sound waves:

$$\Delta \rho = -\rho_0 \frac{\partial s}{\partial x}$$

The gas moves and causes density variations

Momentum...

Newton's law

The fluid in the volume is accelerated by the total force acting on it:

$$\rho \frac{d\mathbf{v}}{dt} = -grad(P) - \rho grad(\phi) + \mathbf{f}_{visc}$$

Fluid moves from high-pressure areas to low-pressure areas. Moving implies that fluid moves in direction of largest change in pressure

External forces that act at a distance; we can suppose that they are conservative (like **gravity** and electricity)

Internal force due to the fact that in a flowing fluid there can also be a shearing stress, and it is called the **viscous** force

Momentum...

Newton's law

The fluid in the volume is accelerated by the total force acting on it:

$$\rho \frac{d\mathbf{v}}{dt} = -grad(P) - \rho\,grad(\phi) + \mathbf{f}_{visc}$$

that can be compared with what we obtained considering for 1D sound waves:

$$\rho_0 \frac{\partial^2 s}{\partial t^2} = -\frac{\partial \Delta P}{\partial x}$$
Pressure variations generate gas motion

Advection

In mathematics and continuum mechanics, including fluid dynamics, the **substantive derivative** (sometimes the **Lagrangian derivative**, **material derivative** or **advective derivative**), written D/Dt, is the **rate of change** of some property of a small parcel of fluid.

Note that if the fluid is moving, the substantive derivative is the rate of change of fluid within the small parcel, hence the other names **advective derivative** and **fluid following derivative**. **Advection** is transport of a some conserved scalar quantity in a vector field.

$$v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} + v_z \frac{\partial f}{\partial z} = \mathbf{v} \cdot \text{grad}(f)$$

Navier-Stokes

Euler equations

Newton's law + **Conservation of matter** - **Viscosity**

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot grad)\mathbf{v} = -\frac{grad(P)}{\rho} - grad(\phi)$$

1.

Using the identity $(\mathbf{v} \cdot grad)\mathbf{v} = (rot\mathbf{v}) \times \mathbf{v} + \frac{1}{2}grad(v^2)$

and defining the **vorticity** as $rot\mathbf{v} = \Omega$

$$\frac{\partial \Omega}{\partial t} + rot(\Omega \times \mathbf{v}) = 0$$

2.

Navier-Stokes

Equation of state

The functional relationship between **density, pressure and temperature**:

$$P=P(\rho,T) \text{ or equivalently, } \rho=\rho(P,T)$$

with T the absolute temperature in Kelvin.

The archetype of an equation of state is that of an ideal gas, $P=\rho RT/M$ where R=8.31 (Joule moles^{-1} K^{-1}) is the universal gas constant and M is the molecular weight (kg/mole).

If the composition of the material changes, then the appropriate equation of state will involve more than three variables, for example the concentration of salt if sea water, or water vapor if air.

An important class of phenomenon may be described by a reduced equation of state having state variables density and pressure alone,

$$P=P(\rho) \text{ or equivalently, } \rho = \rho(P)$$

and the fluid is said to be **barotropic.**

Equation of state - barotropic

The temperature of the fluid will change as pressure work is done on or by the fluid, and yet temperature need not appear as a separate, independent state variable provided conditions approximate one of two limiting cases:

1) If the fluid is a fixed mass of ideal gas, say, that can readily exchange heat with a heat reservoir having a constant temperature, then the gas may remain **isothermal** under pressure changes;

2) the other limit, which is more likely to be relevant, is that heat exchange with the surroundings is negligible because the time scale for significant conduction is very long compared to the time scale (lifetime or period) of the phenomenon. In that event the system is said to be **adiabatic** and in the case of an ideal gas the density and pressure are related by the well-known adiabatic law.

$$\Delta\rho = \frac{\partial\rho}{\partial P}\Delta P$$

that can be compared with what we obtained considering sound waves:

$$\Delta P = \kappa\Delta\rho = c^2\Delta\rho \qquad \text{Density variations cause pressure variations}$$

Incompressible fluids

In many cases of the flow of fluids their density may be supposed invariable, i.e. constant throughout the volume and its motion and we speak of **incompressible flow**

$$\rho = constant$$

Conservation of matter $\qquad$ div (**v**)=0

Euler equation

$$\frac{\partial \mathbf{v}}{\partial t} + \Omega \times \mathbf{v} = -grad\left(\frac{1}{2}v^2 + \frac{P}{\rho} + \phi\right)$$

The conditions under which the fluid can be considered incompressible are:

$$\frac{\partial \rho}{\partial t} << \rho \, div(\mathbf{v}) \Rightarrow \frac{\Delta \rho}{\tau} << \frac{\rho v}{\lambda}$$

i.e. $\qquad \tau >> \dfrac{\lambda}{c}$

$$\Delta \rho = \frac{\Delta P}{c^2} \approx \frac{1}{c^2}\left(\rho \frac{\partial v}{\partial t}\lambda\right) \approx \frac{1}{c^2}\left(\rho \frac{v}{\tau}\lambda\right)$$

i.e. **v<<c**

i.e. the time taken by a sound signal to traverse distances must be small compared with that during which flow changes appreciably

Having considered gravity waves whose length is small compared with the depth of the liquid, let us now discuss the opposite limiting case of waves whose length is large compared with the depth. These are called **long waves**.

Let us examine the propagation of long waves in a channel that is supposed to be along the x-axis, and of infinite length. The cross-section of the channel may have any shape, and may vary along its length. We denote the cross-sectional area of the liquid in the channel by $S = S(x,t)$. The depth and width of the channel are supposed small in comparison with the wavelength.

We shall here consider longitudinal waves, in which the liquid moves along the channel. In such waves the velocity component v_x along the channel is large compared with the components v_y, v_z. We denote v_x by v simply, and omit small terms.

Viscosity

Viscosity is a measure of the resistance of a fluid to deform under shear stress. It is commonly perceived as "thickness", or resistance to pouring. Viscosity describes a fluid's internal resistance to flow and may be thought of as a measure of fluid friction. Thus, water is "thin", having a lower viscosity, while vegetable oil is "thick" having a higher viscosity. All real fluids (except superfluids) have some resistance to shear stress, but a fluid which has no resistance to shear stress is known as an **ideal fluid**
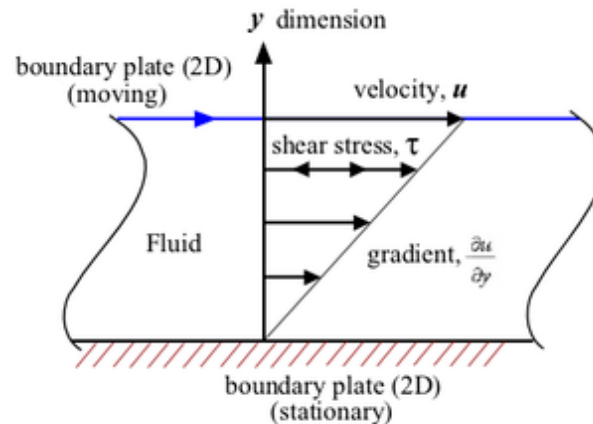
To understand deformation due to shear, picture two flat plates with a fixed spacing, h, between them:
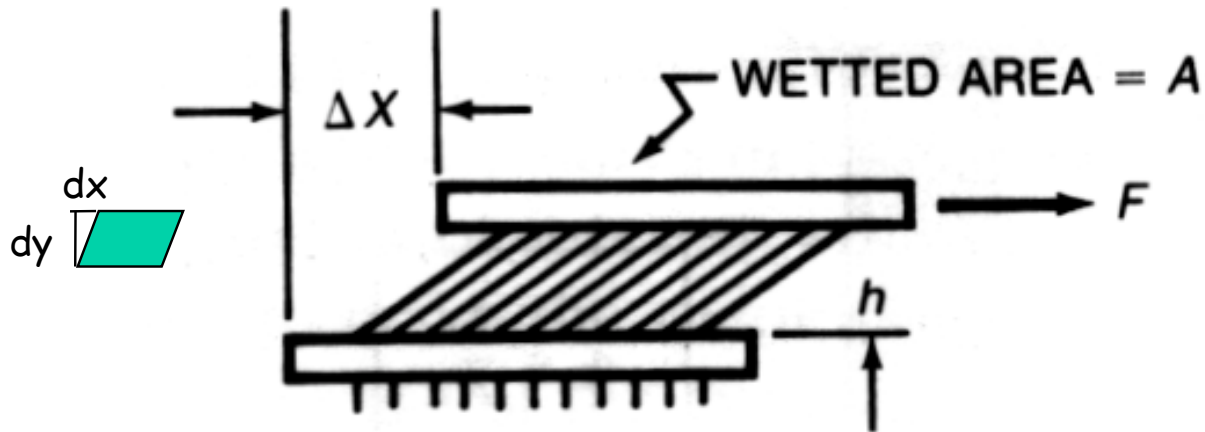
Fluids are qualitatively different from solids in their response to a shear stress. Ordinary fluids such as air and water have no intrinsic configuration, and hence fluids **do not develop a restoring force** that can provide a static balance to a shear stress. When the shear stress is held steady, and assuming that the geometry does not interfere, the **shear deformation rate**, may also be steady or have a meaningful time-average.

