The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

1965-18

9th Workshop on Three-Dimensional Modelling of Seismic Waves Generation, Propagation and their Inversion

22 September - 4 October, 2008

Tsunami Theory (a la Ward)
Lecture 1: Nuts and Bolts

Steven N. Ward

Institute of Geophysics
& Planetary Physics
University of California Santa Cruz
U.S.A.

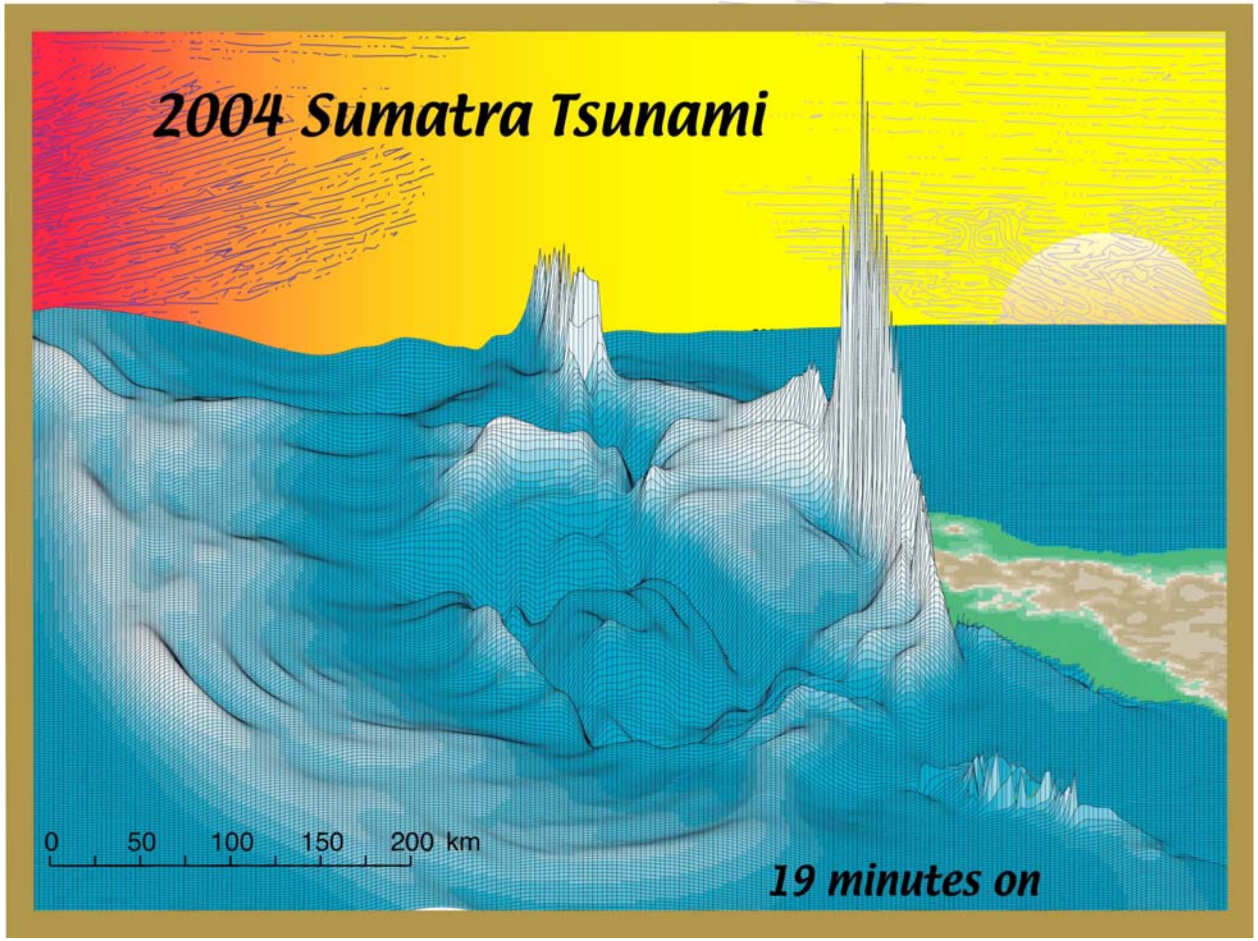
ward@pmc.ucsc.edu

Strada Costiera 11, 34014 Trieste, Italy - Tel. +39 040 2240 111; Fax +39 040 224 163 - sci_info@ictp.it, www.ictp.it

Tsunami Theory (a la Ward)
Lecture 1: Nuts and Bolts

ICTP, Trieste Italy 9/24/08

Steven N. Ward
Institute of Geophysics
and Planetary Physics
University of California
Santa Cruz



1. Formulation

1.1. Fluid dynamics starts with *Euler's equations*

$$\rho(\mathbf{r},t)\frac{D\mathbf{v}(\mathbf{r},t)}{Dt} = \nabla \bullet \mathbf{t}(\mathbf{r},t) + \rho(\mathbf{r},t)\mathbf{F}(\mathbf{r},t) \qquad (1.1.1)$$

and the *continuity equation*

$$\frac{D\rho(\mathbf{r},t)}{Dt} + \rho(\mathbf{r},t)\nabla \bullet \mathbf{v}(\mathbf{r},t) = 0 \qquad (1.1.2)$$

to be solved in fluid volume V.

Here $\rho(\mathbf{r},t)$ is density, $\mathbf{v}(\mathbf{r},t) = \frac{\partial \mathbf{u}(\mathbf{r},t)}{\partial t}$ is velocity, $\mathbf{u}(\mathbf{r},t)$ is displacement, $\mathbf{t}(\mathbf{r},t)$ is the stress tensor, $\mathbf{F}(\mathbf{r},t)$ is body force per unit mass,

and $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}(\mathbf{r},t) \bullet \nabla$

1.2. If *stress linearly relates to strain* and the fluid is *inviscid*, then the non-zero stress tensor elements are pressure p,

$$\mathbf{t}(\mathbf{r},t) = -p(\mathbf{r},t)\mathbf{I} \qquad (1.2.1)$$

and (1.1.1) become the *Navier-Stokes* equations

$$\rho(\mathbf{r},t)\frac{D\mathbf{v}(\mathbf{r},t)}{Dt} = -\nabla p(\mathbf{r},t) + \rho(\mathbf{r},t)\mathbf{F}(\mathbf{r},t) \qquad (1.2.2)$$

1.3. If the motions and body forces are *irrotational*

$$\begin{aligned}\frac{D\mathbf{v}(\mathbf{r},t)}{Dt} &= \frac{\partial\mathbf{v}(\mathbf{r},t)}{\partial t} + \mathbf{v}(\mathbf{r},t) \bullet \nabla\mathbf{v}(\mathbf{r},t) = \frac{\partial\mathbf{v}(\mathbf{r},t)}{\partial t} + \frac{1}{2}\nabla v^2(\mathbf{r},t) - \mathbf{v}(\mathbf{r},t) \times \nabla \times \mathbf{v}(\mathbf{r},t) \\ &= \frac{\partial\mathbf{v}(\mathbf{r},t)}{\partial t} + \frac{1}{2}\nabla v^2(\mathbf{r},t)\end{aligned} \qquad (1.3.1)$$

and

$$\mathbf{F}(\mathbf{r},t) = -\nabla\phi(\mathbf{r},t) = \mathbf{g}(\mathbf{r},t) \qquad (1.3.2)$$

then (1.2.2) become the *Bernoulli equations*

$$\rho(\mathbf{r},t)\frac{\partial\mathbf{v}(\mathbf{r},t)}{\partial t} = -\nabla p(\mathbf{r},t) - \frac{1}{2}\rho(\mathbf{r},t)\nabla v^2(\mathbf{r},t) - \rho(\mathbf{r},t)\nabla\phi(\mathbf{r},t) \qquad (1.3.3)$$

$$\rho(\mathbf{r},t)\frac{\partial \mathbf{v}(\mathbf{r},t)}{\partial t} = -\nabla p(\mathbf{r},t) - \frac{1}{2}\rho(\mathbf{r},t)\nabla v^2(\mathbf{r},t) - \rho(\mathbf{r},t)\nabla\phi(\mathbf{r},t) \qquad (1.3.3)$$

1.4. Although (1.3.3) already uses a linear constitutive law, *we carry the linearization* of (1.3.3) and (1.1.2) all the way through. Following seismological procedures (because I'm a seismologist) let

$$\begin{aligned} \rho(\mathbf{r},t) &= \rho_0(\mathbf{r}) + \rho_1(\mathbf{r},t) \\ p(\mathbf{r},t) &= p_0(\mathbf{r}) + p_1(\mathbf{r},t) \qquad (1.4.1) \\ \phi(\mathbf{r},t) &= \phi_0(\mathbf{r}) + \phi_1(\mathbf{r},t) \end{aligned}$$

where all the sub-0 quantities refer to the undisturbed state and the sub-1 quantities are small perturbations about the initial state. Placing (1,4.1) into (1.3.3) and (1 .1.2) and dropping products of sub-1 quantities (velocity **v** is assumed be of sub-1 size) gives

$$\rho_0(\mathbf{r})\ddot{\mathbf{u}}(\mathbf{r},t) = -\nabla p_1(\mathbf{r},t) + \nabla \bullet [\rho_0(\mathbf{r})\mathbf{u}(\mathbf{r},t)]\nabla\phi_0(\mathbf{r}) - \rho_0(\mathbf{r})\nabla\phi_1(\mathbf{r},t) \qquad (1.4.2)$$

$$\frac{\partial \rho_1(\mathbf{r},t)}{\partial t} + \nabla \bullet [\rho_0(\mathbf{r})\mathbf{v}(\mathbf{r},t)] = 0 \;\Rightarrow \rho_1(\mathbf{r},t) = -\nabla \bullet [\rho_0(\mathbf{r})\mathbf{u}(\mathbf{r},t)] \qquad (1.4.3)$$

$$\rho_0(\mathbf{r})\ddot{\mathbf{u}}(\mathbf{r},t) = -\nabla p_1(\mathbf{r},t) + \nabla \bullet [\rho_0(\mathbf{r})\mathbf{u}(\mathbf{r},t)]\nabla\phi_0(\mathbf{r}) - \rho_0(\mathbf{r})\nabla\phi_1(\mathbf{r},t) \qquad (1.4.2)$$

$$\frac{\partial\rho_1(\mathbf{r},t)}{\partial t} + \nabla \bullet [\rho_0(\mathbf{r})\mathbf{v}(\mathbf{r},t)] = 0 \;\Rightarrow \rho_1(\mathbf{r},t) = -\nabla \bullet [\rho_0(\mathbf{r})\mathbf{u}(\mathbf{r},t)] \qquad (1.4.3)$$

The equations now are expressed in displacement **u** instead of velocity **v**. In obtaining (1.4.2) we used (1.4.3) and assumed that the initial state was *hydrostatic equilibrium*

$$\nabla p_0(\mathbf{r}) = -\rho_0(\mathbf{r})\nabla\phi_0(\mathbf{r}) = \rho_0(\mathbf{r})\mathbf{g}_0(\mathbf{r}) \qquad (1.4.4)$$

Pressure increment $p_1(\mathbf{r},t)$ consists of an elastic term and an advected term

$$p_1(\mathbf{r},t) = -\kappa(\mathbf{r})\nabla\bullet\,\mathbf{u}(\mathbf{r},t) - \mathbf{u}(\mathbf{r},t)\bullet\nabla p_0(\mathbf{r}) \qquad (1.4.5)$$

The $\kappa(\mathbf{r})$ is fluid incompressibility. Equations (1.4.2), (1.4.5) together with

$$\nabla^2\phi_1(\mathbf{r},t) = -4\pi G\nabla\bullet[\rho_0(\mathbf{r})\mathbf{u}(\mathbf{r},t)] \qquad (1.4.6)$$

represent five equations for five unknown functions $\mathbf{u}(\mathbf{r},t)$, $p_1(\mathbf{r},t)$, $\phi_1(\mathbf{r},t)$ to be found in V.

2. Further Simplifications

2.1. Usually for tsunami calculations we take the media to be *homogeneous*

$$\kappa(\mathbf{r}) = \kappa, \quad \rho_0(\mathbf{r}) = \rho_0$$

and gravity to be constant and unchanging

$$\nabla\phi_0(\mathbf{r}) = -g\hat{\mathbf{z}}, \quad \phi_1(\mathbf{r}, t) = 0$$

The four equations of interest now are

$$\begin{aligned} \rho_0 \ddot{\mathbf{u}}(\mathbf{r},t) &= -\nabla p_1(\mathbf{r},t) - g\rho_0 \hat{\mathbf{z}} \nabla \bullet \mathbf{u}(\mathbf{r},t) \\ p_1(\mathbf{r},t) &= -\kappa \nabla \bullet \mathbf{u}(\mathbf{r},t) - g\rho_0 u_z(\mathbf{r},t) \end{aligned} \qquad (2.1.1)$$

or

$$\begin{aligned} \rho_0 \ddot{\mathbf{u}}(\mathbf{r},t) &= -\nabla p_e(\mathbf{r},t) + \rho_0 g[\nabla u_z(\mathbf{r},t) - \hat{\mathbf{z}} \nabla \bullet \mathbf{u}(\mathbf{r},t)] \\ p_e(\mathbf{r},t) &= -\kappa \nabla \bullet \mathbf{u}(\mathbf{r},t) \end{aligned} \qquad (2.1.2)$$

with

$$p_e(\mathbf{r},t) = p_1(\mathbf{r},t) + \mathbf{u}(\mathbf{r},t) \bullet \nabla p_0(\mathbf{r}) = -\kappa \nabla \bullet \mathbf{u}(\mathbf{r},t) \qquad (2.1.3)$$

Four equations (2.1.2a,b) and two seafloor/sea surface boundary conditions are the basis for rigorous tsunami calculations.

SIDE POINT.

Almost all of the "engineering type" of approachs to tsunami are based on (2.1.1) further assuming incompressibility and depth averaging (a.k.a. shallow water or long wave assumption). Let $\mathbf{v}(\mathbf{r},t) = \dot{\mathbf{u}}_h(\mathbf{r},t) + \hat{\mathbf{z}}\dot{u}_z(\mathbf{r},t)$ and **r**=(x,y,z) go to **r**=(x,y)

$$\dot{\mathbf{v}}_h(x,y,t) + \mathbf{v}_h(x,y,t) \bullet \nabla\mathbf{v}_h(x,y,t) = -\rho_0^{-1}\nabla p_1(x,y,t) = g\nabla u_z(x,y,t) \qquad (2.1.1b)$$

where we re-instated the advected part of the velocity change. This, plus the incompressible continuity condition taken at the "surface"

$$\nabla \bullet [H(x,y) + u_z(x,y,t)]\mathbf{v}_h(x,y,t) = \dot{u}_z(x,y,t) \qquad (2.1.1c)$$

represent the "non linear" shallow water equations and are solved by purely numerical means. $H(x,y)$ is the still water depth.