Newton's Law of Viscosity
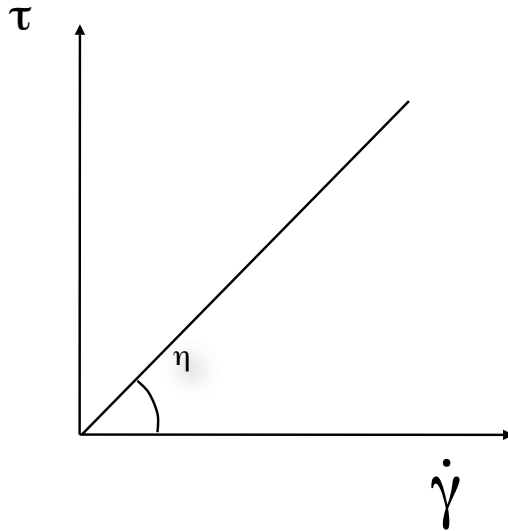
This is called a "**flow curve**"

The proportionality constant is the **viscosity**

Newton's law of viscosity

$$\tau = \eta \frac{dv}{dy} = \eta \frac{d\gamma}{dt}$$

$\therefore$ The deformation of a material is due to stresses imposed to it.

Newtonian fluids Fluids which obey Newton's law:

Shearing stress is linearly related to the rate of shearing strain.

The **viscosity** of a fluid measures its resistance to flow under an applied shear stress.

Newtonian Fluids

Viscosity of Newtonian fluids depends **only** on

temperature and **pressure**, e.g.:

$$\eta\left(T,P\right) = \eta_0\, e^{\left[\frac{\Delta E}{R}\left(\frac{T_0 - T}{T_0 T}\right)\right]} e^{\beta\left(P - P_0\right)}$$

Where: η_o: viscosity at T_o and P_o (reference temperature and pressure)
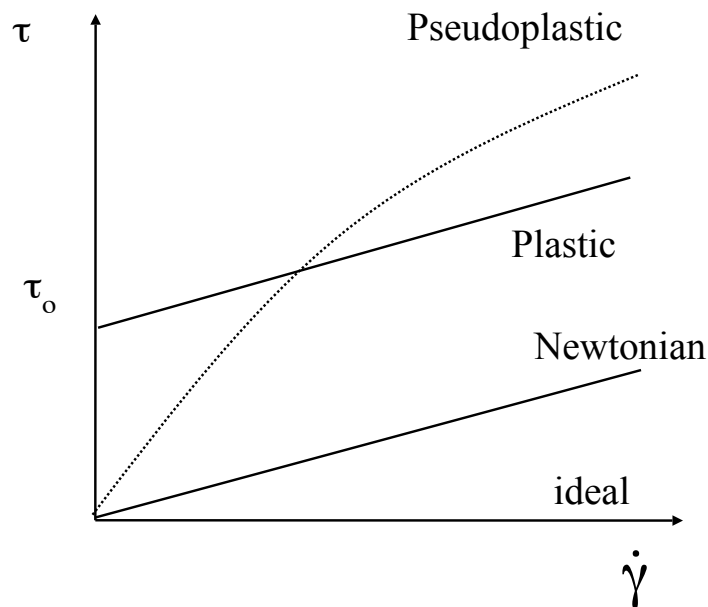
ΔE: activation energy for flow

R: gas constant

β: material property $[m^2/N]$

Plasticity and Yield Stress

The structure of some polymers, especially filled polymers or concentrated suspensions can be sufficiently rigid that it permits the material to withstand a certain level of deforming stress without flowing. The maximum stress that can be sustained without flow is called the "**yield stress**" and this type of behavior is called "**plasticity**"
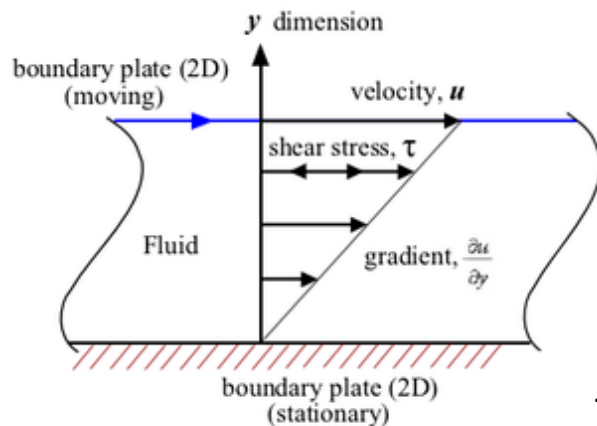
$$\dot{\gamma} = 0 \qquad \text{for } \tau < \tau_o$$
$$\tau = \tau_o + \eta\dot{\gamma} \quad \text{for } \tau \geq \tau_o$$

Viscosity

Viscosity is a measure of the resistance of a fluid to deform under shear stress. It is commonly perceived as "thickness", or resistance to pouring. Viscosity describes a fluid's internal resistance to flow and may be thought of as a measure of fluid friction. Thus, water is "thin", having a lower viscosity, while vegetable oil is "thick" having a higher viscosity. All real fluids (except superfluids) have some resistance to shear stress, but a fluid which has no resistance to shear stress is known as an **ideal fluid**

$$\lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} = \eta \frac{\partial v_x}{\partial y}$$

thus the shear stress is proportional to the **rate of change of shear strain**

$$\eta \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right)$$

In the general case there is another term that depends on other derivatives of the velocity and the general expression is:

$$\eta\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) + \eta'\delta_{ij}\left(div(\mathbf{v})\right)$$

so two constants are required: the "first coefficient of viscosity" or "shear viscosity coefficient" and "second coefficient of viscosity" .

The component of the viscous force per unit volume in the direction of the rectangular coordinate x_j is:

$$(f_{visc})_i = \sum_{j=1,3}\frac{\partial \tau_{ij}}{\partial x_j} = \sum_{j=1,3}\frac{\partial}{\partial x_j}\left[\eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right] + \frac{\partial}{\partial x_i}\left(\eta' div(\mathbf{v})\right)$$

$$f_{visc} = \eta\Delta\mathbf{v} + (\eta + \eta')grad\left(div(\mathbf{v})\right)$$

Navier-Stokes

Navier-Stokes equations

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot grad)\mathbf{v} = -grad(P) - \rho \, grad(\phi) + \\ +\eta \Delta \mathbf{v} + (\eta + \eta')grad\Big(div(\mathbf{v})\Big)$$

and in the incompressible case...