SIDE POINT (continued)

Sometimes (2.1.1b) and (2.1.1.c) are called the non linear shallow water wave equation. Why wave equation? If you take $\nabla\bullet$ of (2.1.1b), the time deriviative of (2.1.1.c) and insert into (2.1.1.b) you get after linearization

$$\ddot{u}_z(x,y,t) = gH(x,y)\nabla^2 u_z(x,y,t) + \dot{\mathbf{v}}_h(x,y,t)\nabla H(x,y) \qquad (2.1.1c)$$

If the ocean was uniform depth (2.1.1.c) would just reduce to the WAVE EQUATION with solutions

$$u_z(x,y,t) = F(t - \frac{R}{\sqrt{gH}})$$

That is, the surface motions of long wave tsunami would travel like an undispersed wave at speed $\sqrt{gH}$

3. Boundary Conditions-Classical Approach

In the linearization above, boundary conditions on deformed surfaces are evaluated on undeformed surfaces S_0. For inviscid fluids, $u_z(\mathbf{r},t)$ and $p_e(\mathbf{r},t)$ are continuous across originally flat laying boundaries between homogenous layers, i.e.

$$\left[u_z(\mathbf{r},t)\right]_-^+ \quad \left[p_e(\mathbf{r},t)\right]_-^+ \quad \text{on } S_0 \qquad (3.1.1\text{-}2)$$

Classical tsunami theory however, instead of solving equations (2.1.2)

$$\rho_0\ddot{\mathbf{u}}(\mathbf{r},t) = -\nabla p_e(\mathbf{r},t) + \rho_0 g[\nabla u_z(\mathbf{r},t) - \hat{\mathbf{z}}\nabla \bullet \mathbf{u}(\mathbf{r},t)]; \quad p_e(\mathbf{r},t) = -\kappa\nabla \bullet \mathbf{u}(\mathbf{r},t) \qquad (2.1.2)$$

with simple boundary conditions (3.1.1-2) rather solves a simpler set of equations

$$\rho_0\ddot{\mathbf{u}}(\mathbf{r},t) = -\nabla p_e(\mathbf{r},t); \quad p_e(\mathbf{r},t) = -\kappa\nabla \bullet \mathbf{u}(\mathbf{r},t) \qquad (3.1.3a,b)$$

with more complex boundary conditions

$$\left[u_z(\mathbf{r},t)\right]_-^+ \quad \left[p_1(\mathbf{r},t)\right]_-^+ = \left[p_e(\mathbf{r},t) + \rho_0 g u_z(\mathbf{r},t)\right]_-^+ \text{ on } S_0 \ (3.1.4\text{-}5)$$

This approximation takes all of gravity's effects in the body of the fluid (note that g does not appear in 3.1.3a,b) and "compresses" them onto boundaries of fluid layers of different density. [Don't confuse the p_1 in (3.1.5) with the p_1 in (1.4.5) as the second term differs in sign.] The effectiveness of the classical approach can be gauged later by comparing the analytical solutions to (3.1.3a,b) and (3.1.4-5) to numerical solutions of (2.1.2) and (3.1.1-2).

4. Two dimensional solutions.

4.1 Let's first solve some tsunami problems in two dimensions. Extensions to three dimensions are straightforward and make heavy use the 2-D results. Let coordinate x be horizontal, coordinate z be directed downward, $\mathbf{g}=g\hat{\mathbf{z}}$, u_y and all $\partial/\partial y$ =0. Equations (3.1.3a,b) become

$$\begin{aligned} \rho_0 \ddot{u}_z(x,z,t) &= -\partial p_e(x,z,t)/\partial z \\ \rho_0 \ddot{u}_x(x,z,t) &= -\partial p_e(x,z,t)/\partial x \\ p_e(x,z,t) &= -\kappa[\partial u_x(x,z,t)/\partial x + \partial u_z(x,z,t)/\partial z] \end{aligned} \qquad (4.1.1)$$

Because we are working with linear equations, we can make use of superposition both in frequency and wavenumber. Let new wavenumber-frequency variables be transforms of space-time variables like

$$f(k,z,\omega) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt\, f(x,z,t) e^{-i(kx-\omega t)}; \qquad (4.1.2a)$$

These are reconstituted by

$$f(x,z,t) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\omega\, f(k,z,\omega) e^{i(kx-\omega t)} \qquad (4.1.2b)$$

With this convention in (4.1.1) $\partial / \partial t \Rightarrow -i\omega$ and $\partial / \partial x \Rightarrow ik$

$$\frac{\partial}{\partial z}\begin{bmatrix} u_z(k,z,\omega) \\ p_e(k,z,\omega) \end{bmatrix} = \begin{bmatrix} 0 & \eta^2/\rho_0\omega^2 \\ \rho_0\omega^2 & 0 \end{bmatrix}\begin{bmatrix} u_z(k,z,\omega) \\ p_e(k,z,\omega) \end{bmatrix} \quad (4.1.3)$$

where $\eta^2 = \eta^2(\omega, k) = k^2 - \rho_0\omega^2/\kappa$ and use was made of

$$u_x(k,z,\omega) = ikp_e(k,z,\omega)/\rho_0\omega^2 \quad (4.1.4)$$

In linear theory, horizontal tsunami motions are not independent, but can be found from p_1 and u_z once they are known. (4.1.3) is exactly the same equations as used in seismic wave propagation in fluid layer.

Given u_z and p_1 and any depth z_0, equations (4.1.3) tell us how to find u_z and p_1 and any other depth z

$$\begin{bmatrix} u_z(k,z,\omega) \\ p_e(k,z,\omega) \end{bmatrix} = \begin{bmatrix} C(z,z_0) & \eta S(z,z_0)/\rho_0\omega^2 \\ \rho_0\omega^2 S(z,z_0)/\eta & C(z,z_0) \end{bmatrix}\begin{bmatrix} u_z(k,z_0,\omega) \\ p_e(k,z_0,\omega) \end{bmatrix} \quad (4.1.7)$$

where $C(z,z_0)=\cosh[\eta(z-z_0)]$ and $S(z,z_0)=\sinh[\eta(z-z_0)]$. We can also write the solutions (4.1.7) in terms of u_z and $p_1(k,z,\omega) = p_e(k,z,\omega) + u_z(k,z,\omega)\rho_0 g$ that are continuous across the undeformed surfaces

$$\begin{bmatrix} u_z(k,z,\omega) \\ p_1(k,z,\omega) \end{bmatrix} = \begin{bmatrix} C(z,z_0) - g\eta S(z,z_0)/\omega^2 & \eta S(z,z_0)/\rho_0\omega^2 \\ \dfrac{\rho_0 S(z,z_0)}{\eta}\left(\omega^2 - g^2\eta^2/\omega^2\right) & C(z,z_0) + g\eta S(z,z_0)/\omega^2 \end{bmatrix}\begin{bmatrix} u_z(k,z_0,\omega) \\ p_1(k,z_0,\omega) \end{bmatrix} \quad (4.1.8)$$

4.2 In all of the cases considered here, we employ the "decoupled" approach that assumes that the tsunami motions do not reach into the elastic space below the ocean. That is, the vertical displacement at the sea floor

$$u_z(k,H,\omega) = \text{always specified} \quad (4.2.1)$$

zero or otherwise. At the sea surface, the linearized boundary condition (3.1.5) says that

$$p_1(k,0,\omega)=0 \quad (4.2.2)$$

With p_1 at the sea surface and u_z at the sea floor specified, we are ready to use (4.1.8) to solve some tsunami excitation problems.

4.3 Tsunami Dispersion relation. In an ocean of depth H, consider the second equation in (4.1.8) at z=0 with (4.2.2) and a rigid bottom condition (4.2.1)

$$0 = [\cosh(\eta H) - g\eta \sinh(\eta H)/\omega^2]p_1(k,H,\omega) \quad (4.3.1)$$

The only way this can hold is if

$$\omega^2 = g\eta(k,\omega)\tanh[\eta(k,\omega)H] \quad (4.3.2)$$

The frequency $\omega(k)$ for a given wavenumber k, or the wavenumber $k(\omega)$ at a given frequency form the tsunami dispersion relationship in a *compressible ocean* of depth H. Although it is not a necessary assumption in our theory, often we take the ocean as *incompressible*. In this case

$$\kappa \Rightarrow \infty \text{ and } \eta = \sqrt{k^2 - \rho_0\omega^2/\kappa} \Rightarrow k \quad (4.3.3)$$

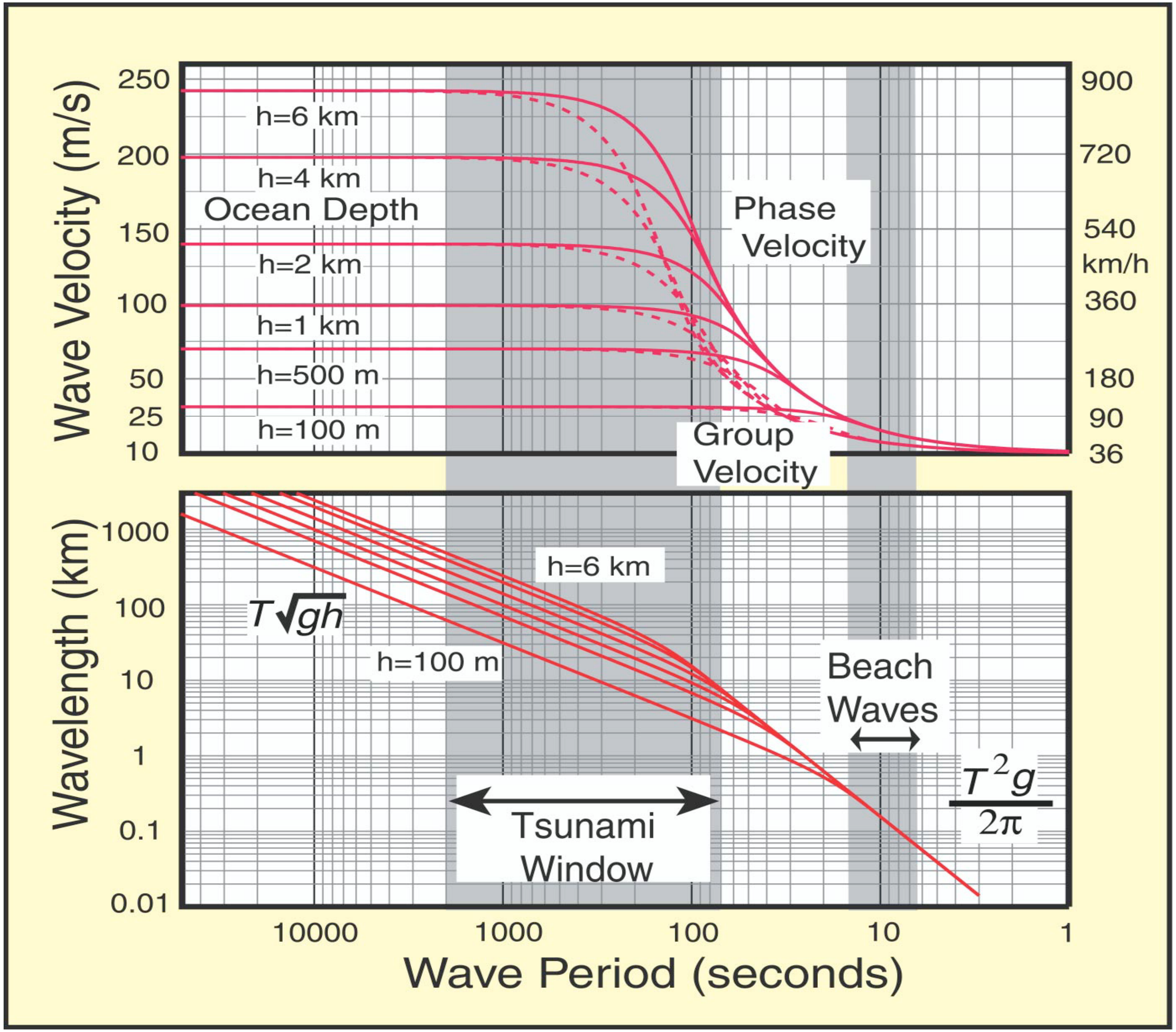
$$\omega^2 = gk\tanh[kH] \quad (4.3.4)$$

Tsunami Modes have NO OVERTONES. Only one frequency satisfies (4.3.4) for each wavenumber.

Note for long waves kH<<1 in incompressible water (4.3.4) reduces to
$\omega^2 \sim gk^2H$ so $c = \omega/k \sim \sqrt{gH}$ as we have already established.

Tsunami waves have far greater period, faster speed, and longer wavelength than familiar beach waves

In deep ocean, a 10m wave rising over 50 km is as flat as Kansas.



Ships at sea don't see tsunami.

4.4 Tsunami Eigenfunctions. Supposing $|u_z|=1$ at the sea surface z=0 and conditions (4.2.2) and (4.3.2), the displacements and pressures at any depth z are from (4.1.8) are

$$u_z(z,\omega)=\frac{k(\omega)g}{\omega^2}\frac{\sinh(k(\omega)(H-z))}{\cosh(k(\omega)H)} \qquad u_x(z,\omega)=-\frac{ik(\omega)g}{\omega^2}\frac{\cosh(k(\omega)(H-z)}{\cosh(k(\omega)H)}$$

$$p_e(z,\omega)=-\rho_0 g\frac{\cosh(k(\omega)(H-z)}{\cosh(k(\omega)H)} \qquad p_1(z,\omega)=-\rho_0 g\frac{\sinh(k(\omega)z))}{\cosh(k(\omega)H)\sinh(k(\omega)H)}$$

(4.4.1)

Clearly, $u_z(H,\omega)=p_1(0,\omega)=0$ as required and $p_e(0,\omega)=-\rho_0 g u_z(0,\omega)$ since $p_1(\omega,z)=p_e(\omega,z)+\rho_0 g u_z(\omega,z)$ for any z. In terms of tsunami "eigenmodes" we just tack on exp(-i[k(ω)x-ωt])

$$u_z(x,z,t,\omega)=\frac{k(\omega)g}{\omega^2}\frac{\sinh(k(\omega)(H-z))}{\cosh(k(\omega)H)}e^{i(k(\omega)x-\omega t)}$$

$$u_x(x,z,t,\omega)=-\frac{ik(\omega)g}{\omega^2}\frac{\cosh(k(\omega)(H-z)}{\cosh(k(\omega)H)}e^{i(k(\omega)x-\omega t)} \qquad (4.4.2)$$

$$p_e(x,z,t,\omega)=-\rho_0 g\frac{\cosh(k(\omega)(H-z)}{\cosh(k(\omega)H)}e^{i(k(\omega)x-\omega t)}$$

$$p_1(x,z,t,\omega)=-\rho_0 g\frac{\sinh(k(\omega)z))}{\cosh(k(\omega)H)\sinh(k(\omega)H)}e^{i(k(\omega)x-\omega t)}$$

Taking the real part of (4.4.2): $u_z \sim \cos(k(\omega)x-\omega t)$ and $u_x \sim \sin(k(\omega)x-\omega t)$. You can see that tsunami motion is a prograde ellipse.