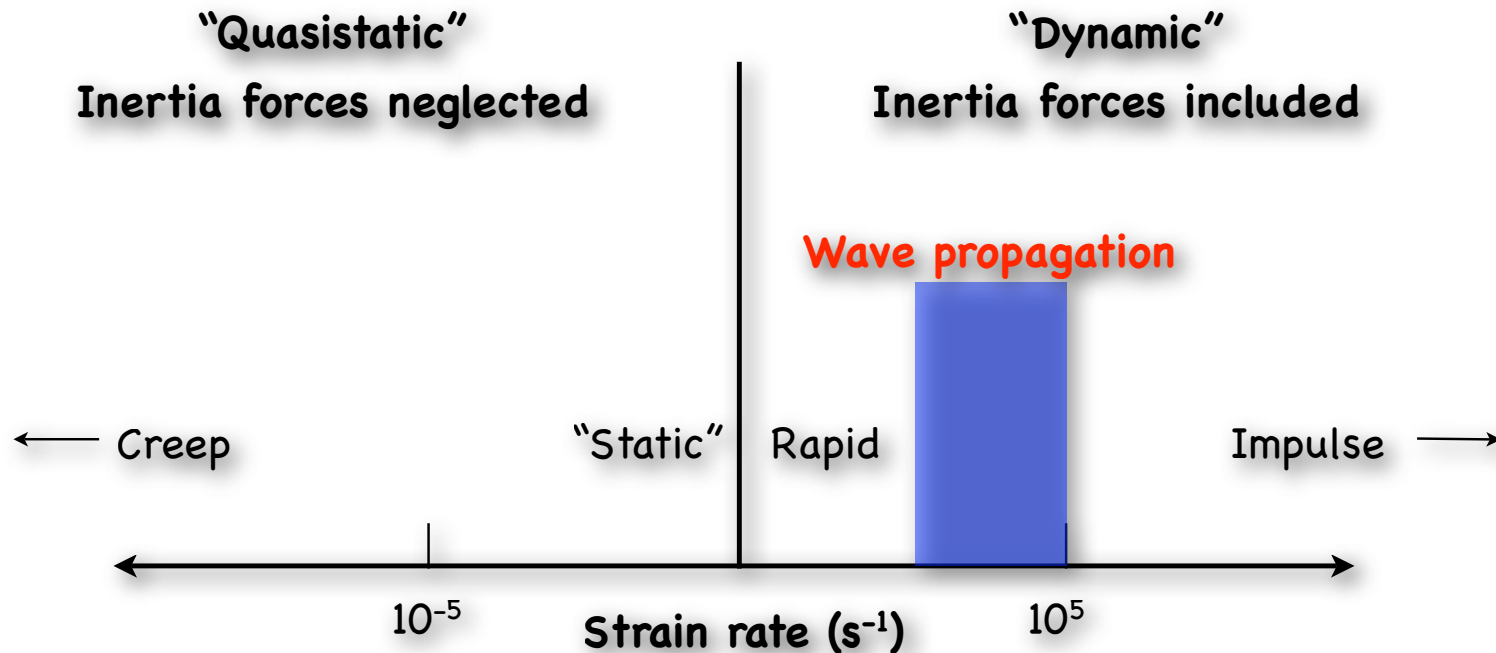$$\frac{\partial \mathbf{\Omega}}{\partial t} + rot(\mathbf{\Omega} \times \mathbf{v}) = \frac{\eta}{\rho}\Delta\mathbf{\Omega}$$

Elasticity...

☑ the study and determination of the response of **continuous**, **perfectly elastic** solids subjected to applications of forces

"Quasistatic"
Inertia forces neglected

"Dynamic"
Inertia forces included

Wave propagation

⟵ Creep   "Static" Rapid   Impulse ⟶

10^{-5} **Strain rate (s^{-1})** 10^{5}

(Seismic) wave propagation

Basic physical concepts
What is a wave?
Discrete and continuous models
Born of wave equation
Dispersion

Basic physical concepts 2
PDE: Poisson, diffusion and wave equation
Navier-Stokes equation
Scattering and diffusion

Basic concepts of EM wavefield

Extinction and **emission** are two main types of the interactions between an electromagnetic radiation field and a medium (e.g., the atmosphere).

Extinction is due to absorption and scattering.

Absorption is a process that **removes** the radiant energy from an electromagnetic field and transfers it to other forms of energy.

Scattering is a process that does not remove energy from the radiation field, but **redirect** it. Scattering can be thought of as absorption of radiant energy followed by re-emission back to the electromagnetic field with negligible conversion of energy, i.e. can be a "source" of radiant energy for the light beams traveling in other directions.
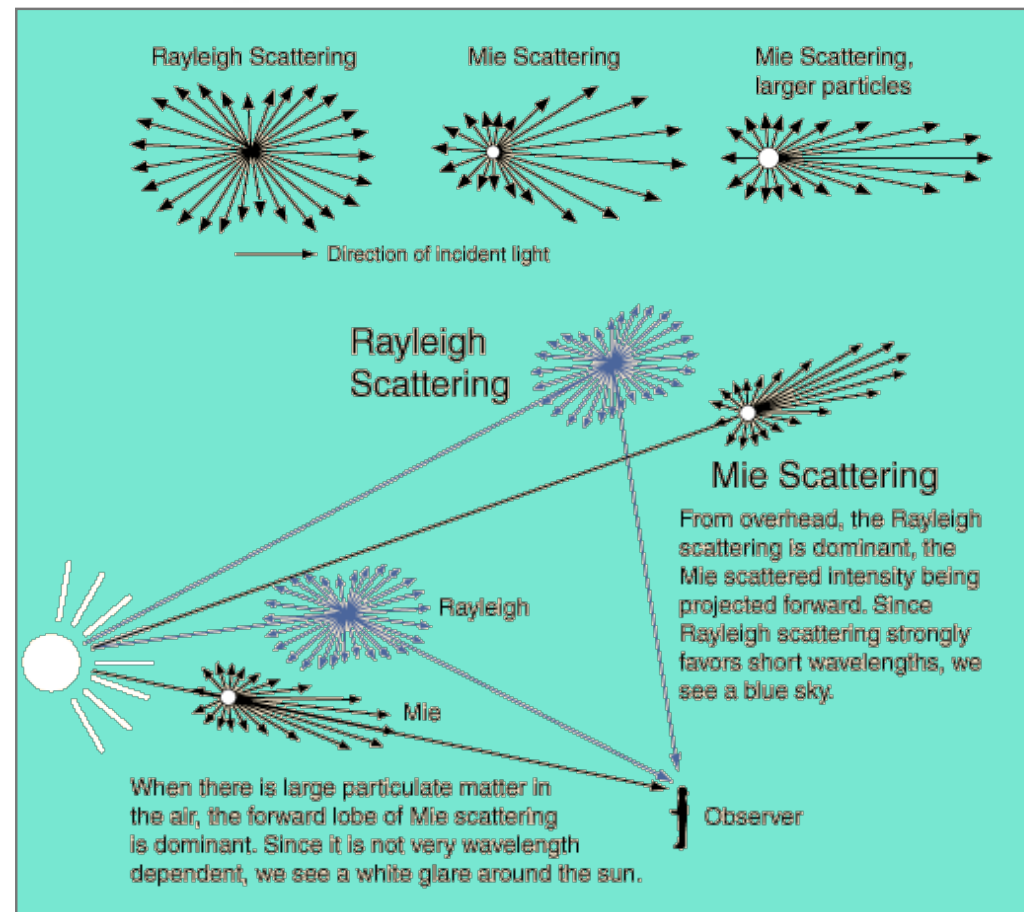
Scattering **occurs at all wavelengths** (spectrally not selective) in the electromagnetic spectrum, for any material whose refractive index is different from that of the surrounding medium (**optically inhomogeneous**).
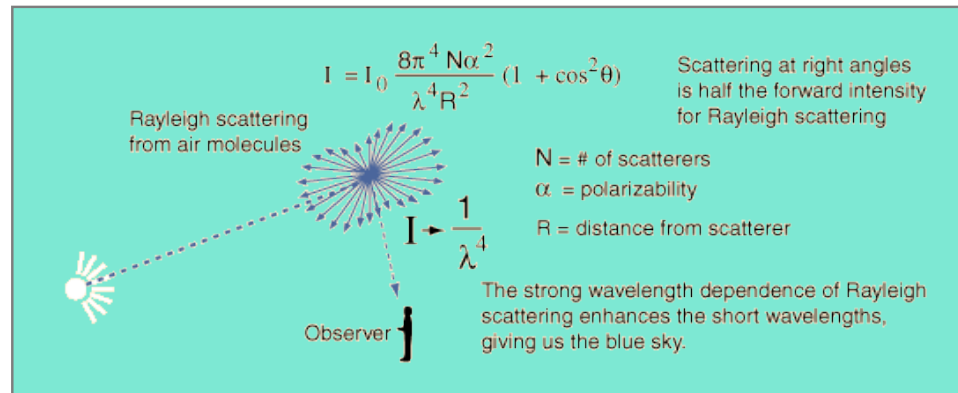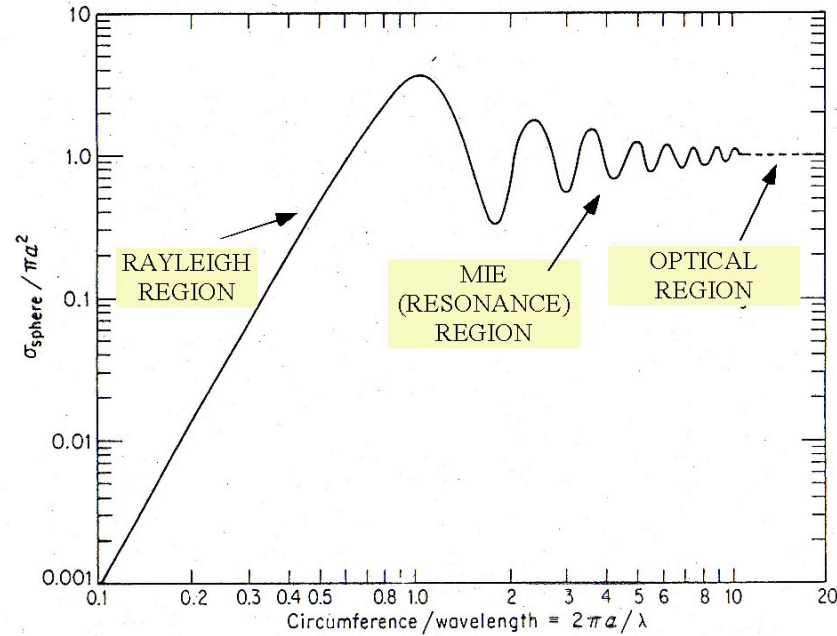
The amount of scattered energy depends strongly on the ratio of: particle size (a) to wavelength (λ) of the incident wave

When **(a < λ/10)**, the scattered intensity on both forward and backward directions are equal. This type of scattering is called **Rayleigh scattering**.

For **(a > λ)**, the angular distribution of scattered intensity becomes more complex with more energy scattered in the forward direction. This type of scattering is called **Mie scattering**

Rayleigh Scattering

Mie Scattering

Mie Scattering, larger particles

Direction of incident light

Rayleigh Scattering

Rayleigh

Mie

When there is large particulate matter in the air, the forward lobe of Mie scattering is dominant. Since it is not very wavelength dependent, we see a white glare around the sun.

Mie Scattering

From overhead, the Rayleigh scattering is dominant, the Mie scattered intensity being projected forward. Since Rayleigh scattering strongly favors short wavelengths, we see a blue sky.
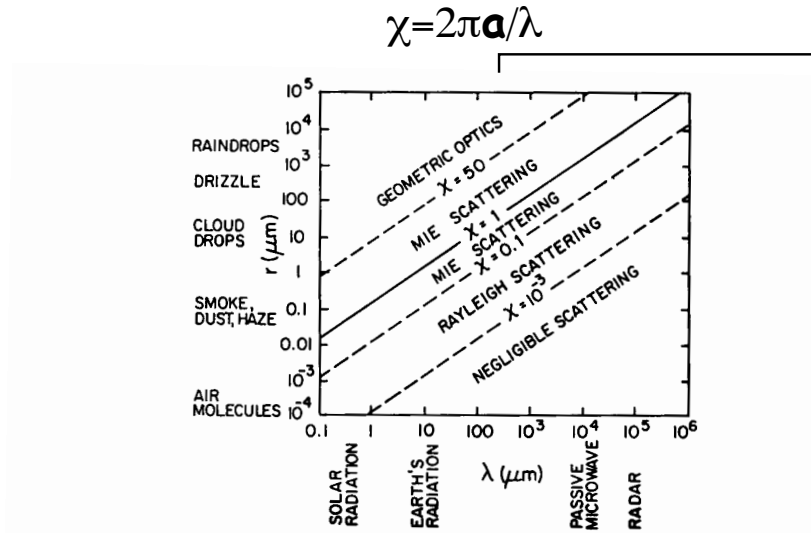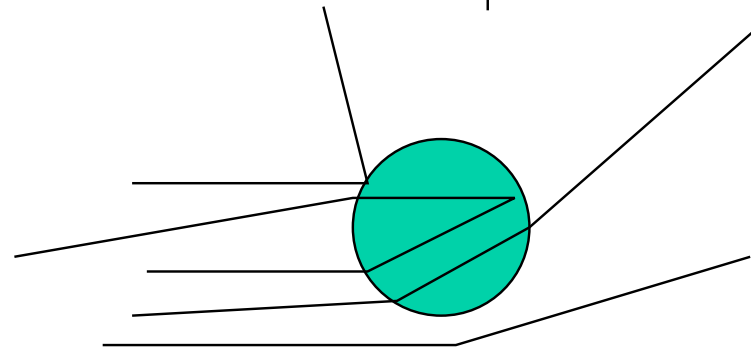
Observer

$$\chi = 2\pi a/\lambda$$

FIGURE 3.18. Scattering regimes. [Adapted from Wallace and Hobbs (1977). Reprinted by permission of Academic Press.]

For **(a >> λ)**, the Scattering characteristics are determined from explicit Reflection, Refraction and Diffraction: **Geometric "Ray" Optics**
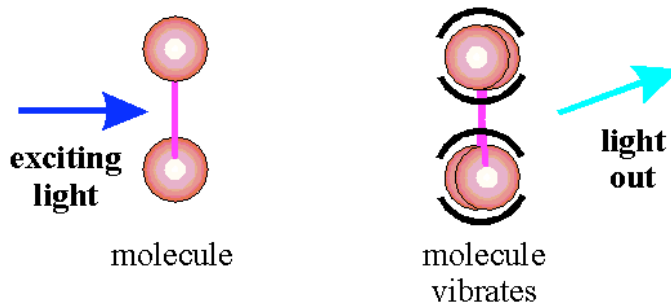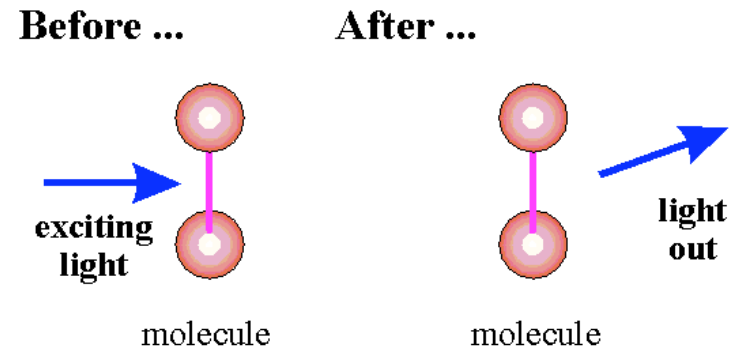
Scattering of EM wavefield (3)

Composition of the scatterer (n) is important!

The interaction (and its redirection) of electromagnetic radiation with matter
May or may not occur with **transfer of energy**, i.e., the scattered radiation has a slightly different or the same wavelength.


Rayleigh scattering - Light out has same frequency as light in, with scattering at many different angles.

Before ...   After ...

exciting light

light out

molecule   molecule

exciting light

light out

molecule   molecule vibrates

Raman scattering - Light is scattered due to vibrations in molecules or optical phonons in solids. Light is shifted by as much as 4000 wavenumbers and exchanges energy with a molecular vibration.

Scattering of EM wavefield (4)

scattering medium

ω_p
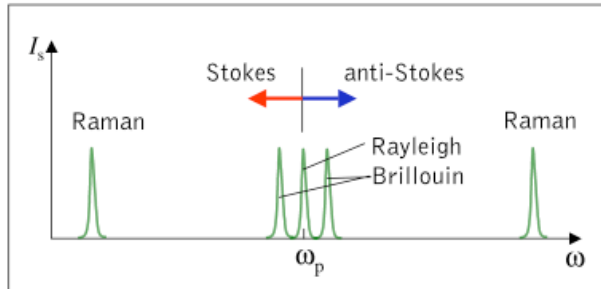
scattered light

spectrally resolved detection
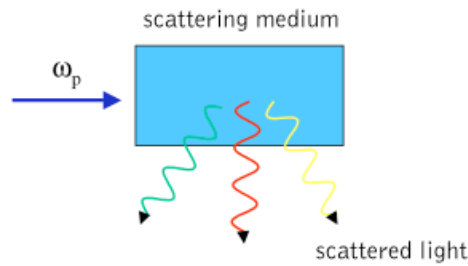
Rayleigh scattering
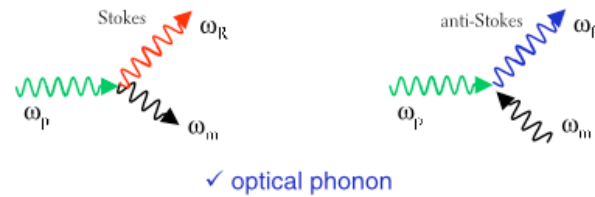 scattering from *nonpropagating* density fluctuations (elastic)
Brillouin scattering
  scattering from *propagating* pressure waves (sound waves, acoustic phonons)
Raman scattering
  interaction of light with vibrational modes of molecules or lattice vibrations of crystals (scattering from optical phonons)
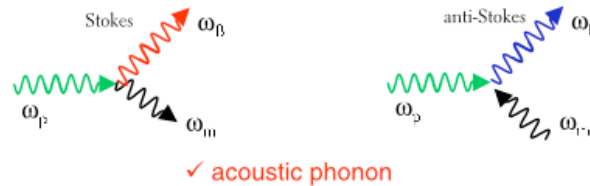
I_s