Vertical Structure of Wave field decomposed by Eigenmodes, propagated individually , then reconstructed at specified receiver point.

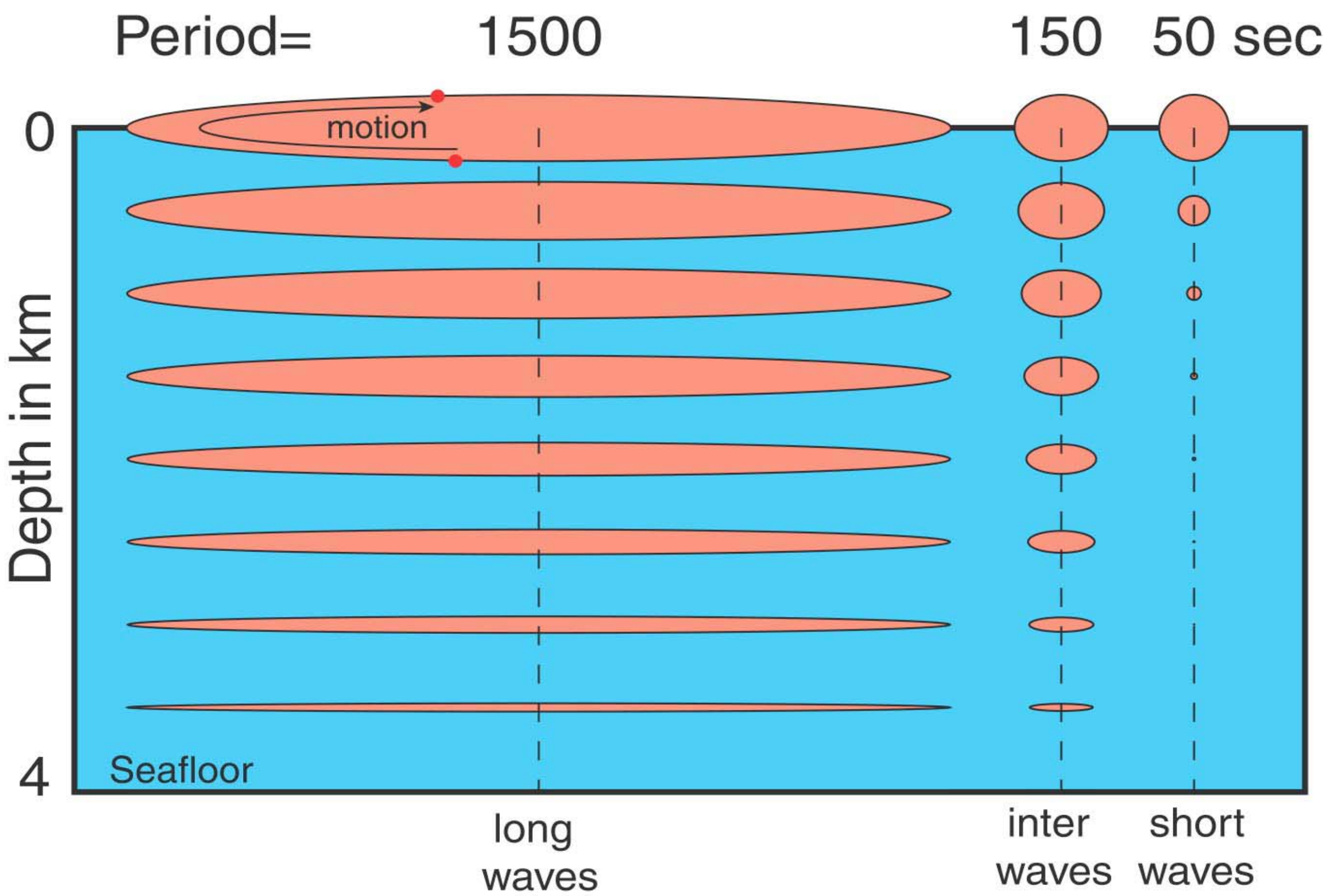
$$u_z(x,z,t,\omega) = \frac{k(\omega)g}{\omega^2}\frac{\sinh(k(\omega)(H-z))}{\cosh(k(\omega)H)}e^{i(k(\omega)x-\omega t)}$$

$$u_x(x,z,t,\omega) = -\frac{ik(\omega)g}{\omega^2}\frac{\cosh(k(\omega)(H-z))}{\cosh(k(\omega)H)}e^{i(k(\omega)x-\omega t)}$$

Unlike regular waves, tsunami reach all the way to the sea bottom.

Generate Bedforms

--You can't out dive a tsunami.



5 Specific 2-D Problems.

<u>5.1 Initial Value Problems at the sea surface</u>. For an asteroid impact tsunami, you might select sea surface displacement to reproduce initial transient cavity shapes given by experiment or by full-blown hydrodynamic simulations of impacts. If so, we specify an **<u>initial vertical surface displacement condition</u>** like

$$u_z(x,0,t=0) = u_z^{top}(x) \quad (5.1.1)$$

and its transform

$$u_z^{top}(k) = \int_{-\infty}^{\infty} dx \; u_z^{top}(x)\, e^{-ikx} \quad (5.1.2)$$

In this case, the eigenmodes (4.4.2) give the evolved tsunami straight away [From now on, I assume an incompressible fluid so $\eta(k(\omega)) = k(\omega)$]

$$\mathbf{u}(x,z,t) = \mathrm{Re}\int_{-\infty}^{\infty} dk \frac{u_z^{top}(k)}{2\pi}\left[\hat{\mathbf{z}}\frac{\sinh(k(H-z))}{\sinh(kH)} - i\hat{\mathbf{x}}\frac{\cosh(k(H-z))}{\sinh(kH)}\right]e^{i(kx-\omega(k)t)}$$

or changing the integration varible to frequency

$$\mathbf{u}(x,z,t) = \mathrm{Re}\int_{-\infty}^{\infty} d\omega \frac{u_z^{top}(k(\omega))}{2\pi u(\omega)}\left[\hat{\mathbf{z}}\frac{\sinh(k(\omega)(H-z))}{\sinh(k(\omega)H)} - i\hat{\mathbf{x}}\frac{\cosh(k(\omega)(H-z))}{\sinh(k(\omega)H)}\right]e^{i(k(\omega)x-\omega t)}$$

OK. We can do our first tsunami simulation already.

$$u_z(x,0,t) = \mathrm{Re}\int_{-\infty}^{\infty} dk\, \frac{u_z^{top}(k)}{2\pi}\, e^{i(kx-\omega(k)t)} \qquad (5.1.3)$$

For vertical motion at the surface, just evaluate this integral at different times.

Piece of Cake.

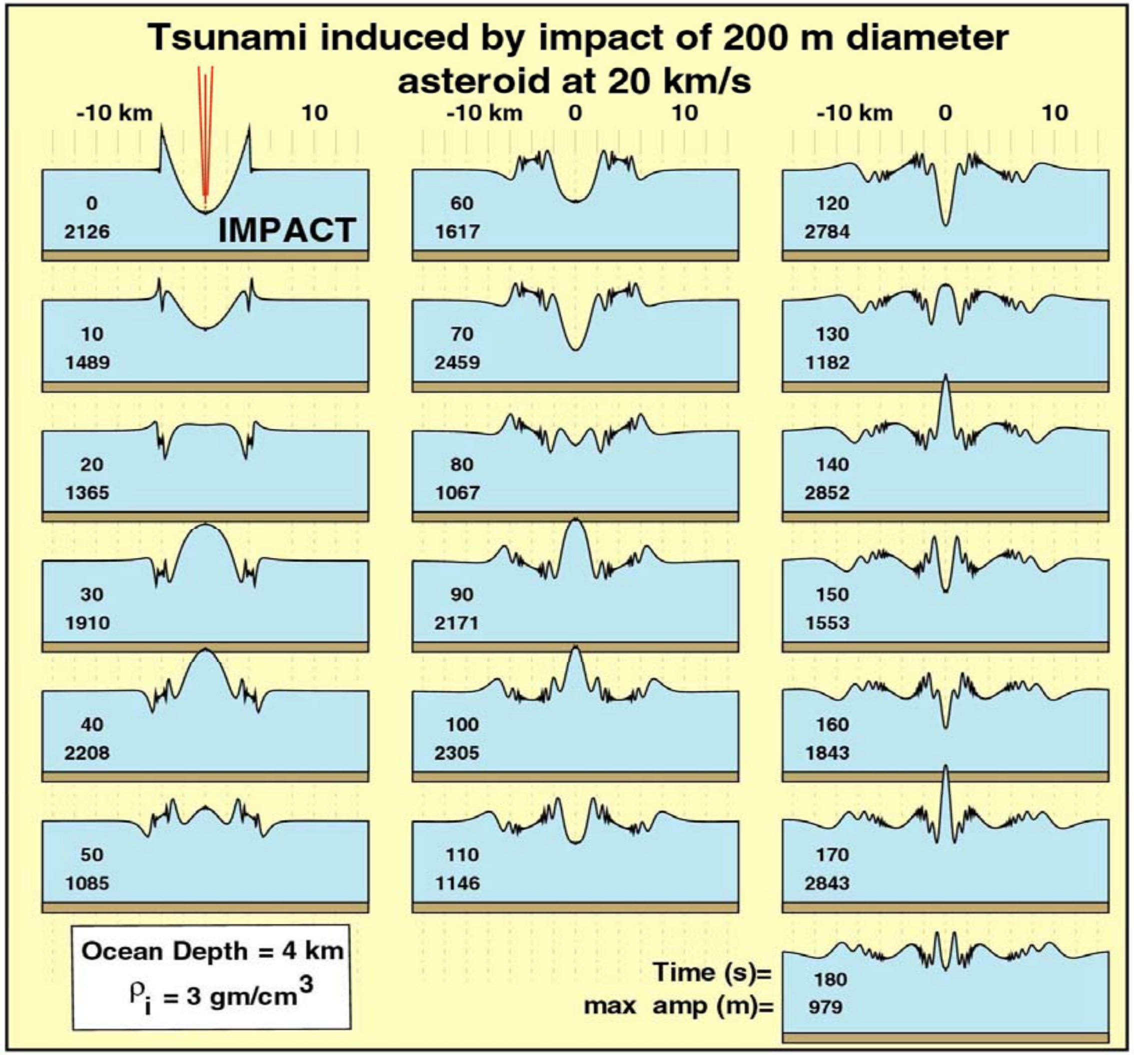
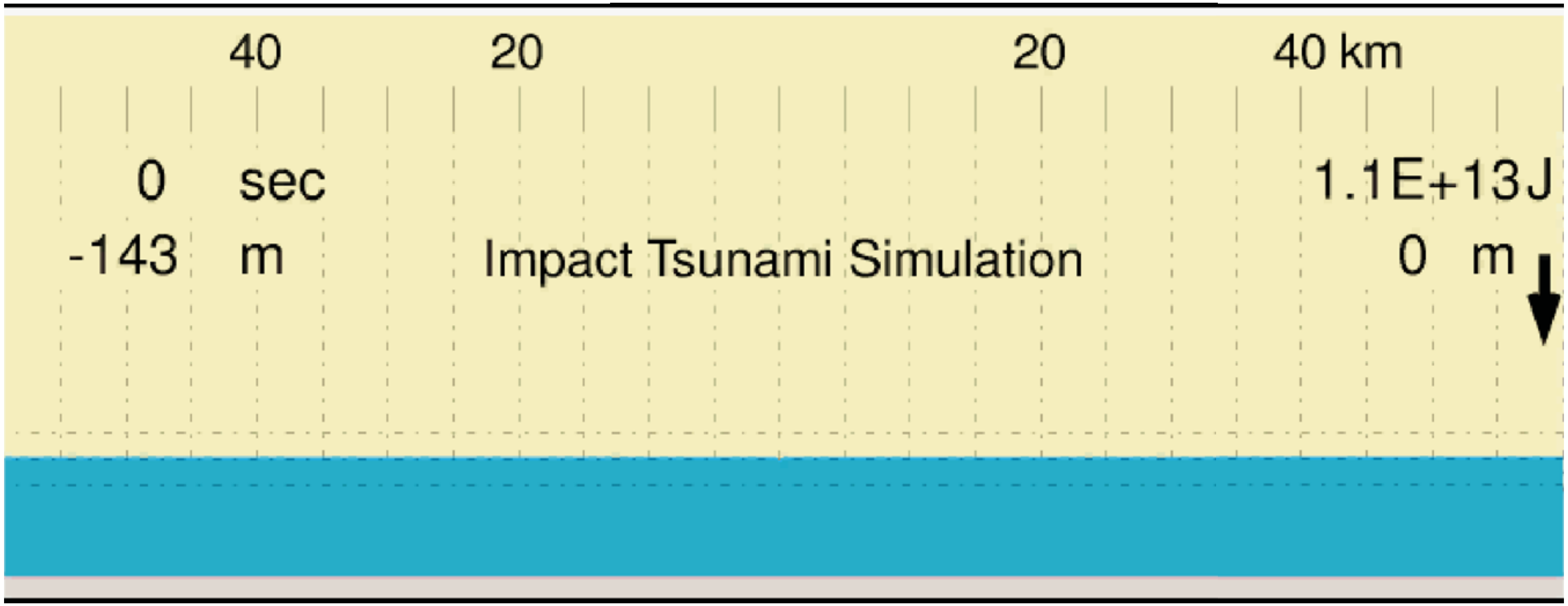


Figure 1. Equation (5.1.3a) evaluated with a parabolic initial displacement of the sea surface. This is my concept of asteroid impact tsunami.