Stokes   anti-Stokes

Raman

Rayleigh
Brillouin

Raman

ω_p ω

Raman scattering

Stokes ω_R

ω_P ω_m

anti-Stokes ω_R

ω_P ω_m

✓ optical phonon

Phonons
  quanta of the ionic displacement field in a solid

phonon dispersion curve

$\omega(k)$

optical

acoustic

k

Brillouin scattering

Stokes ω_β

ω_P ω_m

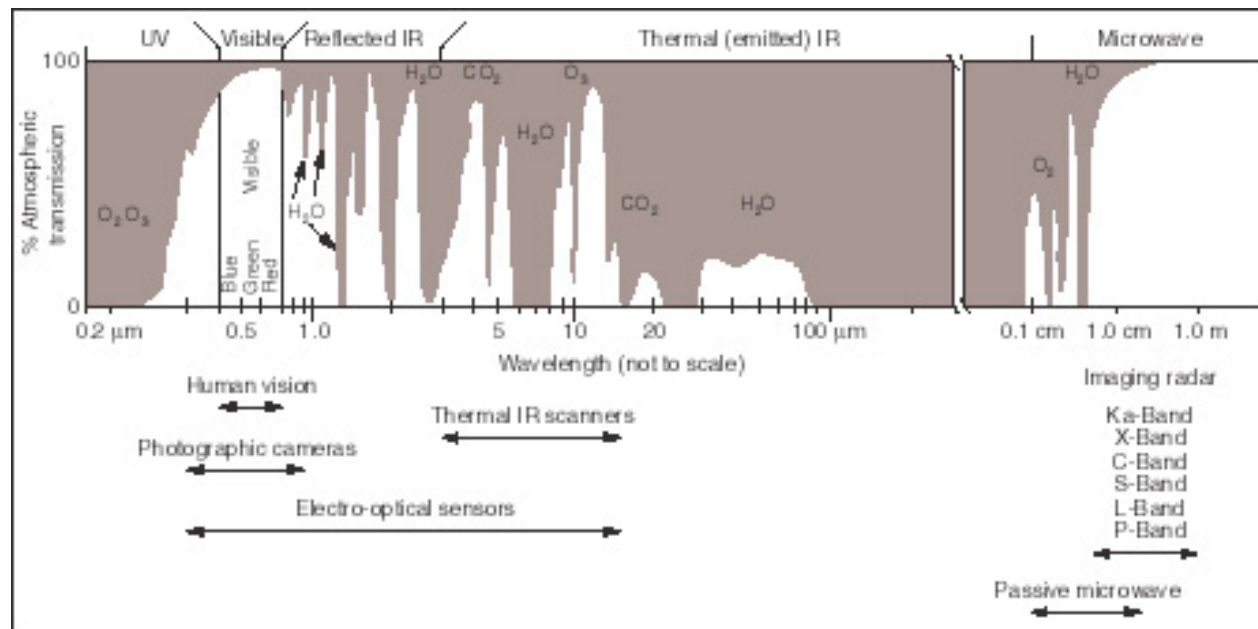anti-Stokes ω_β

ω_P ω_m

✓ acoustic phonon
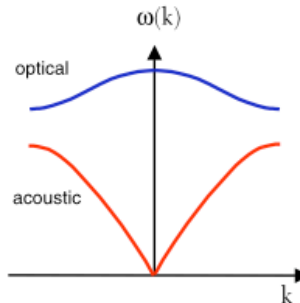
Scattering and **Absorption**

When the photon is absorbed and re-emitted at a different wavelength, this is absorption.

Transmissivity of the Earth's atmosphere

Scattering and **Diffusion**

In single scattering, the properties of the scatterer are important , but multiple scattering erases these effects - eventually **all** wavelengths are scattered in **all** directions.
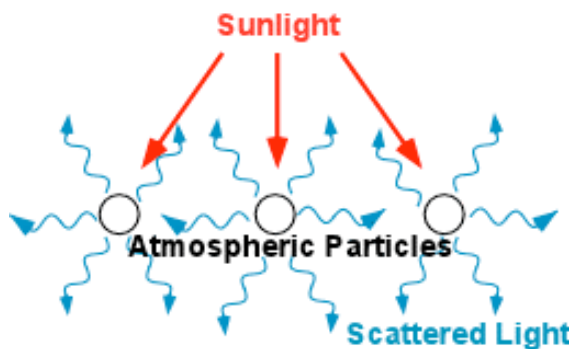
Works for turbid media: clouds, beer foam, milk, etc...

Example: when a solid has a very low temperature, phonons behave like waves (long mean free paths) and heat propagate following ballistic term.
At higher temperatures, the phonons are in a diffusive regime and heat propagate following Maxwell law.

(Seismic) wave propagation

Basic physical concepts
What is a wave?
Discrete and continuous models
Born of wave equation
Dispersion

Basic physical concepts 2
PDE: Poisson, diffusion and wave equation
Navier-Stokes equation
Scattering and diffusion
Application to the seismic wavefield

The governing parameters for the seismic scattering are:

wavelength of the wavefield (or wavenumber k)

λ (10^{0}-10^{6} m)

correlation length, or dimension, of the heterogeneity
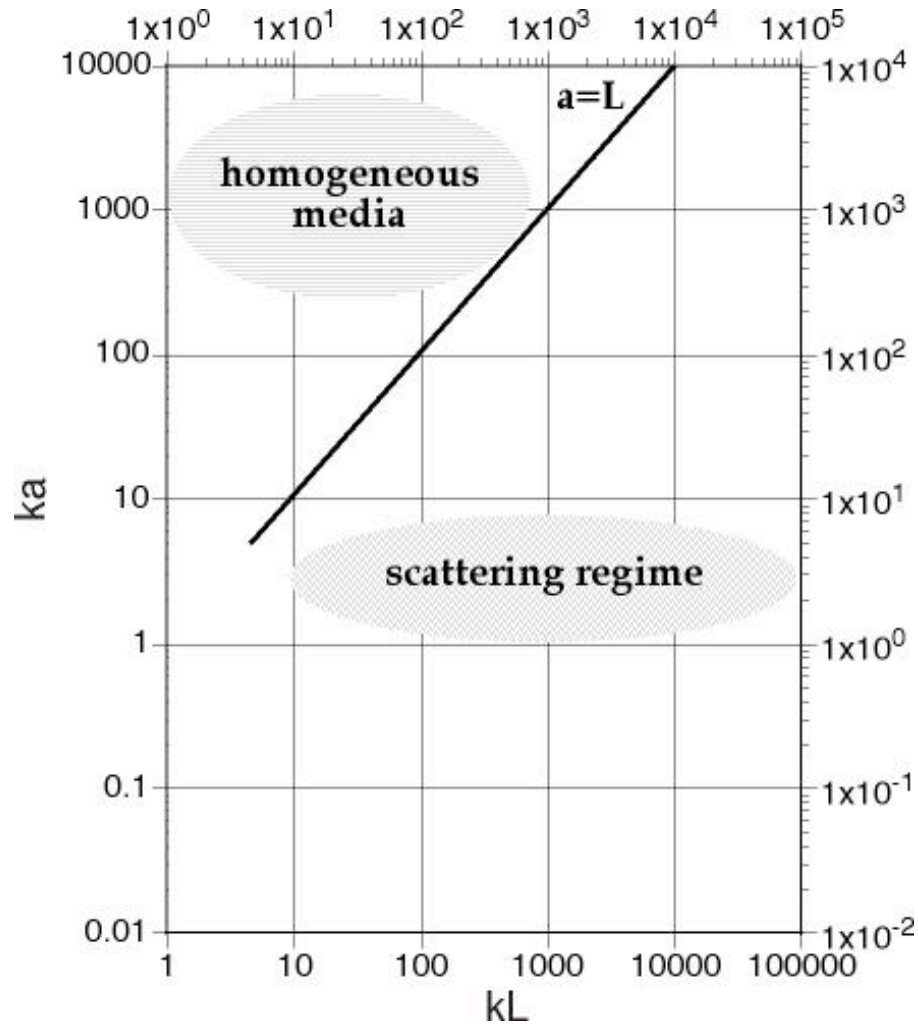
a ($10^{?}$-10^{5} m)

distance travelled in the heterogeneity

L (10^{0}-10^{6} m)

With special cases:
- $a = L$ homogeneous region
- $a \gg \lambda$ ray theory is valid
- $a \approx \lambda$ strong scattering effects

Seismic Scattering (1)

Wave propagation problems can be classified using the parameters just introduced.

This classification is crucial for the choice of technique to calculate synthetic seismograms

(Adapted from Aki and Richards, 1980)

Scattering in a **perturbed** model

Let us consider a **perturbed** model:
reference+perturbation (in elastic parameters)

$$\rho = \rho_0 + \varepsilon\delta\rho \quad \lambda = \lambda_0 + \varepsilon\delta\lambda \quad \mu = \mu_0 + \varepsilon\delta\mu$$

resulting in a velocity perturbation

$$c = c_0 + \varepsilon\delta c$$

solution: **Primary** field + **Scattered** field
$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1(\delta\rho, \delta\lambda, \delta\mu)$$

satisfying equations of motion:
$$\rho_0\ddot{u}_i^0 - \left(\lambda_0 + \mu_0\right)\left(\nabla\cdot\mathbf{u}^0\right)_{,i} - \mu_0\nabla^2\mathbf{u}_i^0 = 0$$

$$\rho_0\ddot{u}_i - \left(\lambda\nabla\cdot\mathbf{u}\right)_{,i} - \left[\mu\left(u_{i,j} + u_{j,i}\right)\right]_{,j} = 0$$

$$\rho_0\ddot{u}_i^1 - \left(\lambda_0 + \mu_0\right)\left(\nabla\cdot\mathbf{u}^1\right)_{,i} - \mu_0\nabla^2\mathbf{u}_i^1 = Q_i$$

How does a point-like perturbation of the elastic parameters affect the wavefield?

Perturbation of the different elastic parameters produce characteristic radiation patterns. These effects are used in diffraction tomography to recover the perturbations from the recorded wavefield.

(Figure from Aki and Richards, 1980)

Type of inhomogeneity	Primary P	
	Scattered P-wave	Scattered S-wave
$\delta\alpha$		
$\nabla(\delta\lambda)$		
$\dfrac{\partial(\delta\mu)}{\partial x_1}$		

Correlation distance

When velocity varies in all directions with a finite scale length, it is more convenient to consider spatial fluctuations

Autocorrelation function (a is the **correlation distance)**:

$$N(\mathbf{r}_1) = \frac{\left\langle \dfrac{\delta c(\mathbf{r})}{c_0(\mathbf{r})} \dfrac{\delta c(\mathbf{r} + \mathbf{r}_1)}{c_0(\mathbf{r} + \mathbf{r}_1)} \right\rangle}{\left\langle \left(\dfrac{\delta c(\mathbf{r})}{c_0(\mathbf{r})} \right)^2 \right\rangle} = \begin{cases} e^{-|\mathbf{r}_1|/a} \\ e^{-(|\mathbf{r}_1|/a)^2} \end{cases}$$

Power Spectra of scattered waves

$$\left\langle |\mathbf{u}_1|^2 \right\rangle \propto \begin{cases} k^4 \left(1 + 4k^2 a^2 \sin^2 \dfrac{\theta}{2} \right)^{-2} \\ k^4 \exp\left(-k^2 a^2 \sin^2 \dfrac{\theta}{2} \right) \end{cases}$$

$\propto k^4$ if $ka \ll 1$ (Rayleigh scattering)
if ka is large (forward scattering)

Wave parameter

Energy loss through a cube of size L (Born approximation)

$$\frac{\Delta I}{I} \propto \begin{cases} k^4 a^3 L\left(1 + 4k^2 a^2\right)^{-1} \\ k^2 aL\left(1 - e^{k^2 a^2}\right)^{-1} \end{cases}$$

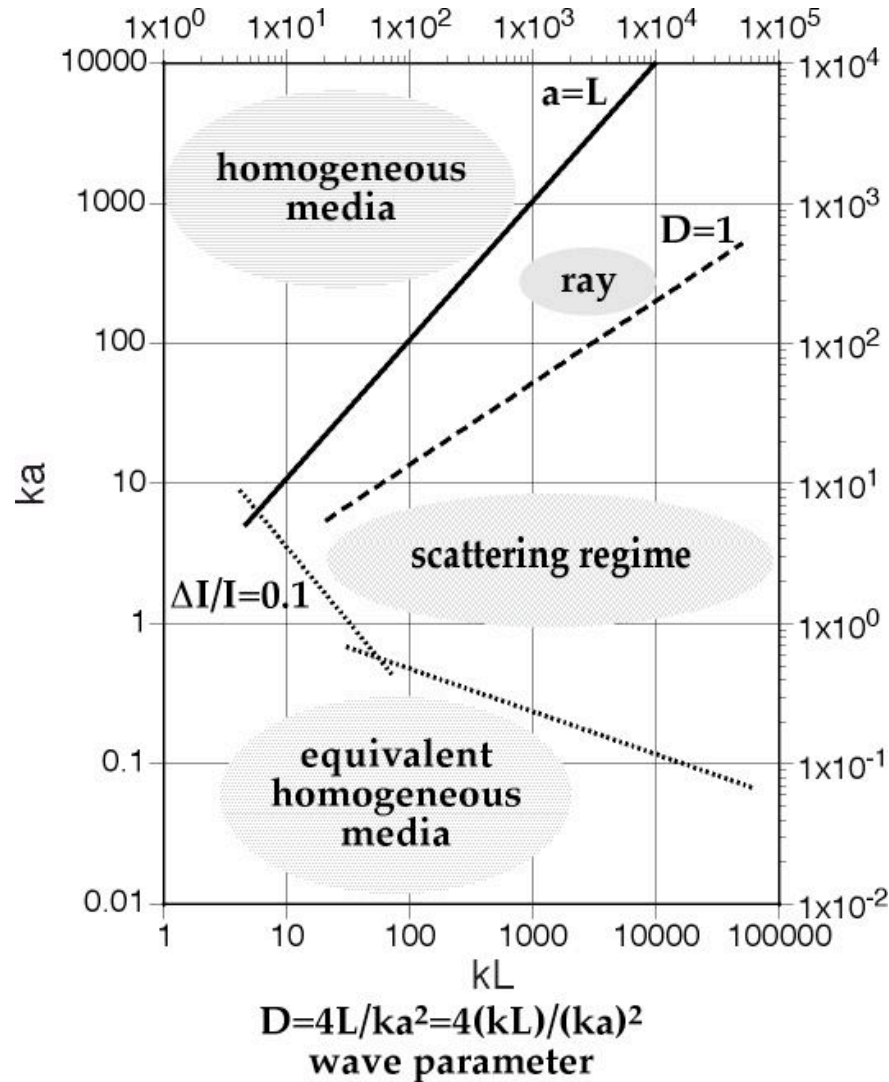but violates the energy conservation law and it is valid if **(<0.1)**

the **perturbations** (P &A) are function of the **wave parameter**:

$$D = \frac{4L}{ka^2}$$

$$D = \begin{cases} 0 & \text{phase perturbation} \\ \infty & \text{phase = amplitude} \end{cases}$$

when D<1, geometric ray theory is valid

Wave propagation problems can be classified using the parameters just introduced.

This classification is crucial for the choice of technique to calculate synthetic seismograms

(Adapted from Aki and Richards, 1980)

From scattering....