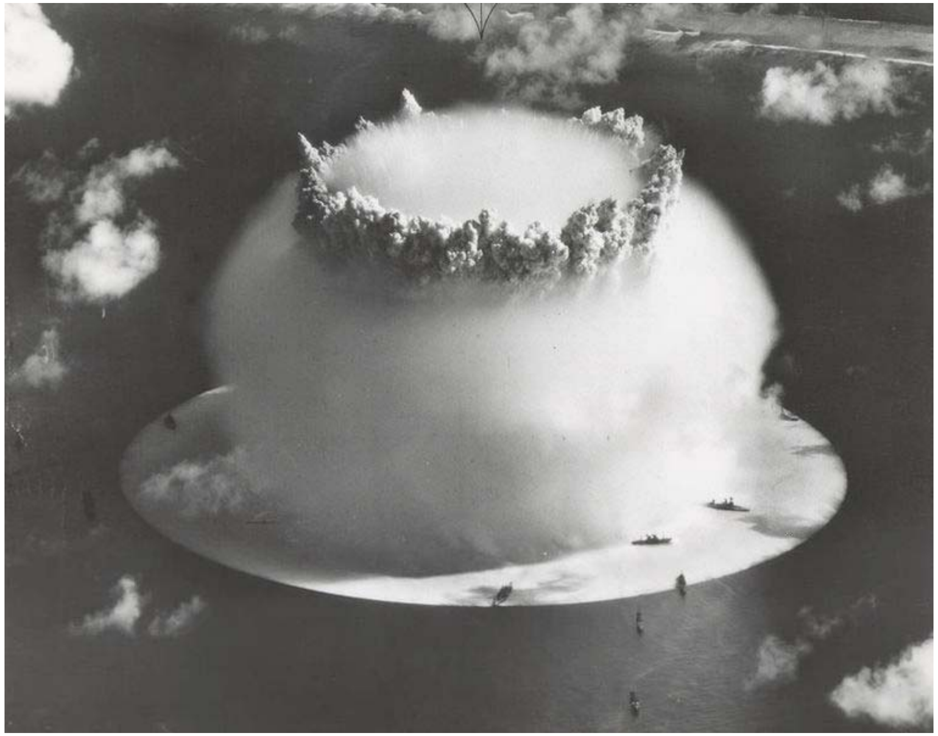


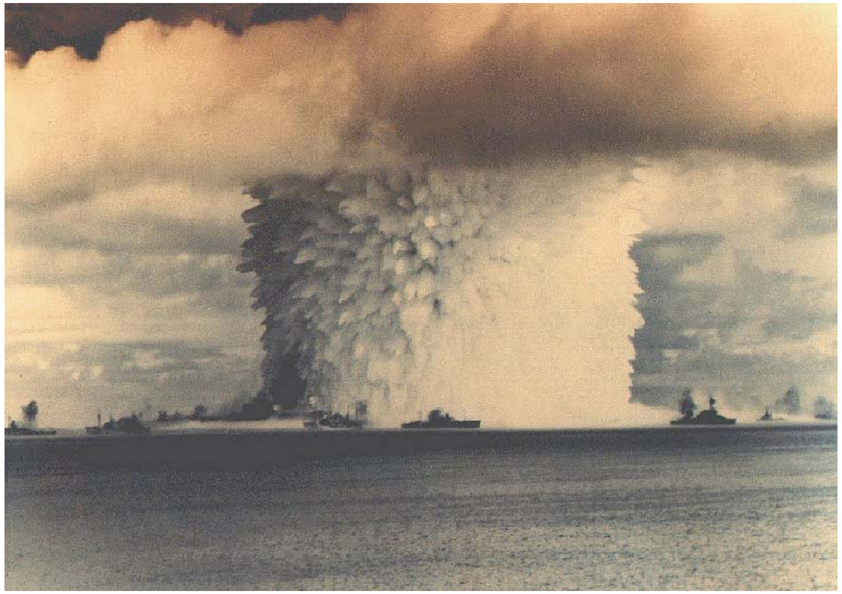
Most Impact tsunami are very dispersive because they fall on the "shoulder" of the dispersion curve.

Long periods travel faster than short periods.

Dispersion reduces tsunami size with distance.

Impact tsunami size falls faster with distance than EQ tsunami.





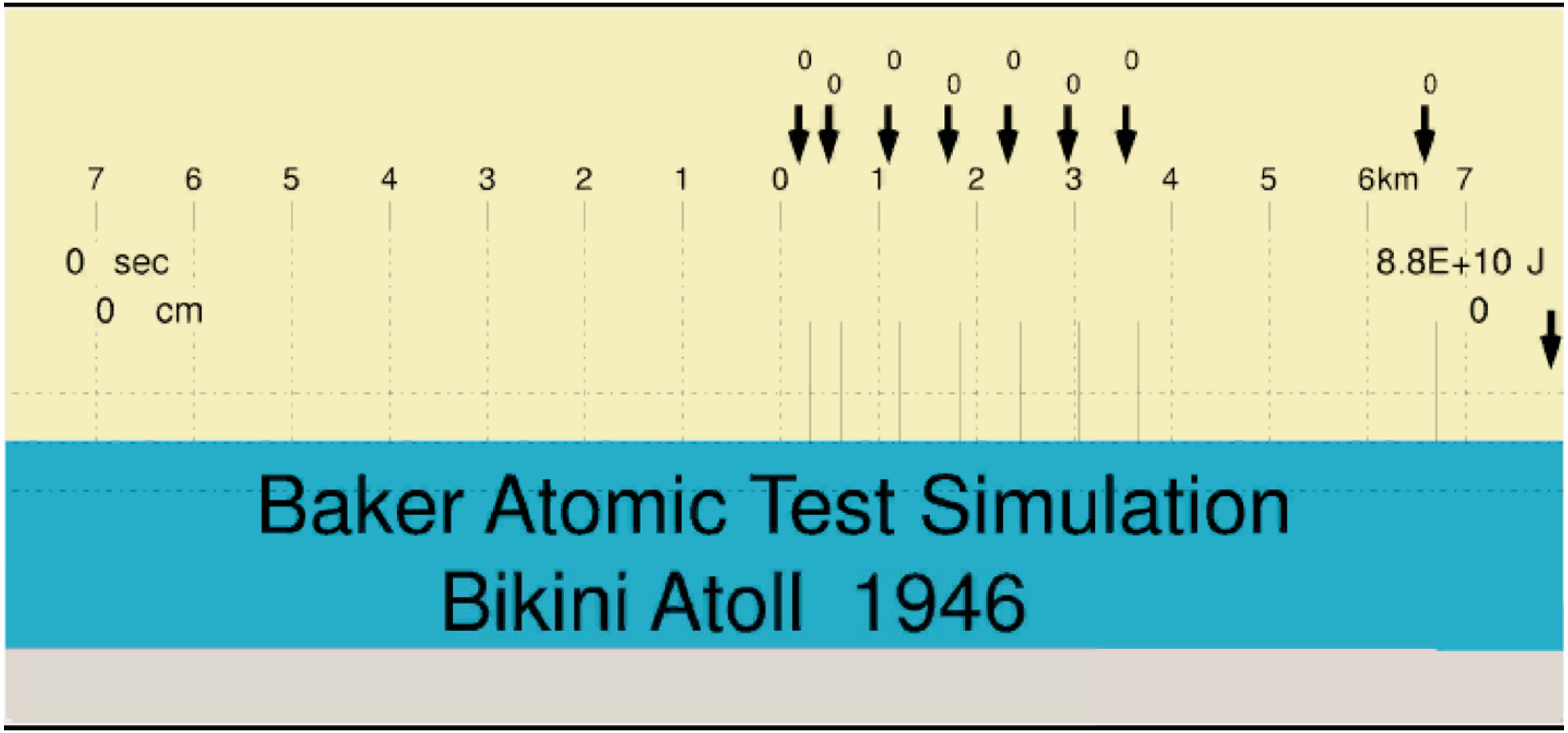
We have some feeling for impact-like effects from nuclear tests.

Baker Test: 1946

Bikini Atoll, Y=23Kt

Cavity Diameter ~ 1000 feet

Local Tsunami Height ~100 feet



How do we know simple linear theory is applicable? Let's scale the initial cavity size to the 23kt Baker Test and run the waves in the same depth of water as done in 1946.

S1 NUMERICAL NUGGET - a.k.a. Nut and Bolt.

The dispersion relation $\omega^2(\mathrm{k}) = \mathrm{gk}\tanh[\mathrm{kh}]$ makes it easy to find frequency ω(k) given a wavenumber k, but what if we are given a frequency ω and need wavenumber k(ω)? Do we try to find solutions to

$$\omega^2 = \mathrm{gk}(\omega)\tanh[\mathrm{k}(\omega)\mathrm{h}] \quad \text{(S1.1)}$$

numerically? Nope, that would take too long. Instead, a first guess at k(ω) is

$$\mathrm{k}_0(\omega) = \hat{\omega}^2\left(\tanh[\hat{\omega}^2\mathrm{h}]^{3/4}\right)^{-2/3}; \qquad \hat{\omega}^2 = \omega^2/g$$

a refined estimate $\mathrm{k}_1(\omega)=\mathrm{k}_0(\omega)+\delta\mathrm{k}(\omega)$ is found by placing this into (S.1.1), linearizing, and solving for δk(ω).

$$\mathrm{k}(\omega) \approx \mathrm{k}_1(\omega) = \mathrm{k}_0(\omega)\left[1 + \frac{\hat{\omega}^2 - \hat{\omega}_0^2}{\mathrm{k}_0^2(\omega)\mathrm{h} + \hat{\omega}_0^2\left(1-\hat{\omega}_0^2\mathrm{h}\right)}\right] \quad \text{(S1.2)}$$

where $\hat{\omega}_0^2 = \mathrm{k}_0(\omega)\tanh[\mathrm{k}_0(\omega)\mathrm{h}]$. I find (S1.2) sufficient for all my purposes, but you could substitute $\mathrm{k}_1(\omega)$ *and* $\hat{\omega}_1^2$ for $\mathrm{k}_0(\omega)$ *and* $\hat{\omega}_0^2$ and evaluate (S1.2) again.

Suppose instead we have some **initial vertical surface velocity condition** like

$$\dot{u}_z(x,0,t=0) = \dot{u}_z^{top}(x) \quad (5.1.5)$$

with its transform

$$\dot{u}_z^{top}(k) = \int_{-\infty}^{\infty} dx\, \dot{u}_z^{top}(x)\, e^{-ikx} \quad (5.1.6)$$

The same reasoning suggests that tsunami the displacement field would be

$$\mathbf{u}(x,z,t) = \mathrm{Re}\int_{-\infty}^{\infty} d\omega \frac{\dot{u}_z^{top}(k(\omega))}{-i\omega 2\pi u(\omega)} \left[\hat{\mathbf{z}} \frac{\sinh(k(\omega)(H-z))}{\sinh(k(\omega)H)} - i\hat{\mathbf{x}} \frac{\cosh(k(\omega)(H-z))}{\sinh(k(\omega)H)} \right] e^{i(k(\omega)x-\omega t)}$$

or (5.1.8)

$$\mathbf{u}(x,z,t) = \mathrm{Re}\int_{-\infty}^{\infty} dk \frac{\dot{u}_z^{top}(k)}{-i\omega(k)2\pi} \left[\hat{\mathbf{z}} \frac{\sinh(k(H-z))}{\sinh(kH)} - i\hat{\mathbf{x}} \frac{\cosh(k(H-z))}{\sinh(kH)} \right] e^{i(kx-\omega(k)t)}$$

(5.1.8) follows from application of $(-i\omega)^{-1}$ in the frequency domain is the same as integration in the time domain. (5.1.3) and (5.1.8) are in fact independent solutions so that they may be combined like

$$\mathbf{u}(x,0,t) = \mathrm{Re}\int_{-\infty}^{\infty} d\omega \frac{\left\{u_z^{top}(k(\omega)) + i\dot{u}_z^{top}(k(\omega))/\omega\right\}}{2\pi u(\omega)} \left[\hat{\mathbf{z}} \frac{\sinh(k(\omega)(H-z))}{\sinh(k(\omega)H)} - i\hat{\mathbf{x}} \frac{\cosh(k(\omega)(H-z))}{\sinh(k(\omega)H)} \right] e^{i(k(\omega)x-\omega t)}$$

or (5.1.9)

$$\mathbf{u}(x,0,t) = \mathrm{Re}\int_{-\infty}^{\infty} dk \frac{\left\{u_z^{top}(k) + i\dot{u}_z^{top}(k)/\omega(k)\right\}}{2\pi} \left[\hat{\mathbf{z}} \frac{\sinh(k(\omega)(H-z))}{\sinh(k(\omega)H)} - i\hat{\mathbf{x}} \frac{\cosh(k(\omega)(H-z))}{\sinh(k(\omega)H)} \right] e^{i(kx-\omega(k)t)}$$

(5.1.9) states the ***tsunami initial value problem.*** It says "Given the vertical displacement and vertical velocity OF THE SEA SURFACE AT ANY ONE TIME, (5.1.9) can be used to find the displacement and velocity of the sea AT ANY DEPTH AT ANY TIME LATER.

This is important. For landslide sources for instance, if you can specify sea surface conditions just once after the slide is done, then you can use (5.1.9) to propagate the waves anytime further.

Too (5.1.9) explains why workers who employ "initial static lumps of water" as tsunami sources can't correctly model many situations. Given a fixed lump, different selections of initial velocity can give totally different tsunami motions.