Multiple scattering process leads to **attenuation** (spatial loss non a true dissipative one) and **energy mean free path**
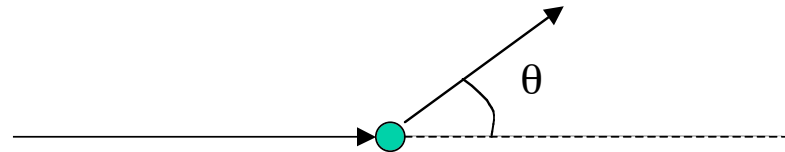
$\sigma(\theta)$ is the **differential scattering cross-section** and after a wave has travelled a distance x, the energy is reduced by an amount of

$$e^{-\Sigma x} \quad \Sigma = \int_{-1}^{+1}\sigma(\cos\theta)d\cos\theta$$

and the **average path length** between scattering events is

$$l = \int_{0}^{\infty} e^{-\Sigma x}dx = \frac{1}{\Sigma}$$

Towards **random** media

forward scattering tendency

$$\Sigma' = \int_{-1}^{+1} (\cos\theta)\sigma(\cos\theta)d\cos\theta \begin{cases} > 0 \text{ forward} \\ \approx 0 \text{ isotropic} \\ < 0 \text{ backward} \end{cases}$$

Multiple scattering randomizes the phases of the waves adding a diffuse (incoherent) component to the average wavefield.
Statistical approaches can be used to derive **elastic radiative transfer equations**

Diffusion constants
use the definition of a diffusion (transport) mean free path

$$d = \frac{cl^*}{3} \quad l^* = \frac{1}{\Sigma - \Sigma'} \text{ (acoustic)}$$

$$d = \frac{1}{1 + 2K^3}\left(\frac{c_p l_p^*}{3} + 2K^2 \frac{c_s l_s^*}{3}\right) \text{ (elastic)}$$

for non-preferential scattering l* coincides with energy mean free path, l
for enhanced forward scattering l*>l

Experiments for ultrasound in materials can be applied to seismological problems…

Scattering in random media

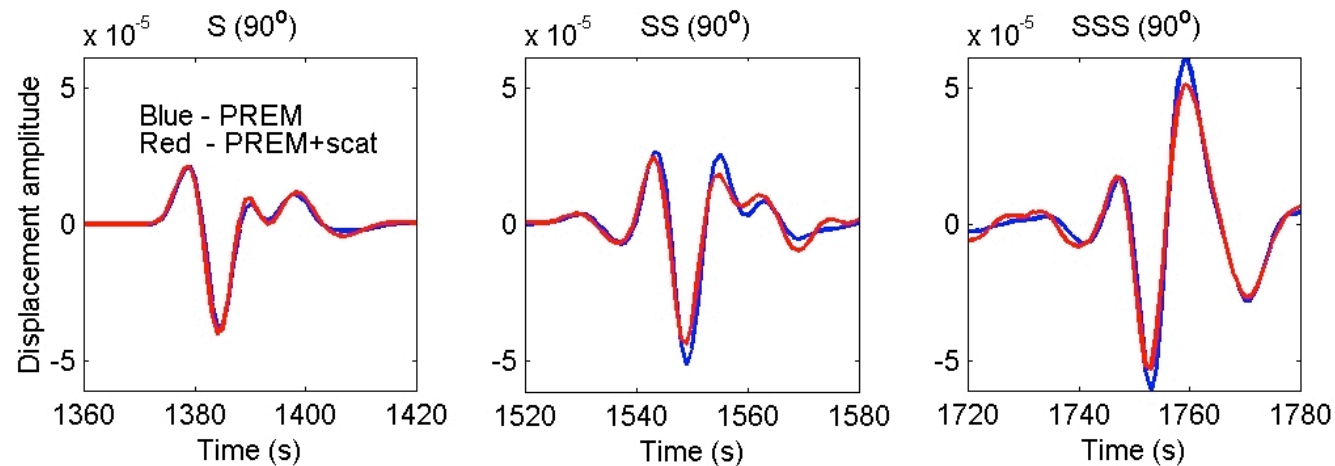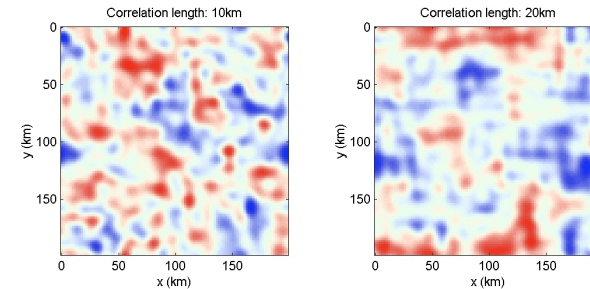Correlation length: 10km

Correlation length: 20km
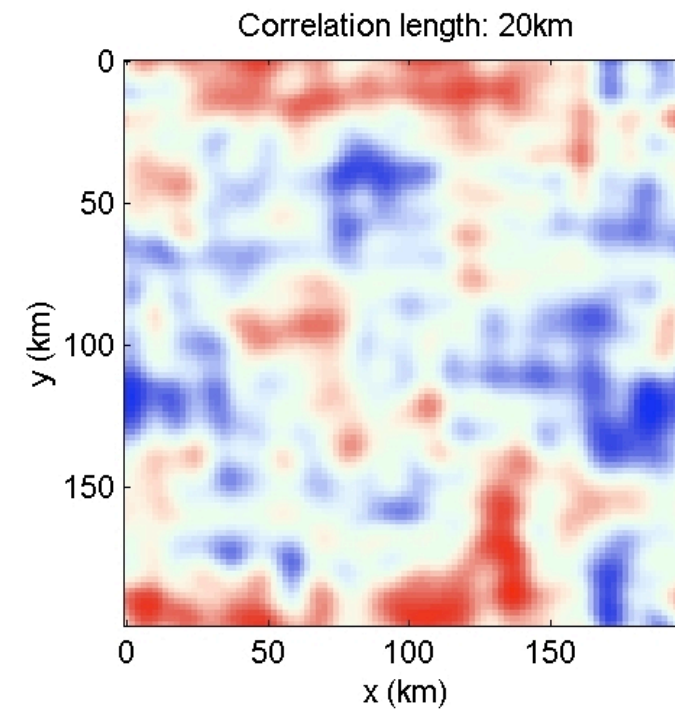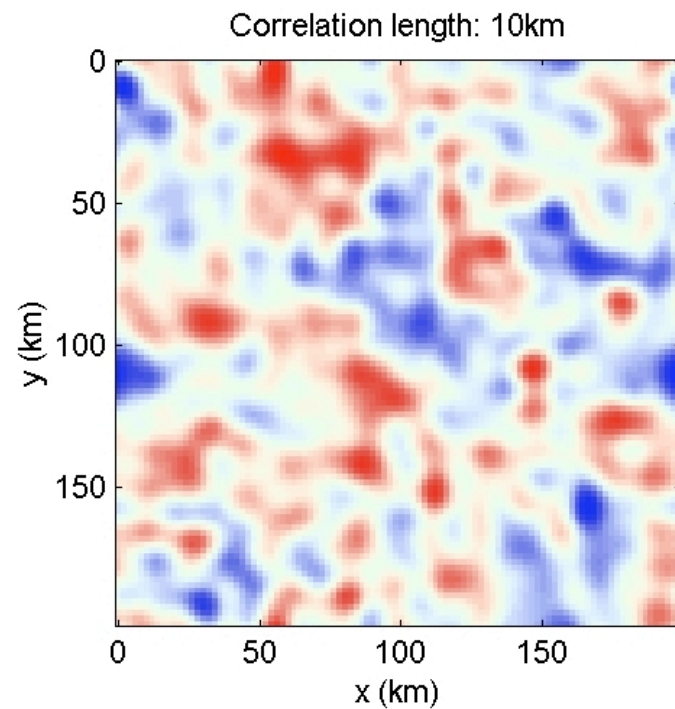
How is a propagating wavefield affected by random heterogeneities?
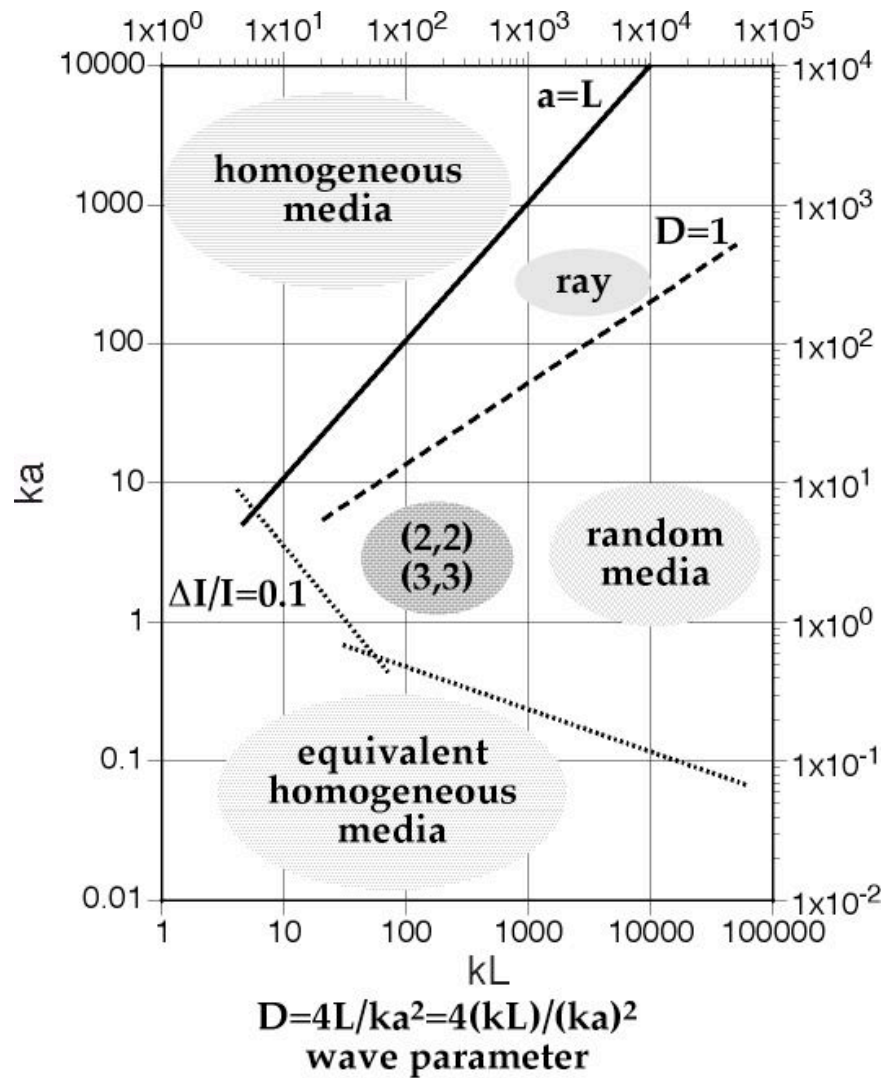
Synthetic seismograms