5.2 Finite duration sources. Suppose now that we have some **vertical surface displacement condition** that takes place over a *finite period of time* t>0 like

$$u_z(x,0,t) = u_z^{top}(x,t)\,H(t) \qquad (5.2.1)$$

The convolution theorem tells us how to form the tsunami fields given (5.1.3)

$$\mathbf{u}(x,z,t) = \mathrm{Re}\int_{-\infty}^{\infty} d\omega \frac{1}{2\pi u(\omega)}\left[\hat{\mathbf{z}}\frac{\sinh(k(\omega)(H-z))}{\sinh(k(\omega)H)} - i\hat{\mathbf{x}}\frac{\cosh(k(\omega)(H-z))}{\sinh(k(\omega)H)}\right] e^{i(k(\omega)x-\omega t)}$$

$$\times\int_{-\infty}^{\infty} dx_0 \int_0^t dt_0 \;\; \dot{u}_z^{top}(x_0,t_0)\, e^{-i(k(\omega)x_0-\omega t_0)}$$

or (5.2.2)

$$\mathbf{u}(x,z,t) = \mathrm{Re}\int_{-\infty}^{\infty} dk \frac{1}{2\pi}\left[\hat{\mathbf{z}}\frac{\sinh(k(H-z))}{\sinh(kH)} - i\hat{\mathbf{x}}\frac{\cosh(k(H-z))}{\sinh(kH)}\right] e^{i(kx-\omega(k)t)}$$

$$\times\int_{-\infty}^{\infty} dx_0 \int_0^t dt_0 \;\; \dot{u}_z^{top}(x_0,t_0)\, e^{-i(kx_0-\omega(k)t_0)}$$

Be aware of the time differentiation of the surface condition in (5.2.2).

$$\mathbf{u}(x,z,t) = \mathrm{Re}\int_{-\infty}^{\infty} dk \frac{1}{2\pi}\left[\hat{\mathbf{z}}\frac{\sinh(k(H-z))}{\sinh(kH)} - i\hat{\mathbf{x}}\frac{\cosh(k(H-z))}{\sinh(kH)}\right] e^{i(kx-\omega(k)t)}$$
$$\times \int_{-\infty}^{\infty} dx_0 \int_0^t dt_0 \;\; \dot{u}_z^{top}(x_0,t_0)\, e^{-i(kx_0-\omega(k)t_0)} \qquad (5.2.2)$$

Sometimes, $u_z^{top}(x,t)$ can be simplified such that one or both of the sub-0 integrals above can be done by hand. For instance, for a propagating source all the time histories of uplift at different points might be the same within a constant factor, only delayed in time.

$$u_z^{top}(x,t) = u_z^{top}(x) S(t - t(x)); \;\; S(t) = 0 \text{ if } t < 0$$

$$\int_{-\infty}^{\infty} dx_0 \int_0^t dt_0 \;\; \dot{u}_z^{top}(x_0,t_0)\, e^{-i(k(\omega)x_0 - \omega t_0)} = \int_{-\infty}^{\infty} dx_0\, u_z^{top}(x_0) e^{-i(k(\omega)x_0 - \omega t(x_0))} \int_0^{t-t(x_0)} dt_0 \;\; \dot{S}(t_0)\, e^{i\omega t_0}$$
(5.2.3)

If S(t) was a step function, the last integral would equal 1 for $t>t(x_0)$ and 0 for $t<t(x_0)$. If S(t) was a ramp function, the last integral would equal $\min[1, (t- t(x_0))/T_R]$ for $t>t(x_0)$ and 0 for $t<t(x_0)$.

5.3 Initial Value Problems at the seafloor. To model a submarine earthquake or landslide, you might select seafloor vertical displacement to follow a certain uplift history. In this case we'd like the tsunami from an **initial vertical bottom displacement condition** like

$$u_Z(x,H,t=0) = u_Z^{bot}(x) \quad (5.3.1)$$

and its transform

$$u_Z^{bot}(k) = \int_{-\infty}^{\infty} dx\ u_Z^{bot}(x)\ e^{-ikx} \quad (5.3.2)$$

In this problem, you can't just plug in the eigenmodes (4.4.2) like we did for asteroid impacts because u_z in the eigenmodes vanish at the seafloor. There is no way to match (5.3.2). To solve this problem, we have to go all the way back to (4.1.8), now with (5.3.2) and (3.1.5)

$$\begin{bmatrix} u_z(k,0,\omega) \\ 0 \end{bmatrix} = \begin{bmatrix} C(0,H) - gkS(0,H)/\omega^2 & kS(0,H)/\rho_0\omega^2 \\ \dfrac{\rho_0 S(0,H)}{k}\left(\omega^2 - g^2k^2/\omega^2\right) & C(0,H) + gkS(0,H)/\omega^2 \end{bmatrix} \begin{bmatrix} u_z^{bot}(k,\omega) \\ p_1(k,H,\omega) \end{bmatrix} \quad (5.3.3)$$

where $C(z,z_1) = \cosh[k(z-z_1)]$, $T(z,z_1) = \tanh[k(z-z_1)]$, etc.

Solve the second equation of (5.3.3) first for pressure at the seafloor

$$P_1(k, H, \omega) = -\frac{\rho_0 g \omega^2 \left(1 - \omega^2 T(H,0)/kg\right)}{\omega^2 - \omega^2(k)} u_z^{bot}(k, \omega) \qquad (5.3.4)$$

then substitute into the first equation of (5.3.30 at any depth z to find displacement

$$u_z(k, z, \omega) = \frac{\begin{bmatrix} \omega^2\left(C(H,0)C(z,H) + S(z,H)S(H,0)\right) \\ -gk\left(C(z,H)S(H,0) + C(H,0)S(z,H)\right) \end{bmatrix}}{C(H,0)[\omega^2 - \omega^2(k)]} u_z^{bot}(k, \omega) \qquad (5.3.5)$$

Take care here to distinguish general frequency ω from the characteristic frequency ω(k). (5.3.5) is our first landslide tsunami. All we need to do is transform it back to time and space by (4.1.2b). As formulated above, $u_z^{bot}(k,\omega)$ actually is any function of time. If we want it to be a fixed initial uplift then

$$u_z^{bot}(k, \omega) = \frac{u_z^{bot}(k)}{-i\omega} \qquad (5.3.6)$$

The $(-i\omega)^{-1}$ is the time transform of a step function, thus

$$u_z(k, z, \omega) = \frac{\begin{bmatrix} i\omega\left(C(H,0)C(z,H) + S(z,H)S(H,0)\right) \\ -(igk/\omega)\left(C(z,H)S(H,0) + C(H,0)S(z,H)\right) \end{bmatrix}}{C(H,0)[\omega^2 - \omega^2(k)]} u_z^{bot}(k) \qquad (5.3.7)$$

We apply the inverse time transform, making use of the residue theorem to evaluate the simple poles that lay at ω=0, ω=±ω(k). Here's a useful table of transforms.

$$\frac{i\omega}{\omega^2-\omega^2(\mathrm{k})} \Leftrightarrow \cos[\omega(\mathrm{k})\mathrm{t}]\mathrm{H(t)}; \qquad \frac{\omega}{\omega^2-\omega^2(\mathrm{k})} \Leftrightarrow -i\cos[\omega(\mathrm{k})\mathrm{t}]\mathrm{H(t)}$$

$$\frac{1}{\omega(\omega^2-\omega^2(\mathrm{k}))} \Leftrightarrow \frac{-i\cos[\omega(\mathrm{k})\mathrm{t}]\mathrm{H(t)}}{\omega^2(\mathrm{k})}+\frac{i\mathrm{H(t)}}{\omega^2(\mathrm{k})}; \qquad \frac{i}{\omega^2-\omega^2(\mathrm{k})} \Leftrightarrow \frac{\cos[\omega(\mathrm{k})\mathrm{t}]\mathrm{H(t)}}{\omega^2(\mathrm{k})}-\frac{\mathrm{H(t)}}{\omega^2(\mathrm{k})} \qquad (5.3.8)$$

In the time domain (5.3.7) is

$$\mathrm{u_z(k,z,t)} = \frac{\left[\begin{array}{l}\left(C(\mathrm{H},0)C(\mathrm{z},\mathrm{H})+S(\mathrm{z},\mathrm{H})S(\mathrm{H},0)\right)\cos[\omega(\mathrm{k})\mathrm{t}] \\ +\dfrac{C(\mathrm{H},0)}{S(\mathrm{H},0)}\left(C(\mathrm{z},\mathrm{H})S(\mathrm{H},0)+C(\mathrm{H},0)S(\mathrm{z},\mathrm{H})\right)\left[-\cos[\omega(\mathrm{k})\mathrm{t}]+1\right]\end{array}\right]}{C(\mathrm{H},0)}\mathrm{u_z^{bot}(k)H(t)} \qquad (5.3.9)$$

$$= \frac{-S(\mathrm{z},\mathrm{H})}{C(\mathrm{H},0)S(\mathrm{H},0)}\mathrm{u_z^{bot}(k)}\cos[\omega(\mathrm{k})\mathrm{t}]\mathrm{H(t)}+\left(C(\mathrm{z},\mathrm{H})+\frac{S(\mathrm{z},\mathrm{H})}{T(\mathrm{H},0)}S(\mathrm{z},\mathrm{H})\right)\mathrm{u_z^{bot}(k)H(t)}$$

All we have to do now is the inverse wavenumber transform. For t>0 we have

$$\mathbf{u}(\mathrm{x,z,t}) = \mathrm{Re}\int_{-\infty}^{\infty}\mathrm{dk}\frac{\mathrm{u_z^{bot}(k)}}{2\pi\cosh(\mathrm{kH})}\left[\hat{\mathbf{z}}\frac{\sinh(\mathrm{k(H-z)})}{\sinh(\mathrm{kH})}-i\hat{\mathbf{x}}\frac{\cosh(\mathrm{k(H-z)})}{\sinh(\mathrm{kH})}\right]e^{i(\mathrm{kx}-\omega(\mathrm{k})\mathrm{t})}$$

$$+\int_{-\infty}^{\infty}\mathrm{dk}\frac{\mathrm{u_z^{bot}(k)}}{2\pi}\left[\hat{\mathbf{z}}\left\{\cosh(\mathrm{k(h-z)})-\frac{\sinh(\mathrm{k(h-z)})}{\tanh(\mathrm{kh})}\right\}-i\hat{\mathbf{k}}\left\{\sinh(\mathrm{k(h-z)})-\frac{\cosh(\mathrm{k(h-z)})}{\tanh(\mathrm{kh})}\right\}\right]e^{i\mathrm{kx}} \qquad (5.3.10\mathrm{a,b})$$

$$\mathbf{u}(x,z,t) = \mathrm{Re}\int_{-\infty}^{\infty} dk \frac{u_z^{bot}(k)}{2\pi\cosh(kH)}\left[\hat{\mathbf{z}}\frac{\sinh(k(H-z))}{\sinh(kH)} - i\hat{\mathbf{x}}\frac{\cosh(k(H-z))}{\sinh(kH)}\right] e^{i(kx-\omega(k)t)}$$

$$+\int_{-\infty}^{\infty} dk \frac{u_z^{bot}(k)}{2\pi}\left[\hat{\mathbf{z}}\left\{\cosh(k(h-z)) - \frac{\sinh(k(h-z))}{\tanh(kh)}\right\} - i\hat{\mathbf{k}}\left\{\sinh(k(h-z)) - \frac{\cosh(k(h-z))}{\tanh(kh)}\right\}\right] e^{ikx} \qquad (5.3.10a,b)$$

Equation (5.3.10) is the full 2-D tsunami wave solution for an instantaneous uplift of the sea bottom. It looks messy, but it is not so bad.

Consider integral (5.3.10b), see that time does not appear. This is a static field that does not propagate. Too, at the surface z=0 the vertical component of the static field vanishes.

At z=H the static term reduces to the vertical displacement at the sea floor as it should.

Recall that the propagating eigenmodes in (5.3.10a) have zero vertical motion at the sea floor. In a word, (5.3.10b), is needed to match the boundary conditions, but it is usually not of much interest.

Consider the surface vertical displacement of the propagating tsunami from an instantaneous bottom vertical bottom disturbance

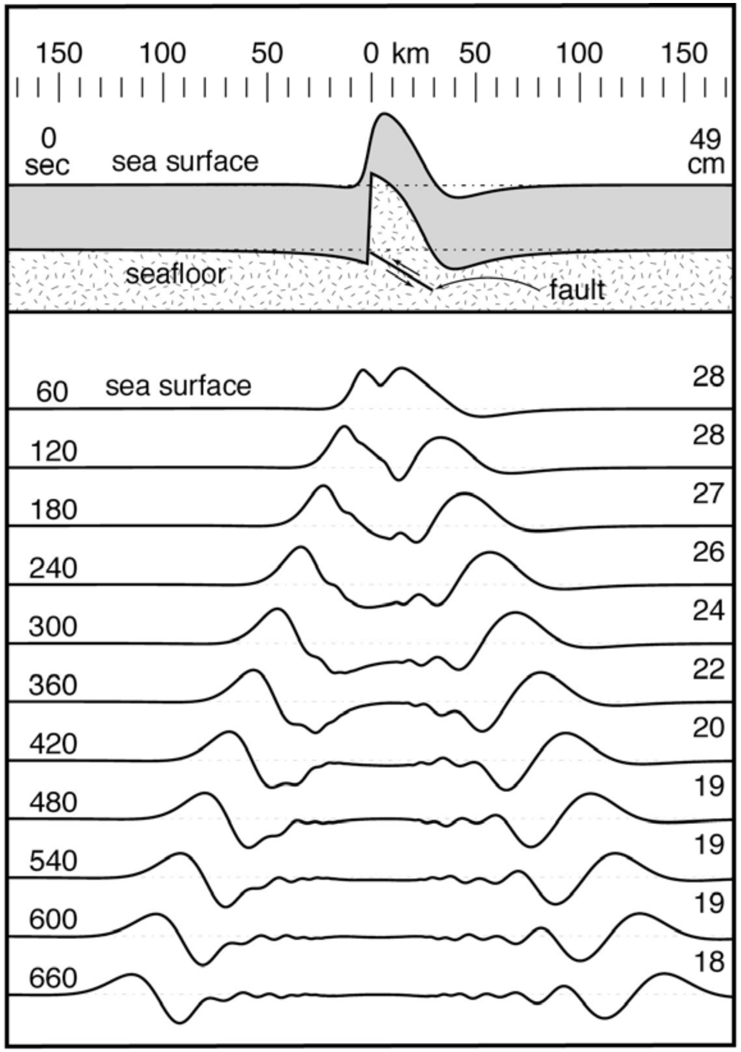
$$u_z(x,0,t) = \mathrm{Re}\int_{-\infty}^{\infty} dk \frac{u_z^{bot}(k)\, e^{i(kx-\omega(k)t)}}{2\pi \cosh(kH)} \qquad (5.3.11)$$

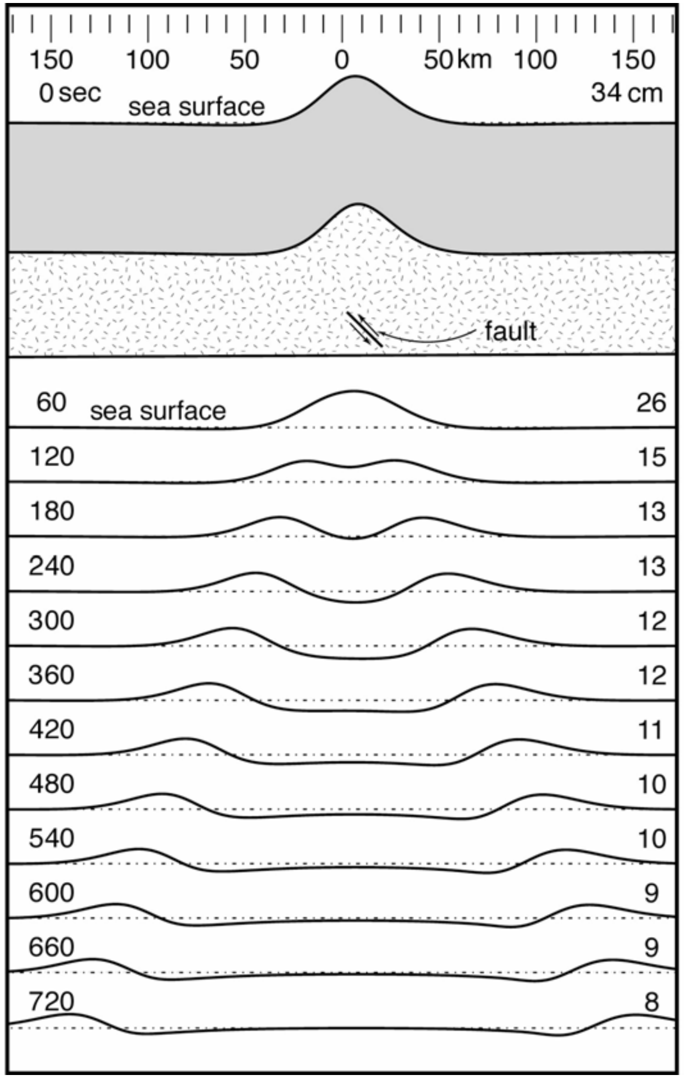
You can't get much simpler than this!

It is remarkable to me that the propagating tsunami from a bottom displacement is identical to the tsunami from the same surface displacement (5.1.3)

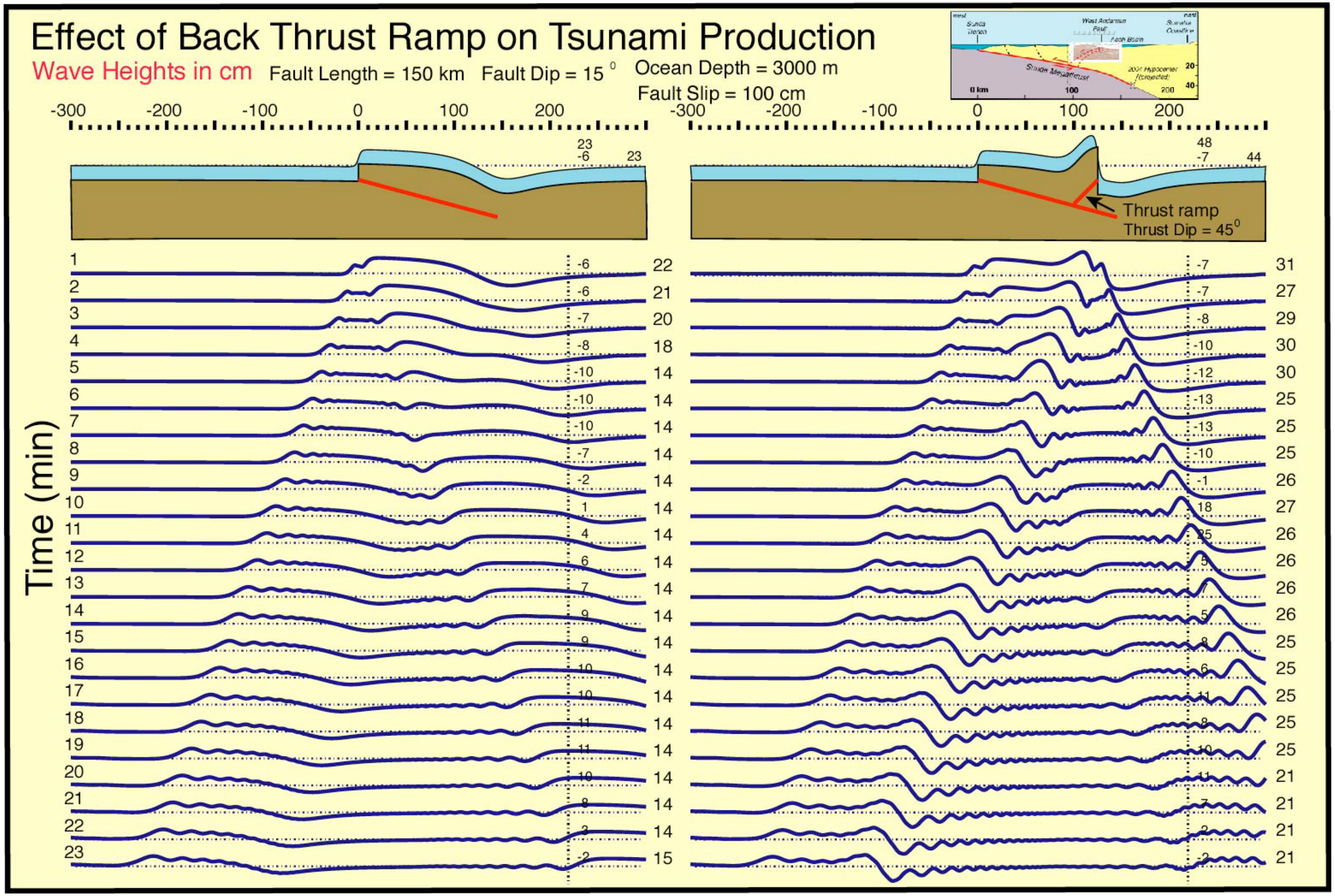
$$u_z(x,0,t) = \mathrm{Re}\int_{-\infty}^{\infty} dk \frac{u_z^{top}(k)}{2\pi} e^{i(kx-\omega(k)t)} \qquad (5.1.3)$$

except for the cosh(kH) term in the former. This term acts as a low pass filter. Short wavelength elements in the bottom uplift history, don't show up in the tsunami field at the surface. This helps us in the modeling because small details in landslides usually are not important.



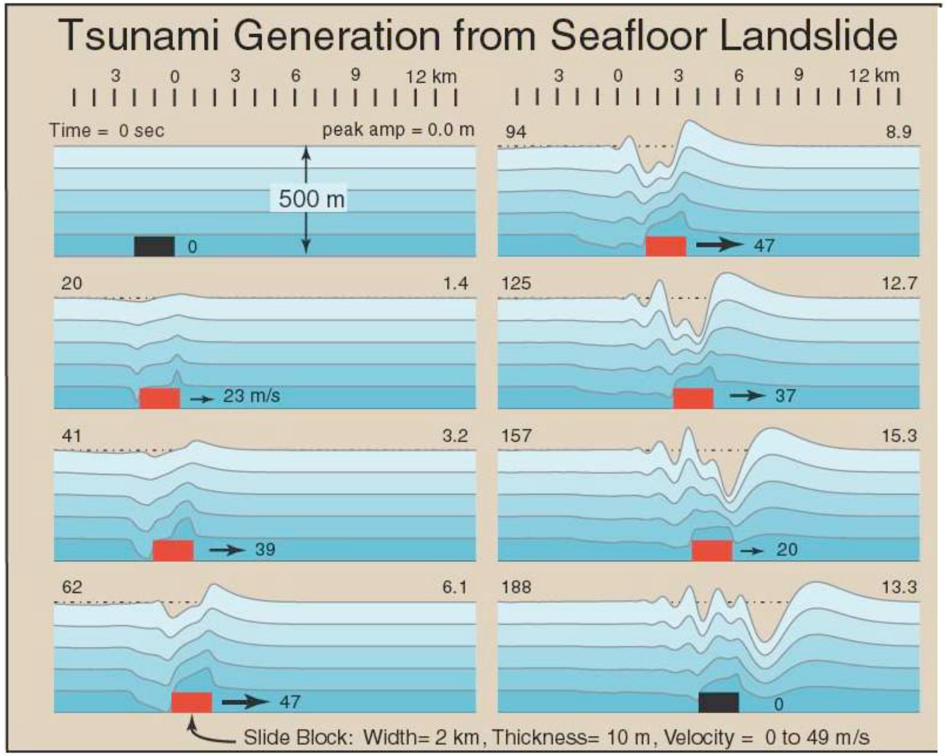


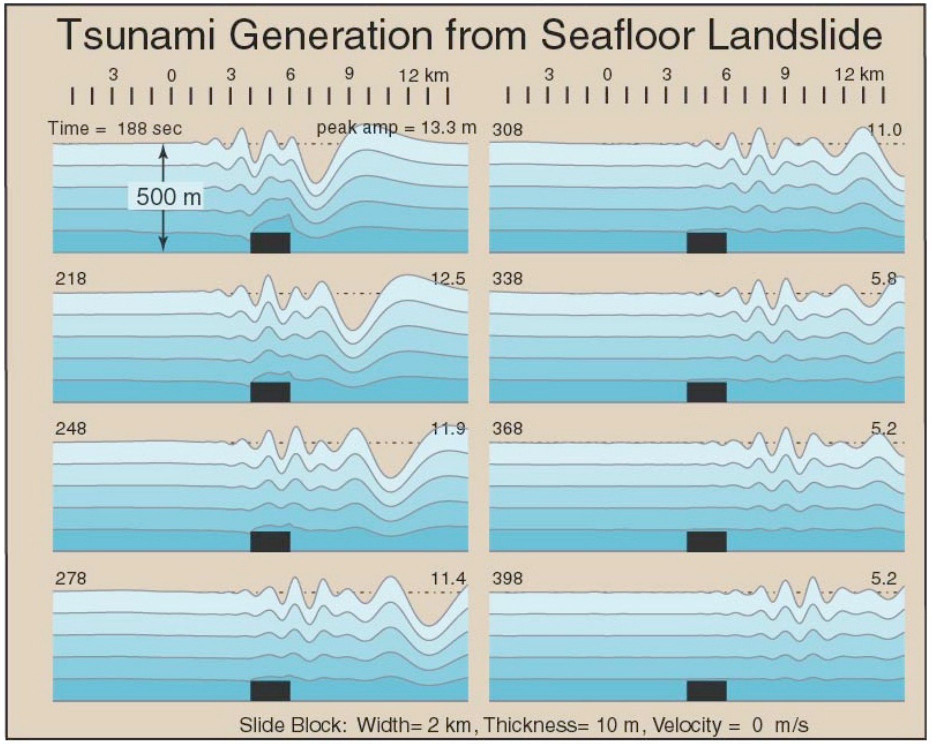
From this 2D theory (5.3.11) you can already put in the uplift from 2D dislocations and investigate the tsunami produced by quakes



2D tsunami theory applied to 2004 Sumatra subduction zone quake.

2D tsunami theory applied to landslides.





Because we use linear theory landslides are just a sequence of uplift sources distributed in space and time.

At front of slide bits of ocean bottom are moved up. At the back bits are moved down

$$\mathbf{u}(x,z,t) = \int_{-\infty}^{\infty} dk \, \frac{e^{ikx}}{2\pi\cosh(kH)}\left[\hat{\mathbf{z}}\frac{\sinh(k(H-z))}{\sinh(kH)} - i\hat{\mathbf{x}}\frac{\cosh(k(H-z))}{\sinh(kH)}\right]$$

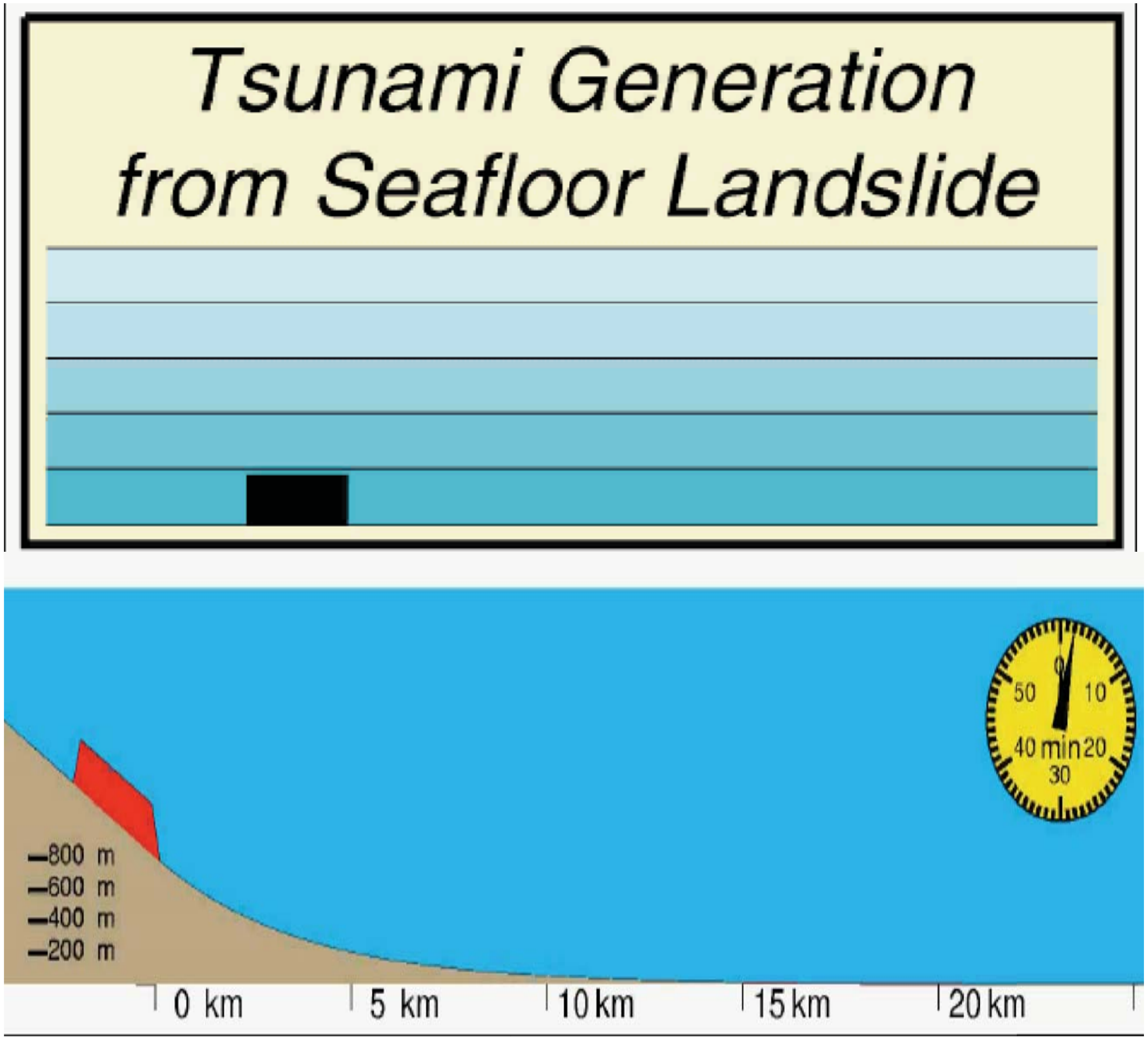
$$\times \int_{-\infty}^{\infty} dx_0 \, e^{-ikx_0} \int_0^t dt_0 \;\; \dot{u}_z^{bot}(x_0,t_0)\;\cos[\omega(k)(t - t_0)]$$

Movie Version In 2D

As material moves along the seafloor, the water must go UP, DOWN, or AROUND. This makes tsuanmi waves.