Synthetic seismograms for a global model with random velocity perturbations.

When the wavelength is long compared to the correlation length, scattering effects are difficult to distinguish from intrinsic attenuation.

Seismic Scattering Classification

D=4L/ka²=4(kL)/(ka)²
wave parameter

Wave propagation problems can be classified using the parameters just introduced.

This classification is crucial for the choice of technique to calculate synthetic seismograms

(Adapted from Aki and Richards, 1980)

Small perturbations of a **stable** equilibrium point $\longrightarrow$ **Linear restoring force** $\longrightarrow$ **Harmonic Oscillation**
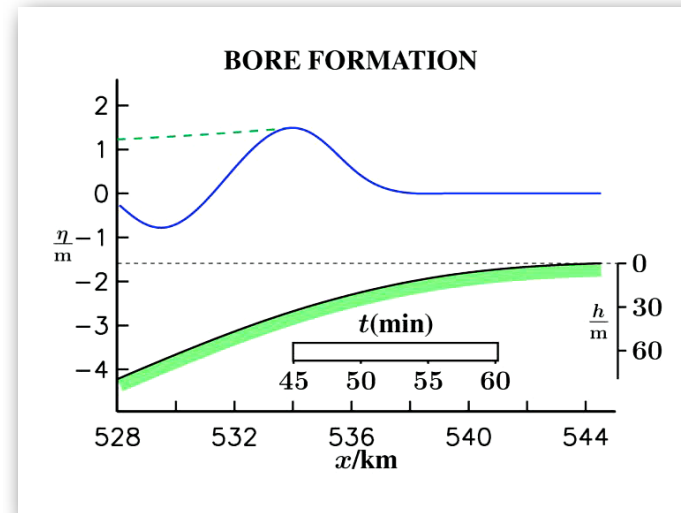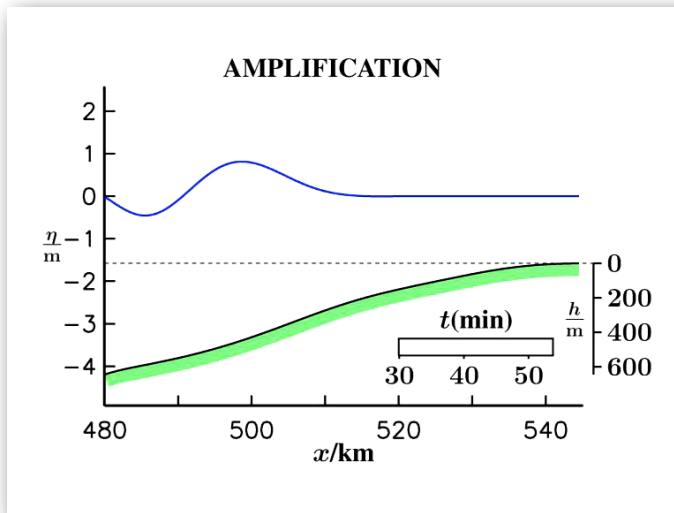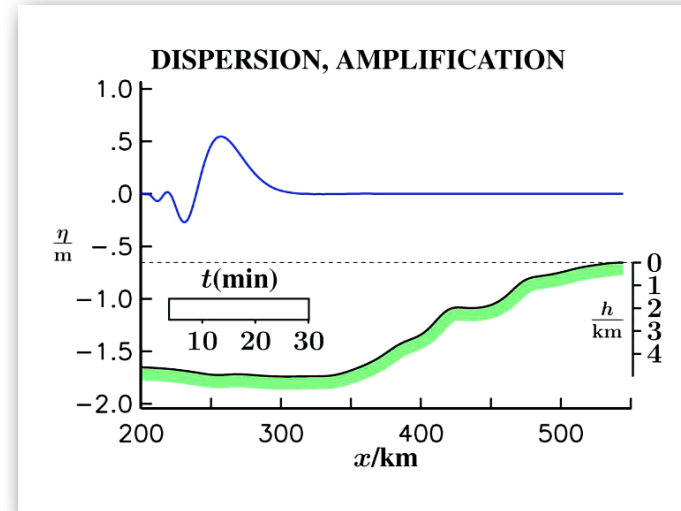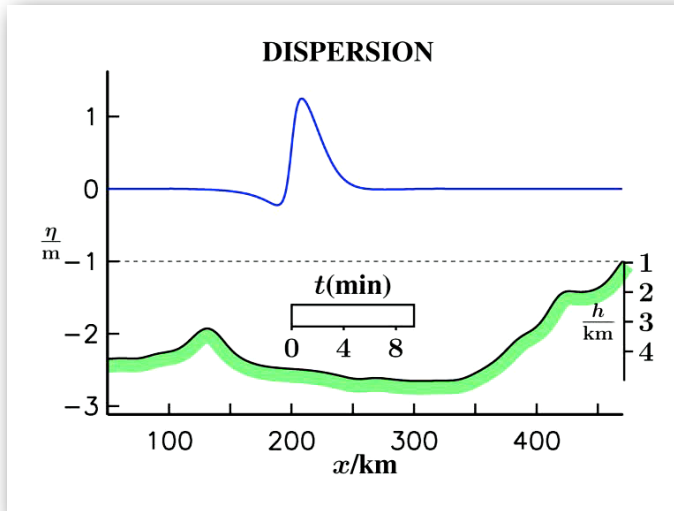
Coupling of harmonic oscillators $\longrightarrow$ the disturbances can propagate, superpose, stand, and **be dispersed**

WAVE: organized **propagating imbalance**, satisfying differential equations of motion

Organization can be destroyed, when interference is destructive

non linearity

strong scattering

Turbulence

Diffusion

Exceptions

Solitons

Phonons

Tsunami physics

Dispersion & Non linearity

The dynamics of water waves in shallow water is described mathematically by the Korteveg – de Vries (KdV) equation

u=u(x,t) measures the elevation at time t and position x, i.e. the height of the water above the equilibrium level
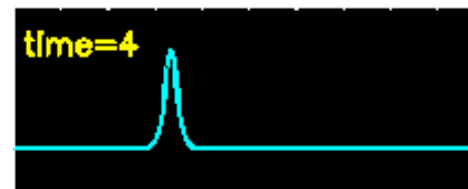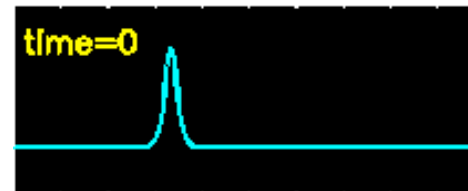
Dispersive term

$$u_t + u_{xxx} = 0$$

Nonlinearity

$$u_t + u\, u_x = 0$$

KdV

$$u_t + u_{xxx} + u\, u_x = 0$$

time=2

time=0

time=4

Selected References

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Scales, J., and Snieder, R., 1999. What is a wave?, Nature, 401, 739-740.

Snieder, R., 2002. General theory of elastic wave scattering , in Scattering and Inverse Scattering in Pure and Applied Science, Eds. Pike, R. and P. Sabatier, Academic Press, San Diego, 528-542.

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