Landslide tsunami can be very directional depending on speed.

Whether a solid block or a disintegrating gravel pile, the process is the same.



$$\mathbf{u}(x,z,t) = \int_{-\infty}^{\infty} dk \, \frac{e^{ikx}}{2\pi \cosh(kH)} \left[\hat{\mathbf{z}} \frac{\sinh(k(H-z))}{\sinh(kH)} - i\hat{\mathbf{x}} \frac{\cosh(k(H-z))}{\sinh(kH)} \right]$$

$$\times \int_{-\infty}^{\infty} dx_0 \, e^{-ikx_0} \int_0^t dt_0 \;\; \dot{u}_z^{bot}(x_0,t_0) \; \cos[\omega(k)(t - t_0)]$$

6. Passage to 3-D

(8 slides till break)

6.1 Once we have 2-D tsunami fields, passage to 3-D is fairly easy. Here are the steps:

1) position x, and wave number k to go vectors $\mathbf{k}=k_x\hat{\mathbf{x}}+k_y\hat{\mathbf{y}}$, $k=|\mathbf{k}|$, $\mathbf{r}=x\hat{\mathbf{x}}+y\hat{\mathbf{y}}$

2) product kx goes to $\mathbf{k}\bullet\mathbf{r}$

3) 1-D integrals over wave number and position now cover all 2-D space

4) the unit vector $\hat{\mathbf{x}}$ goes to $\hat{\mathbf{k}}$

With these, the propagating 2-D tsunami from an arbitrary bottom landslide

$$\mathbf{u}(x,z,t)=\int_{-\infty}^{\infty}dk\,\frac{e^{ikx}}{2\pi\cosh(kH)}\left[\hat{\mathbf{z}}\frac{\sinh(k(H-z))}{\sinh(kH)}-i\hat{\mathbf{x}}\frac{\cosh(k(H-z))}{\sinh(kH)}\right]$$
$$\times\int_{-\infty}^{\infty}dx_0\;e^{-ikx_0}\int_0^t dt_0\;\dot{u}_z^{bot}(x_0,t_0)\;\cos[\omega(k)(t-t_0)] \tag{6.1.1}$$

becomes in 3-D

$$\mathbf{u}(x,y,z,t)=\int_{\mathbf{k}}d\mathbf{k}\,\frac{e^{i\mathbf{k}\bullet\mathbf{r}}}{4\pi^2\cosh(kH)}\left[\hat{\mathbf{z}}\frac{\sinh(k(H-z))}{\sinh(kH)}-i\hat{\mathbf{k}}\frac{\cosh(k(H-z))}{\sinh(kH)}\right]$$
$$\times\int_{\mathbf{r}_0}d\mathbf{r}_0\;e^{-i\mathbf{k}\bullet\mathbf{r}_0}\int_0^t dt_0\;\dot{u}_z^{bot}(\mathbf{r}_0,t_0)\;\cos[\omega(k)(t-t_0)] \tag{6.1.2}$$

where $d\mathbf{k}=dk_x dk_y$, $d\mathbf{r}_0=dx_0dy_0$.

There are millions of ways we can recast (6.1.2). Taking the origin of co-ordinates at a representative location in the source region, one rewrite is

$$\mathbf{u}(\mathbf{r},t)=\sum_{m=-\infty}^{m=-\infty}\int_0^{\infty}\frac{k\,dk}{2\pi\cosh(kh)}\left[\hat{\mathbf{z}}\frac{\sinh(k(h-z))}{\sinh(kh)}-\frac{\cosh(k(h-z))}{\sinh(kh)}[\hat{\mathbf{r}}k^{-1}\partial/\partial r+\hat{\theta}(kr)^{-1}\partial/\partial\theta]\right]J_m(kr)e^{im\theta}$$

$$\times\int_{A(t)}d\mathbf{r}_0J_m(kr_0)e^{-im\theta_0}\int_0^t dt_0\dot{u}_z^{bot}(\mathbf{r}_0,t_0)\cos(\omega(k)(t-t_0))\qquad(6.1.3)$$

In (6.1.3), the $J_m(x)$ are cylindrical Bessel functions, θ is azimuth angle measured from x toward y and r=|**r**|. Again We have here a sum of 1-D integrates over wavenumber. Not 2-D

Assumed to be known is $\dot{u}_z^{bot}(\mathbf{r}_0,t_0)$, the time derivative of the vertical displacement of the seafloor. $\dot{u}_z^{bot}(\mathbf{r}_0,t_0)$ integrated over the source area A(t) and source duration drives the tsunami.

The advantage of (6.1.3) is that the second integral does not depend on receiver position **r** so it can be done once separately. Also, the sum over azimuthal order is often limited to just a few terms.

Example: Instantaneous point moment tensor at depth d in halfspace.

We can do second integral in (6.1.3) exactly. Only m=-2 to 2 appear in azimuthal sum

$$u_z^{surf}(\mathbf{r},t) = \int_0^\infty k\,dk \frac{\cos\omega(k)t}{2\pi\cosh(kh)}\left[A\Delta u \mathbf{M}_{ij}\varepsilon_{ij}\right] \quad (6.1.4)$$

where

$$\varepsilon_{xx} = -\frac{1}{4}\left(\frac{\mu}{\lambda+\mu} - kd\right)\left[J_0(kr) - J_2(kr)\cos 2\theta\right]e^{-kd}$$

$$\varepsilon_{yy} = -\frac{1}{4}\left(\frac{\mu}{\lambda+\mu} - kd\right)\left[J_0(kr) + J_2(kr)\cos 2\theta\right]e^{-kd}$$

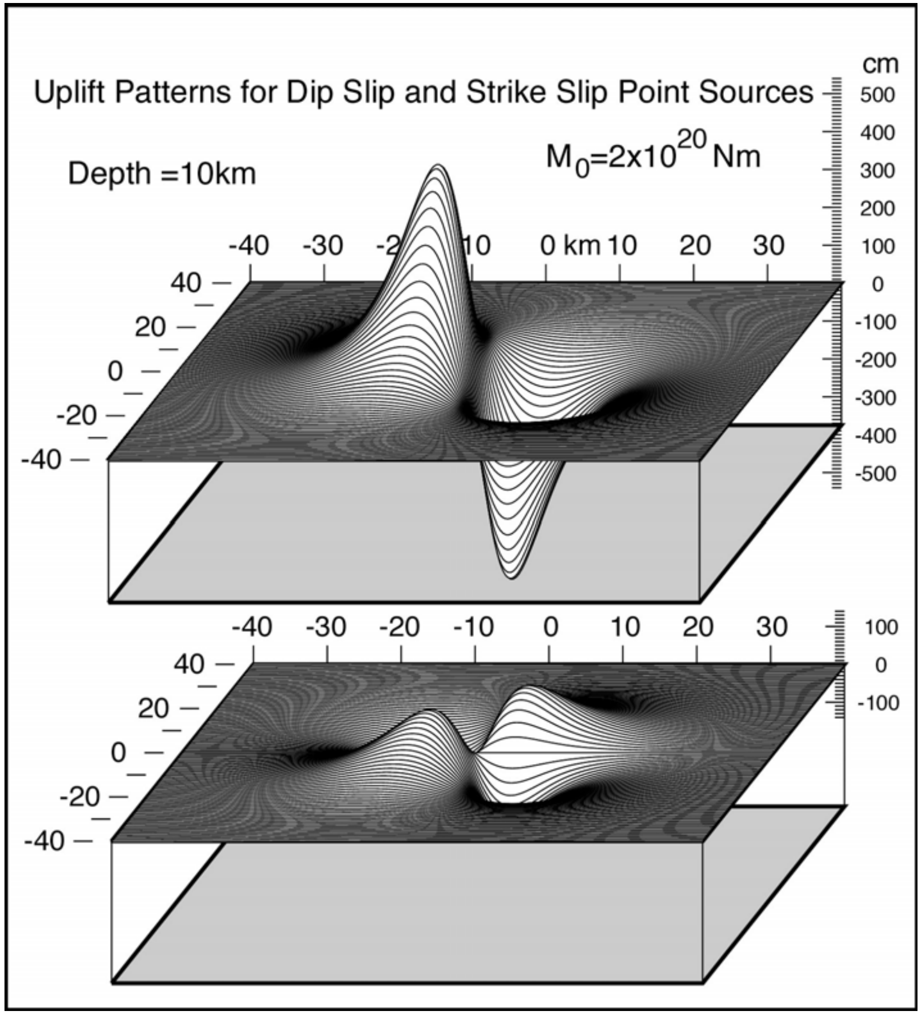
$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{4}\left(\frac{\mu}{\lambda+\mu} - kd\right)\left[J_2(kr)\sin 2\theta\right]e^{-kd}$$

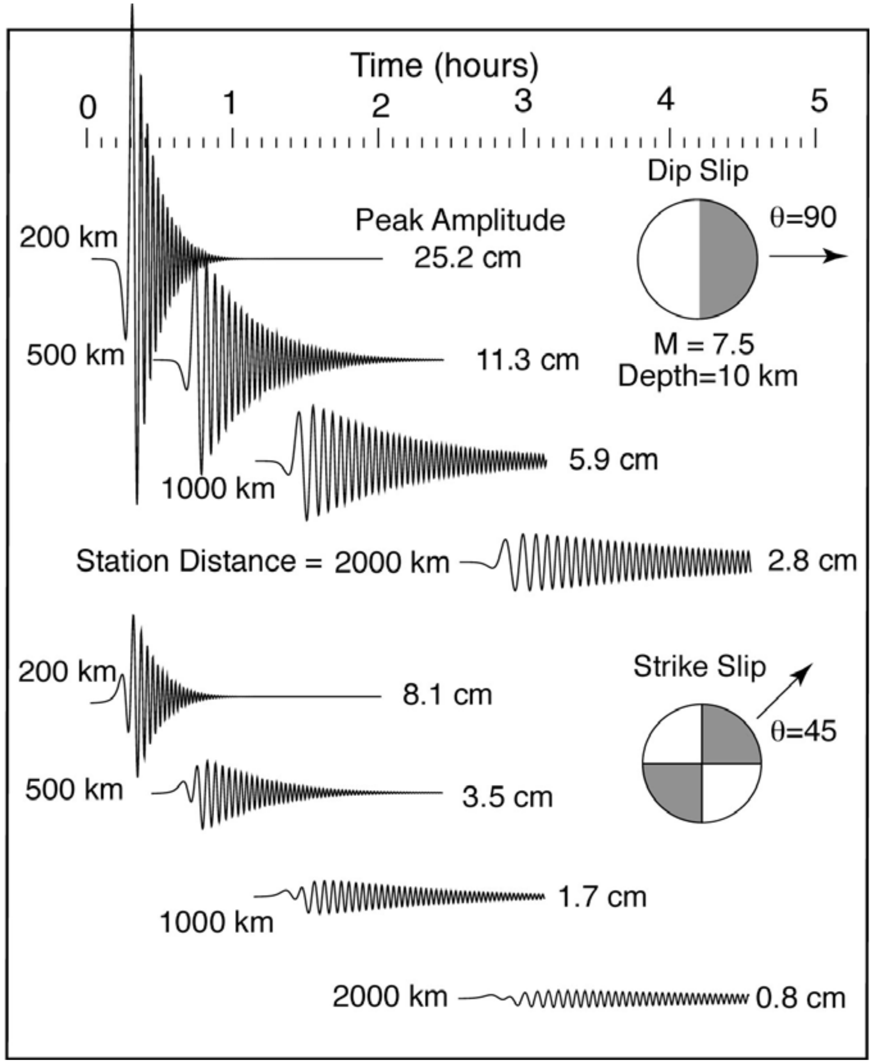
$$\varepsilon_{zz} = -\frac{1}{2}\left(\frac{\mu}{\lambda+\mu} + kd\right)\left[J_0(kr)\right]e^{-kd}$$

$$\varepsilon_{xz} = \varepsilon_{xz} = \frac{kd}{2}\left[J_1(kr)\cos\theta\right]e^{-kd}$$

$$\varepsilon_{yz} = \varepsilon_{zy} = \frac{kd}{2}\left[J_1(kr)\sin\theta\right]e^{-kd}$$

The six elements of symmetric tensor $M_{jk} = (\hat{\mathbf{a}}_j \hat{\mathbf{n}}_k + \hat{\mathbf{n}}_j \hat{\mathbf{a}}_k)$ (10) capsulize the mechanism of the earthquake. In (10), $\hat{\mathbf{n}}$, $\hat{\mathbf{a}}$ are the fault normal and slip vector. Note as d increases, all the strains (and tsunami) decrease.



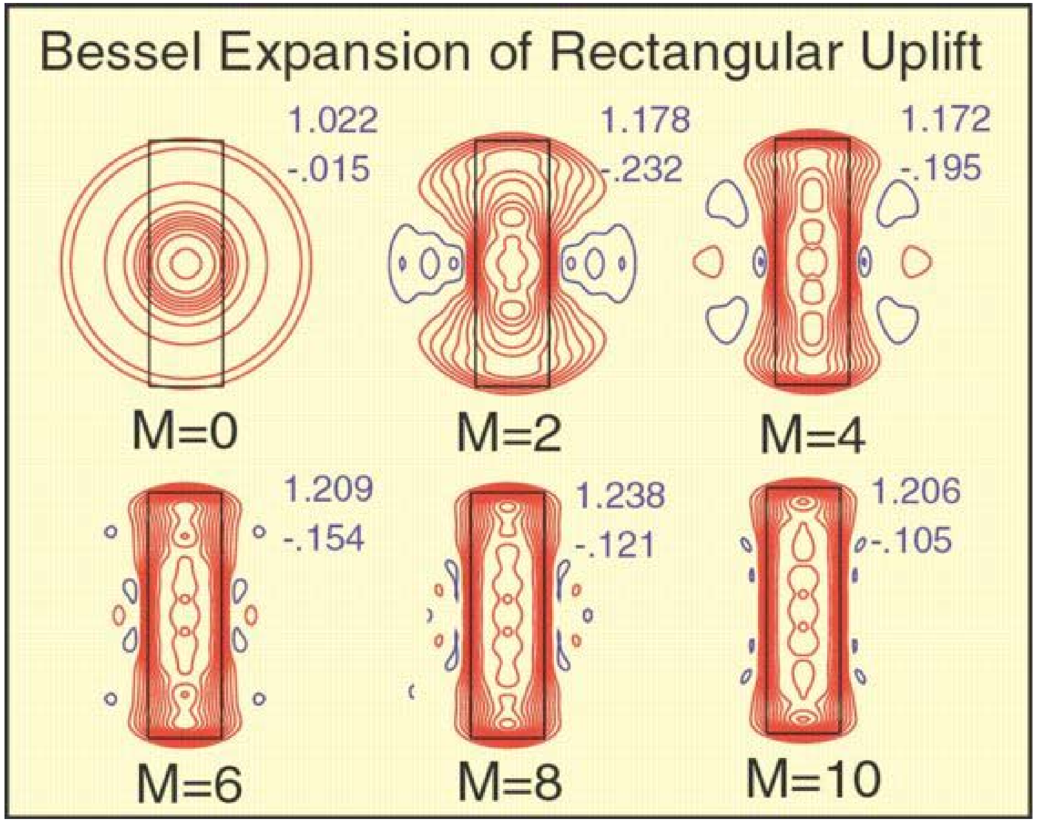


Point source uplift (left) and tsunami (right) for Dip Slip and Strike Slip Cases. Dip slip tsunami ~3 times smaller for fixed moment.

For Finite Faults you can either add point sources, or return to (6.1.4) and integrate over uplift.

$$u_z(x,y,t) = \sum_{m=-\infty}^{m=-\infty} \int_0^{\infty} \frac{k\,dk}{2\pi \cosh(kh)} J_m(kr) e^{im\theta} \cos(\omega(k)(t - t_0 - \tau/2)) \frac{\sin(\omega(k)\tau/2)}{-\omega(k)T_R/2} \Big|_{\tau=0}^{\tau=\min(t-t_0,T_R)} \int_{Area} d\mathbf{r}_0 U(\mathbf{r}_0) J_m(kr_0) e^{-im\theta_0}$$

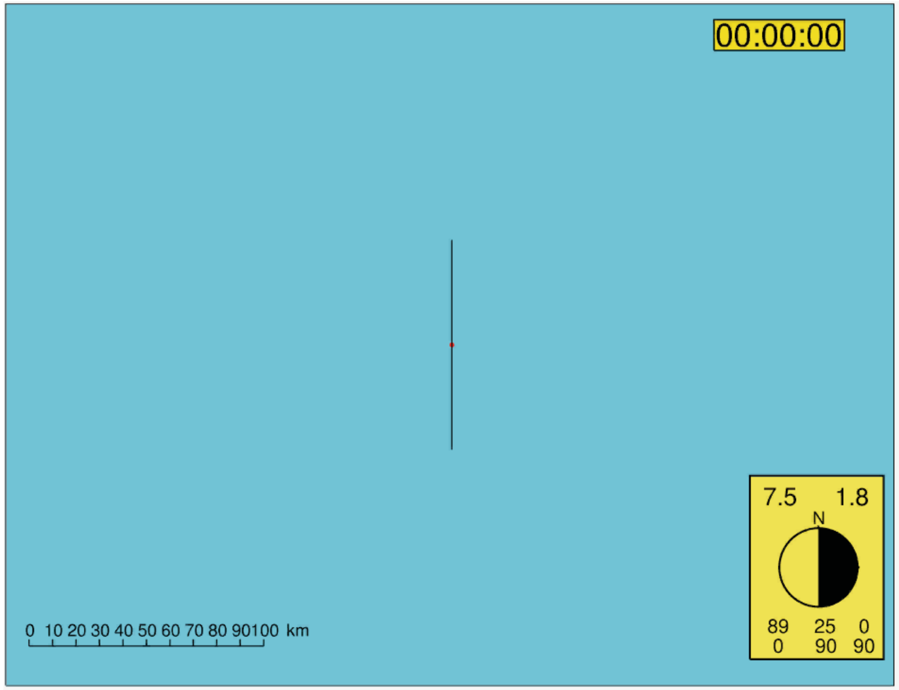
eg. Tsunami from "Piston source" uplifting U(**r**) meters over time T_R starting at t_0



Same formula would be used to make earthquake tsunami. Earthquake uplift is a piston source but with a more complex shape

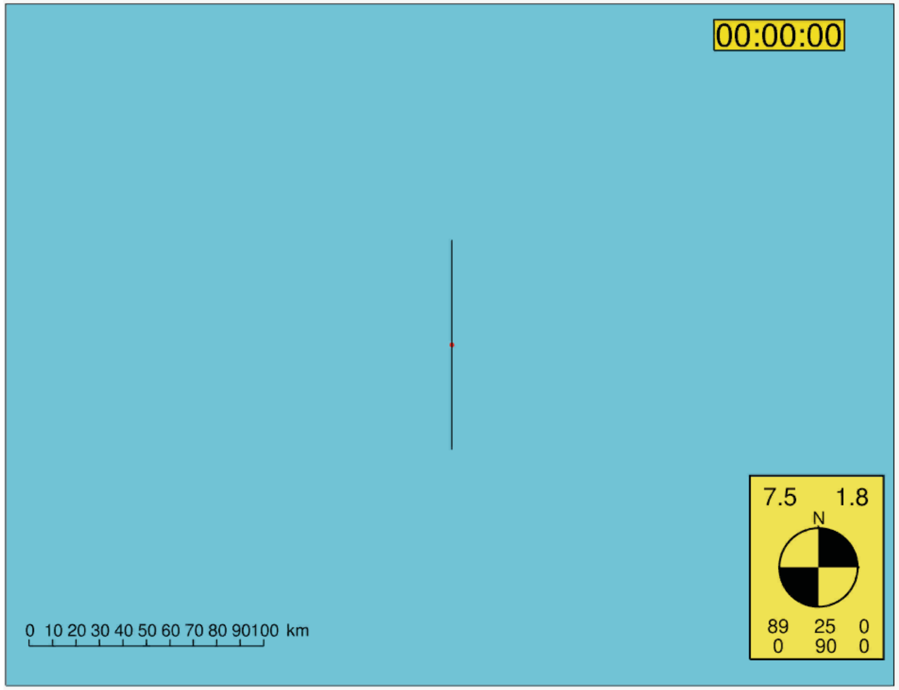
How big of tsunami depends on how much seafloor uplift there is. The amount of uplift depends mostly on the magnitude of the quake and the “Style” of faulting. Dip Slip or Strike Slip

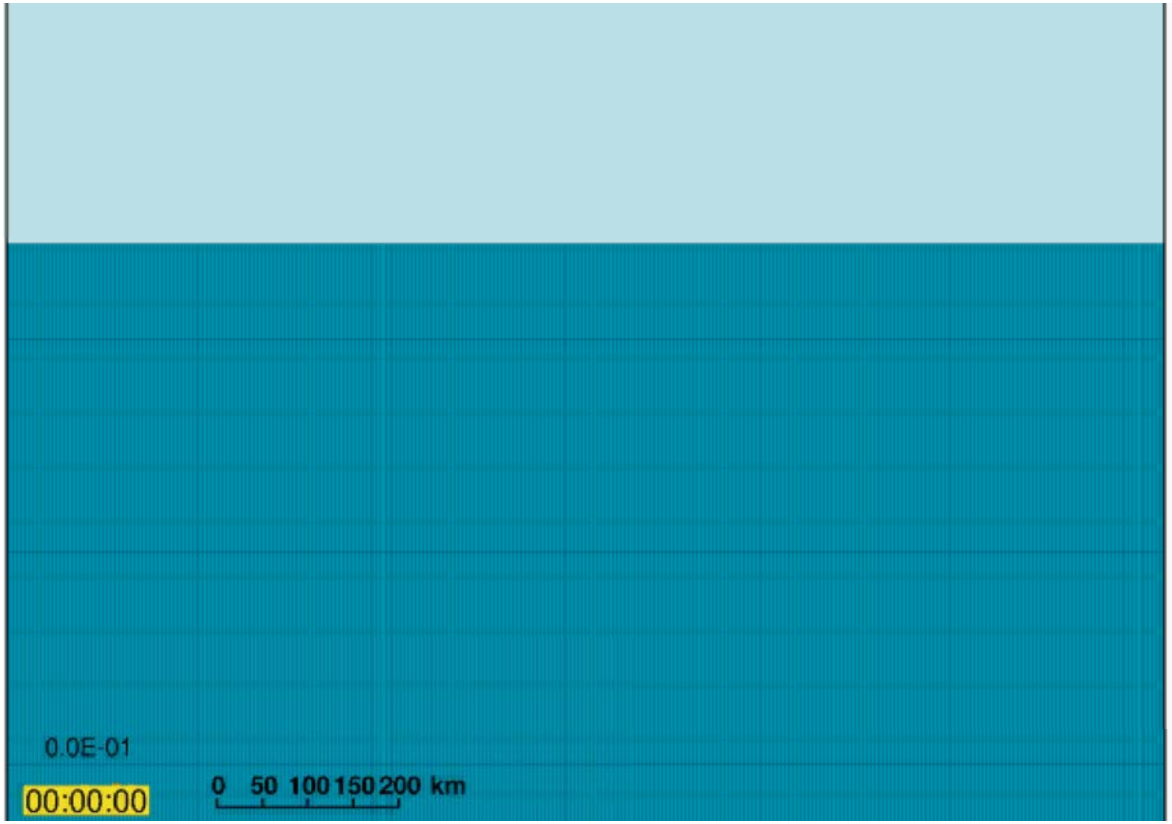
Dip Slip on vertical fault here. One side up, one side down. Ditto for the tsunami. Peak amp 49cm



Strike Slip on vertical fault here. Four lobes now. Ditto for the tsunami. Peak amp just 4cm!

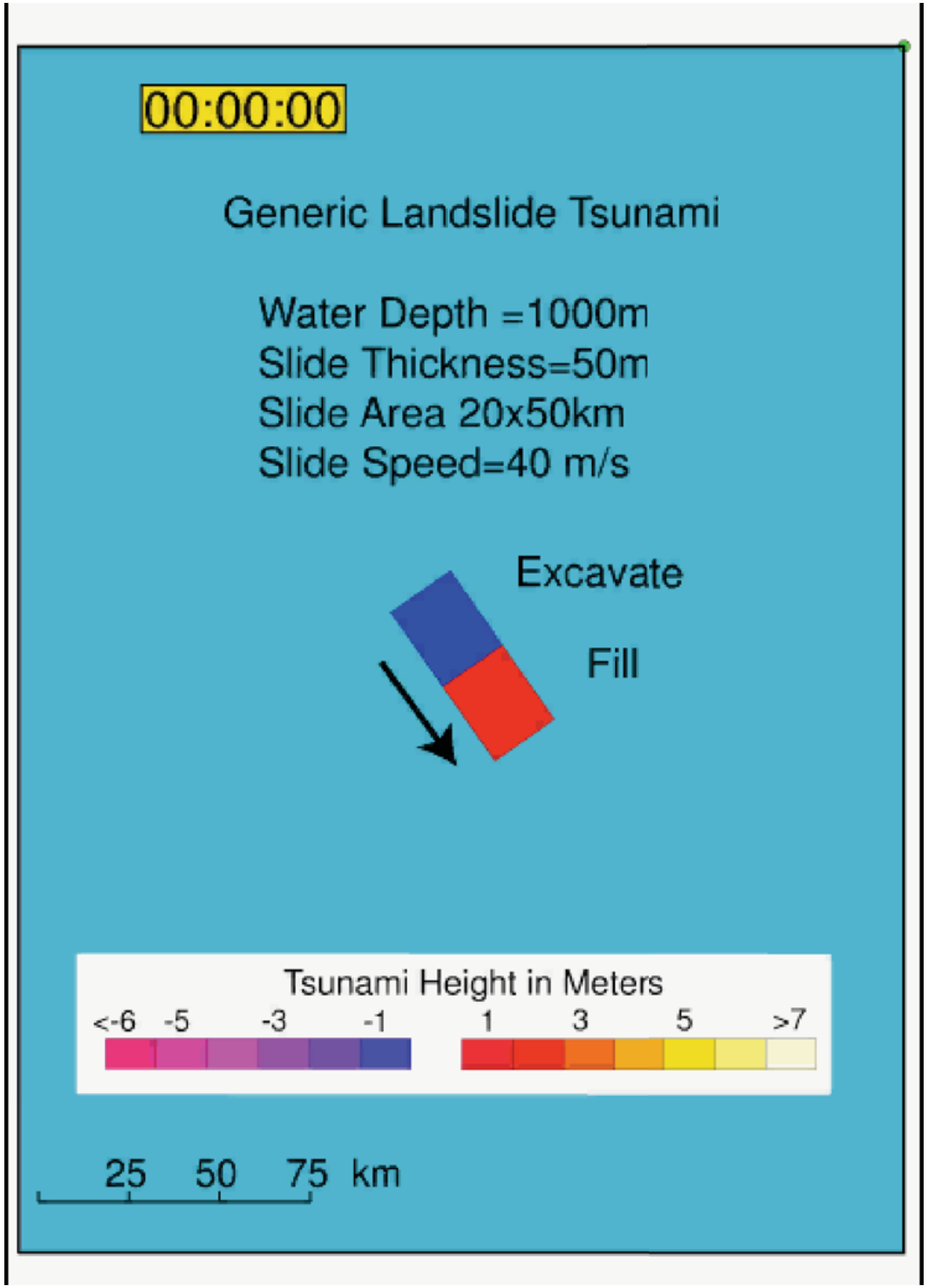
“Perpendicular to strike”





Generic 3D Asteroid Impact Tsunami

You can really see how strong dispersion is in this case. Instead if $r^{-1/2}$ waves decay more like $r^{-0.9}$ or so. How fast tsunami waves decay is a critical aspect of tsunami hazard forecasts.



Generic 3D Landslide Tsunami

Move Material from Blue area to Red area.

Strongest waves tend to go forward.

Positive Leading Wave Forward

Negative Leading Wave Back.

Things get more interesting in variable depth oceans.

Time for Lunch?